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Stability of quasiminimizers of the p-Dirichlet integral with varying p on metric spaces

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Abstract

We prove a stability result, with respect to the varying exponent p, for a family of quasiminimizers of the p-Dirichlet energy functional on a doubling metric measure space. In addition, we prove global higher integrability for upper gradients of quasiminimizers with fixed boundary data, provided that the boundary data belong to a slightly better Newtonian space.

1. *Introduction*

One of the most important elliptic variational problems studied in Euclidean spaces is to minimize the p-energy functional. This is equivalent to solving the p-harmonic equation. In a general metric measure space it is not clear what the counterpart to the p-harmonic equation is. However, in such a space, the variational approach to p-harmonic functions is available; it is possible to define p-harmonic functions as minimizers of the Dirichlet integral. The basic reason is that the Sobolev spaces on a metric measure space can be defined without the notion of partial derivatives; see, for example, [**8**, **23**]. Direct methods in the calculus of variations are also available and one can prove existence results for the Dirichlet problem; see, for example, [**2**, **24**].

A class of functions closely related to p-harmonic functions are *quasiminimizers*. A function u is called a quasiminimizer if it minimizes the Dirichlet functional up to some multiplicative constant K; that is,

$$\int |Du|^p \, dx \leqslant K \int |Dv|^p \, dx$$

among all functions v that have the same boundary values. The notion of quasiminimizers was introduced by Giaquinta and Giusti in [**7**] as a tool for unified treatment of variational integrals, elliptic equations and systems, obstacle problems and quasiregular mappings. In the setting of metric spaces, the approach via quasiminimizers is particularly useful, as the Euler equation for the p-Dirichlet energy integral does not need to exist.

In recent years several papers have been published considering quasiminimizers in the setting of doubling metric measure spaces supporting a Poincaré inequality; see, for example, [**4**, **3**, **15**, **16**]. All notions of metric measure spaces that appear here are explained in Section 2 below. (Local) Hölder continuity for quasiminimizers has been proved by Kinnunen and Shanmugalingam [**16**]. In [**15**], Kinnunen and Martio studied non-linear potential theory for quasiminimizers. Boundary continuity for quasiminimizers on a bounded set Ω with fixed boundary data was examined by Björn [**4**].

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Assume (X, μ, d) to be a complete, locally linearly connected (LLC), doubling metric measure space that supports a weak $(1, p)$-Poincaré inequality for some $p > 1$. Our main result is the following.

THEOREM 1.1. *Let Ω be an open and bounded subset of X such that $X \setminus \Omega$ is of positive p-capacity and uniformly p-fat. Let $w \in N^{1,s}(\Omega)$ for some $s > p$. Assume that $p = \lim_{i\to\infty} p_i$, and let $(u_i)_{i=1}^{\infty}$ be a sequence of K-quasiminimizers of the p_i-energy in Ω with boundary data w. If*

$$u_i \longrightarrow u \quad \mu\text{-a.e. in } \Omega,$$

then u is a K-quasiminimizer of the p-energy integral in Ω with boundary data w.

Note that since quasiminimizers do not provide unique solutions to the Dirichlet problem, in general, even if p does not vary, they may not converge. It was shown by Kinnunen and Martio that the class of (local) quasiminimizers, for p fixed, is closed under monotone convergence, provided that the limit function is bounded.

Formulated for quasiminimizers, our result is new also in the Euclidean setting. However, Li and Martio examined a quasilinear elliptic operator and proved in [**18**] a corresponding convergence result for solutions of an obstacle problem in a bounded subset Ω of $\mathbb{R}^n$. Later they proved a similar result for a double obstacle problem; see [**19**]. Both cases require a measure or a capacity thickness condition on the complement of Ω. Lindqvist considered stability of solutions to $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$, that is, minimizers to the corresponding variational problem with varying p. The problem is solved for a bounded subset of $\mathbb{R}^n$ in [**20**]. In [**21**] Lindqvist studies stability with respect to p of the p-harmonic eigenvalue problem. Here a question on regularity of the set Ω arises.

Roughly speaking, proofs of stability results can often be divided into two parts. First, with a higher integrability tool, such as the Gehring lemma, it is shown that the functions are better integrable than *a priori* assumed. Then it remains to prove the stability by using this information. The techniques that are needed in the second part vary. For instance, Li and Martio are dealing with an equation and thus they are able to exploit the monotonicity of the operator, whereas we have only the properties of quasiminimizers to hand.

We start by proving the global higher integrability of upper gradients of quasiminimizers in the setting of Theorem 1.1.

THEOREM 1.2. *Let Ω be an open and bounded subset of X such that $X \setminus \Omega$ is of positive p-capacity and uniformly p-fat. Let $w \in N^{1,s}(\Omega)$ for some $s > p$.*

If $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p-energy integral in Ω with boundary data w, then there exists a $\delta_0 = \delta_0(p) \leqslant s - p$ ($\delta_0 > 0$) such that $g_u \in L^{p+\delta}(\Omega)$ for all $\delta < \delta_0$ ($\delta > 0$) and

$$\left(\int_\Omega g_u^{p+\delta} d\mu\right)^{1/(p+\delta)} \leqslant c\left[\left(\int_\Omega g_u^{p} d\mu\right)^{1/p} + \left(\int_\Omega g_w^{p+\delta} d\mu\right)^{1/(p+\delta)}\right],$$

where c depends only on p and on the constants related to the space and to the domain Ω.

One standard, yet non-trivial, assumption in the metric setting is that the space satisfies a weak $(1, q)$-Poincaré inequality for some $q < p$, where p is the natural exponent associated with the problem studied. However, as shown by Keith and Zhong [**12**], the Poincaré inequality is a self-improving property. In quite general spaces a weak $(1, p)$-Poincaré implies a weak $(1, q)$-Poincaré for some $q < p$. The same holds also for a p-fatness condition, that is, a capacity thickness property of a set. We refer the reader to Paragraphs 2.1.2 and 2.1.7, respectively.

The paper is organized as follows. In Section 2 we fix the general setup and we present basic facts about analytic tools used in a metric setting. Most of the results are stated without proofs; in some cases we add the proof for the reader's convenience. Since no single sufficient reference exists, we decided to collect all the definitions needed, in Subsection 2.1. For more detail we refer to [**4**, **2**, **5**, **8**, **9**, **11**, **13**, **15**, **16**, **23**, **24**]. The reader familiar with metric measure spaces may omit this part. Section 3 contains the proof of Theorem 1.2, and Section 4 the proof of the stability result.

2. *Preliminaries*

We remind the reader that throughout the paper p is a real number such that $1 < p < \infty$.

Our notation is standard. We assume that a ball comes always with a centre and a radius; that is, $B = B(x, r) = \{y \in X : d(x, y) < r\}$ with $0 < r < \infty$. We denote

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u\, d\mu,$$

and when there is no possibility of confusion, we denote by λB a ball with the same centre as B but λ times its radius.

Throughout the paper we assume (X, μ, d) to be a complete metric space equipped with a Borel regular measure μ satisfying $0 < \mu(B) < \infty$ for all balls B of X. We will assume that the measure is *doubling*; that is, there exists a constant $c_d > 0$ such that for every ball B in X

$$\mu(2B) \leqslant c_d \mu(B).$$

We refer to this property by calling (X, d, μ), or briefly X, a *doubling metric measure space*. A doubling metric measure space that is complete is always *proper*; that is, its closed and bounded subsets are compact. In addition, we will assume that X is an LLC space.

Unless otherwise mentioned, all constants depend only on the constants of the space X, that is, the doubling constant and the constant of the Poincaré inequality. We allow dependence on the domain Ω and on its characteristic constants,which are clear in each context. Constants may also depend on the quasiminimality constant K.

2.1. *Basic definitions*

2.1.1. *Upper gradients.* Let u be a real-valued function on X. A non-negative Borel measurable function g on X is said to be an *upper gradient* of u if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leqslant \int_\gamma g\, ds. \tag{2.1}$$

If the above property fails only for a set of paths that is of zero p-modulus (see, for example, [**11**, Section 2.3] for the definition of the p-modulus of a family of paths), then g is said to be a *p-weak upper gradient* of u. We recall that if $1 \leqslant p < \infty$, every function u that has a p-integrable p-weak upper gradient has a *minimal p-integrable p-weak upper gradient* denoted by g_u.

It is important to notice that for every $c \in \mathbb{R}$ the minimal p-weak upper gradient satisfies $g_u = 0$ μ-almost everywhere on the set $\{x \in X : u(x) = c\}$.

2.1.2. *Poincaré inequality.* We say that the space supports a *weak $(1, q)$-Poincaré inequality* if there exist $c > 0$ and $\tau \geqslant 1$ such that

$$\fint_B |u - u_B|\, d\mu \leqslant cr \left(\fint_{\tau B} g^q\, d\mu \right)^{1/q}$$

for all balls $B(x,r)$ in X and all pairs $\{u,g\}$, where u is a locally integrable function on X and g is a q-weak upper gradient of u. A result of [9] shows that in a doubling measure space a weak $(1,q)$-Poincaré inequality implies a weak (t,q)-Poincaré inequality for some $t>q$ and possibly a new τ; that is, there exist $c'>0$ and $\tau'\geqslant 1$ such that

$$\left(\fint_B |u-u_B|^t\,d\mu\right)^{1/t} \leqslant c'r\left(\fint_{\tau' B} g^q\,d\mu\right)^{1/q}, \tag{2.2}$$

where

$$\begin{cases} 1\leqslant t\leqslant Qq/(Q-q) & \text{if } q<Q,\\ 1\leqslant t & \text{if } q\geqslant Q,\end{cases}$$

for all balls B in X, and $Q=\log_2 c_d$.

We assume that X supports a weak $(1,p)$-Poincaré inequality. In a complete doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality, there exists a $q<p$ $(q>1)$ such that the space supports a weak $(1,q)$-Poincaré inequality by a result in [12]. Increasing q if necessary, we may additionally assume that $p\in(q,q^*)$, where $q^*=qQ/(Q-q)<\infty$.

2.1.3. *Newtonian spaces.* We define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p-integrable functions u on X that have a p-integrable p-weak upper gradient g on X. This space is equipped with the seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)}=\|u\|_{L^p(X)}+\inf\|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients of u. We define the equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u\sim v$ if

$$\|u-v\|_{\tilde{N}^{1,p}(X)}=0.$$

The *Newtonian space* $N^{1,p}(X)$ is then defined to be the space $\tilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)}=\|u\|_{\widetilde{N}^{1,p}(X)}.$$

2.1.4. *Capacity.* The *p-capacity* of a set $E\subset X$ is defined by

$$C_p(E)=\inf_u\|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u\in N^{1,p}(X)$ such that $u\geqslant 1$ on E. We say that a property holds *p-quasi everywhere* (p-q.e.) if the set of points for which the property fails is of zero p-capacity.

Let Ω be a bounded subset of X and let $E\subset\subset\Omega$; that is, E is compactly contained in Ω. We define the *relative p-capacity* of E with respect to Ω by

$$\operatorname{cap}_p(E,\Omega)=\inf_u\int_\Omega g_u^p\,d\mu,$$

where the infimum is taken over all $u\in N^{1,p}(\Omega)$ such that $u\geqslant 1$ on E and $u=0$ on $X\setminus\Omega$ p-quasi everywhere. Lemma 2.2 in Subsection 2.2 shows that in a doubling metric measure space supporting a weak Poincaré inequality, the measure and the capacities are comparable.

2.1.5. *Newtonian spaces with zero boundary values.* Let Ω be an arbitrary subset of X. We define $N_0^{1,p}(\Omega)$ to be the set of functions $u\in N^{1,p}(X)$ that are zero on $X\setminus\Omega$ p-quasi everywhere. The space $N_0^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{N_0^{1,p}(\Omega)}=\|u\|_{N^{1,p}(\Omega)}.$$

There are several approaches to defining Newtonian spaces with zero boundary values. In general these approaches imply different spaces, but it can be shown that for a wide class of metric spaces the definitions agree. Let us present two other definitions based on Lipschitz functions.

Define $\mathrm{Lip}_0^{1,p}(\Omega)$ to be the collection of all Lipschitz functions in $N^{1,p}(X)$ that vanish on $X \setminus \Omega$ and let $\mathrm{Lip}_{C,0}^{1,p}(\Omega)$ be the collection of functions in $\mathrm{Lip}_0^{1,p}(\Omega)$ that have compact support in Ω. Let $H_0^{1,p}(\Omega)$ be the closure of $\mathrm{Lip}_0^{1,p}(\Omega)$ in the norm of $N^{1,p}(X)$, and let $H_{C,0}^{1,p}(\Omega)$ be the closure of $\mathrm{Lip}_{C,0}^{1,p}(\Omega)$ in the norm of $N^{1,p}(X)$. If X is a proper, doubling metric measure space supporting a $(1,p)$-Poincaré inequality and Ω is an open subset of X, then

$$H_{C,0}^{1,p}(\Omega) = H_0^{1,p}(\Omega) = N_0^{1,p}(\Omega).$$

The subject is discussed and the equality is proved in [**24**].

2.1.6. *Quasiminimizers.* Let Ω be an open subset of X. Let $w \in N^{1,p}(\Omega)$. We say that $u \in N^{1,p}(\Omega)$ is a *quasiminimizer* of the p-energy integral in Ω with boundary data w, if $u - w \in N_0^{1,p}(\Omega)$ and there exists a constant $K > 0$ such that for all open $\Omega' \subset\subset \Omega$ and all $\phi \in N_0^{1,p}(\Omega')$ we have

$$\int_{\overline{\Omega'}} g_u^p \, d\mu \leqslant K \int_{\overline{\Omega'}} g_{u+\phi}^p d\mu. \tag{2.3}$$

Quasiminimizers can be defined in several equivalent ways. For example, the integral can be taken just over Ω' instead of its closure. Also, requiring Ω' to be compactly contained in Ω is not necessary. As for test functions, it is possible to use compactly supported Lipschitz functions or $\phi \in N^{1,p}(\Omega)$ such that $\operatorname{supp} \phi \subset\subset \Omega$ instead of $N_0^{1,p}(\Omega)$-functions. Also, in these cases the integral in (2.3) can be taken over the support of ϕ or the set $\{\phi \neq 0\}$. For further discussion and the equivalence proof, see [**1**].

2.1.7. *LLC-property and p-fatness.* The local linear connectivity, that is, the LLC-property of X, means that there exist constants $C \geqslant 1$ and $r_0 > 0$ such that for all balls B in X with radius at most r_0, every pair of points in the annulus $2B \setminus \overline{B}$ can be connected by a curve lying in the annulus $2CB \setminus C^{-1}\overline{B}$. Notice that the definition of LLC that we assume here is the same as in [**5**], and is stronger than the definition in [**11**]. It can be shown that, for example, Ahlfors s-regular Loewner spaces supporting a weak $(1,s)$-Poincaré inequality satisfy both versions of LLC; see [**11**].

In general, the Poincaré inequality does not imply LLC (not even the weak version). However, besides the Loewner case above, this is known to be true, for example, in complete spaces with a doubling measure that satisfies

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \leqslant c\Big(\frac{r}{R}\Big)^s$$

for all $x \in X$ and $0 < r \leqslant R$. With these assumptions a $(1,p)$-Poincaré inequality implies LLC for all $p \leqslant s$; see [**17**, Theorem 3.3] and [**9**, Proposition 4.5].

We say that the set $E \subset X$ is uniformly p-fat if there exist constants $c_f > 0$ and $r_0 > 0$ such that for all $x \in E$ and $0 < r < r_0$, we have

$$\operatorname{cap}_p(E \cap B(x,r); B(x,2r)) \geqslant c_f \operatorname{cap}_p(B(x,r); B(x,2r)).$$

If X is a proper, LLC, doubling metric measure space supporting a $(1,q)$-Poincaré inequality for some $1 < q < p$, and Ω is an open and bounded subset of X such that $\operatorname{cap}_p(X \setminus \Omega) > 0$ and $X \setminus \Omega$ is uniformly p-fat, then [**5**, Theorem 1.2] says that $X \setminus \Omega$ is also uniformly p_0-fat for some $p_0 < p$.

2.2. *Preliminary results*

Here we collect some basic facts concerning the properties of capacity, Newtonian spaces and Sobolev–Poincaré-type inequalities in the metric setting.

We start with an upper gradient lemma. Its proof follows the same method as does the proof of [15, Lemma 2.4].

LEMMA 2.1. *Suppose that $u, v \in N^{1,p}(X)$ and that η is a Lipschitz continuous function in X with $0 \leqslant \eta \leqslant 1$. Let g_u, g_v and g_η be the p-weak upper gradients of u, v and η, respectively. Define $w = u + \eta(v - u)$. Then*

$$g_w \leqslant (1-\eta)g_u + \eta g_v + |v-u| g_\eta$$

μ-almost everywhere in X.

The next lemma provides an estimate for the capacity of a ball and shows that capacities cap_p and C_p are essentially equivalent. For the proof, see [4].

LEMMA 2.2. *Let X be a doubling metric measure space supporting a weak $(1, q)$-Poincaré inequality and let $E \subset B = B(x_0, r)$ with $0 < r < \operatorname{diam} X/6$. There exists a $c > 0$ such that*

$$\frac{\mu(E)}{cr^q} \leqslant \mathrm{cap}_q(E, 2B) \leqslant \frac{c\mu(B)}{r^q} \tag{2.4}$$

and

$$\frac{C_q(E)}{c(1+r^q)} \leqslant \mathrm{cap}_q(E, 2B) \leqslant 2^{q-1}\left(1 + \frac{1}{r^q}\right) C_q(E).$$

The following proposition is a capacity version of the Sobolev–Poincaré inequality. The proof is a straightforward generalization of the Euclidean case; nevertheless we present it here for the reader's convenience. One can also see [4] for a proof of the appropriate Poincaré inequality.

PROPOSITION 2.3. *Let X be a doubling metric measure space supporting a weak $(1, q)$-Poincaré inequality and u a q-quasicontinuous function in $N^{1,q}(X)$. Then there exists a $c > 0$ such that for all balls B in X and $S = \{x \in \frac{1}{2}B \colon u(x) = 0\}$ the inequality*

$$\left(\fint_B |u|^t \, d\mu\right)^{1/t} \leqslant \left(\frac{c}{\mathrm{cap}_q(S, B)} \int_{\tau' B} g_u^q \, d\mu\right)^{1/q} \tag{2.5}$$

holds for t, and τ' are as in (2.2).

Proof. If $u_B = 0$ then the assertion follows from the (t, q)-Poincaré inequality (2.2) and (2.4). We may thus assume that $u_B = 1$. Take a Lipschitz cut-off function η such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $\frac{1}{2}B$, $\operatorname{supp} \eta \subset B$ and $g_\eta \leqslant c/r$. Then $\phi = -\eta(u - u_B) \in N_0^{1,q}(B)$ and $\phi = 1$ on S. Therefore

$$\mathrm{cap}_q(S, B) \leqslant \int_B g_\phi^q \, d\mu.$$

Since $g_\phi \leqslant |u - u_B| g_\eta + \eta g_u$ μ-a.e., we have

$$\mathrm{cap}_q(S, B) \leqslant \frac{c}{r^q} \int_B |u - u_B|^q \, d\mu + c \int_B g_u^q \, d\mu.$$

The space X supports the (q, q)-Poincaré inequality, so we obtain

$$\mathrm{cap}_q(S, B) \leqslant c \int_{\tau' B} g_u^q \, d\mu,$$

and therefore

$$1 = u_B \leqslant \left(\frac{c}{\operatorname{cap}_q(S,B)} \int_{\tau' B} g_u^q \, d\mu \right)^{1/q}.$$

We can now estimate

$$\begin{aligned}\left(\fint_B |u|^t \, d\mu \right)^{1/t} &\leqslant c \left(\fint_B |u - u_B|^t \, d\mu \right)^{1/t} + c u_B \\ &\leqslant \left(\frac{c}{\operatorname{cap}_q(S,B)} \int_{\tau' B} g_u^q \, d\mu \right)^{1/q},\end{aligned}$$

by the (t, q)-Poincaré inequality and (2.4). □

The next lemma is a Sobolev-type inequality for Newtonian functions with zero boundary values. For a proof see [**4**] or [**16**].

LEMMA 2.4. *Let $1 < p < \infty$, and let X let be a doubling metric measure space supporting a weak $(1, q)$-Poincaré inequality for some $1 < q < p$. Moreover, let $u \in N_0^{1,q}(B)$, and let the radius r of B be at most* $\operatorname{diam} X/3$. *Then*

$$\left(\fint_B |u|^t \, d\mu \right)^{1/t} \leqslant cr \left(\fint_B g_u^q \, d\mu \right)^{1/q},$$

where t is as in (2.2).

Next we present some useful results concerning Newtonian spaces with zero boundary values. Proposition 2.5 provides a characterization for $N_0^{1,p}$-functions by means of the Hardy inequality. Lemma 2.6 gives a sufficient condition for a sequence of $N_0^{1,p}$-functions to converge to a $N_0^{1,p}$-function. Finally, Lemma 2.7 shows that $N_0^{1,p}$ can be presented as an intersection of $N^{1,p}$ and of zero Newtonian spaces with lower exponents. For a proof of the following proposition, see [**5**].

PROPOSITION 2.5. *Let X be a proper, doubling,* LLC *metric measure space supporting a weak $(1, q)$-Poincaré inequality for some $1 < q < p$, and suppose that Ω is a bounded domain in X such that $X \setminus \Omega$ is uniformly p-fat. Then there is a constant $c(\Omega, p) > 0$ such that a function $u \in N^{1,p}(X)$ is in $N_0^{1,p}(\Omega)$ if and only if*

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p d\mu \leqslant c \int_\Omega g_u(x)^p \, d\mu. \tag{2.6}$$

Note that the constant c in the above proposition formally depends on p. However, if p varies inside a bounded interval, then the arguments in the proof of Proposition 2.5 show that the appropriate constants are uniformly bounded. For this reason, since in our case all exponents vary inside a bounded interval (q, q^*) we omit the dependence of the constant on p.

LEMMA 2.6. *In the setting of Proposition* 2.5, *let $(u_i)_{i=1}^\infty$ be a bounded sequence in $N_0^{1,p}(\Omega)$. If $u_i \to u$ μ-a.e., then $u \in N_0^{1,p}(\Omega)$.*

Lemma 2.6 is formulated in [**13**] for (X, d, μ) doubling without further requirements and for Ω open such that $X \setminus \Omega$ satisfies a measure thickness assumption. In general a measure thickness condition is stronger than a fatness assumption. However, the lemma follows also from Proposition 2.5 and the fact that $u \in N^{1,p}(\Omega)$ is in $N_0^{1,p}(\Omega)$ if

$$\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)}$$

is in $L^p(\Omega)$ for an open Ω and $1 < p < \infty$. See [**5**, **13**].

The assertion of the next proposition is not trivial but depends on the set Ω. Even in $\mathbb{R}^n$ some type of thickness assumption on the domain is needed; see [**10**]. Li and Martio show in [**18**] that, for example, p-fatness of $\mathbb{R}^n \setminus \Omega$ suffices for (2.7) to hold. The same result exists in the metric case and the proof follows from Proposition 2.5.

Proposition 2.7. *Let X be a proper, doubling,* LLC *metric measure space supporting a weak $(1, q)$-Poincaré inequality for some $1 < q < p$, and suppose that Ω is a bounded domain in X such that $X \setminus \Omega$ is uniformly p-fat. Then*

$$N_0^{1,p}(\Omega) = N^{1,p}(\Omega) \cap \bigcap_{s<p} N_0^{1,s}(\Omega). \tag{2.7}$$

Proof. The inclusion '⊂' in (2.7) is clear as Ω is bounded. It remains to prove the case '⊃'. Since $X \setminus \Omega$ is uniformly p-fat, it is also $(p - \varepsilon)$-fat for all $\varepsilon > 0$ small enough, as discussed in Paragraph 2.1.7. Consequently,

$$\int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)}\right)^{p-\varepsilon} d\mu \leqslant c \int_\Omega g_u^{p-\varepsilon}\, d\mu \tag{2.8}$$

for all $\varepsilon > 0$ small enough, by Proposition 2.5. We now show that (2.6) holds also for p. Indeed,

$$\begin{aligned}\int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)}\right)^{p} d\mu &= \lim_{\varepsilon\to 0} \int_\Omega \left(\frac{|u|}{\operatorname{dist}(z, X \setminus \Omega)}\right)^{p-\varepsilon} d\mu \\ &\leqslant \lim_{\varepsilon\to 0} c \int_\Omega g_u^{p-\varepsilon}\, d\mu \\ &= c \int_\Omega g_u^{p}\, d\mu,\end{aligned}$$

and the assertion follows by Proposition 2.5. □

3. *Quasiminima — higher integrability of upper gradients*

The local regularity of quasiminimizers (that is, Hölder continuity) was studied by Kinnunen and Shanmugalingam in [**16**]. In particular, they proved the following Caccioppoli-type inequality.

Theorem 3.1 (Caccioppoli inequality). *Let Ω be an open subset of X. If $u \in N^{1,p}(\Omega)$ is a quasiminimizer of the p-energy integral in Ω then there exists a $c > 0$ such that for all $x \in \Omega$ and $0 < r < R$ so that $B(x, R) \subset \Omega$ we have*

$$\int_{B(x,r)} g_u^p\, d\mu \leqslant \frac{c}{(R-r)^p} \int_{B(x,R)} |u - u_{B(x,R)}|^p\, d\mu. \tag{3.1}$$

We prove the global higher integrability of upper gradients of quasiminimizers. The proof follows a similar method to that of the Euclidean proof of Kilpeläinen and Koskela in [**14**] for solutions of p-harmonic-type equations. The growth of integrability is achieved in a standard way by application of the Gehring lemma (its proof in the metric setting may be found, for example, in [22] or [25]). Note that the lemma holds in all doubling metric measure spaces.

THEOREM 3.2 (Gehring lemma). *Let $s \in [s_0, s_1]$, where $s_0, s_1 > 1$ are fixed. Let $g \in L^s_{\mathrm{loc}}(X)$ and $f \in L^{s_1}_{\mathrm{loc}}(X)$ be non-negative functions. Assume that there exists a constant $b > 1$ such that for every ball $B \subset \sigma B \subset X$ the inequality*

$$\fint_B g^s \, d\mu \leqslant b \left[\left(\fint_{\sigma B} g \, d\mu \right)^s + \fint_{\sigma B} f^s \, d\mu \right]$$

holds for some $\sigma > 1$. Then there exists an $\varepsilon_0 = \varepsilon_0(s_0, s_1, c_d, \sigma, b) > 0$ such that $g \in L^{\tilde{s}}_{\mathrm{loc}}(X, \mu)$ for $\tilde{s} \in [s, s + \varepsilon_0)$ and moreover

$$\left(\fint_B g^{\tilde{s}} \, d\mu \right)^{1/\tilde{s}} \leqslant c \left[\left(\fint_{\sigma B} g^s \, d\mu \right)^{1/s} + \left(\fint_{\sigma B} f^{\tilde{s}} \, d\mu \right)^{1/\tilde{s}} \right]$$

for $c = c(s_0, s_1, c_d, \sigma, b)$.

Proof of Theorem 1.2. Recall that X is an LLC space that supports a weak $(1, q)$-Poincaré inequality for some $1 < q < p$. Since $p \in (q, q^*)$, the space also supports a weak (p, q)-Poincaré inequality (see Paragraph 2.1.2). Remember also that $X \setminus \Omega$ is uniformly p-fat.

As mentioned in Paragraph 2.1.7, if $X \setminus \Omega$ is uniformly p-fat, then $X \setminus \Omega$ is also uniformly p_0-fat for some $p_0 < p$. If $p_0 < q$, then we can increase it in order to have $q = p_0$ and to be able to use the (p, p_0)-Poincaré inequality. If $p_0 \geqslant q$ then the (p, p_0)-Poincaré inequality follows from the Hölder inequality.

Choose a ball B_0 in X such that $\Omega \subset\subset B_0 \subset 2B_0$. Fix $r > 0$ and let $B = B(x_0, r)$ be a ball such that $4\lambda B \subset 2B_0$, where λ is the multiplicative coefficient of radius in the (p, p_0)-Poincaré inequality.

If $2\lambda B \subset \Omega$ then by the Caccioppoli estimate (3.1), the doubling condition and the (p, p_0)-Poincaré inequality we have

$$\begin{aligned} \left(\fint_B g_u^p \, d\mu \right)^{1/p} &\leqslant \frac{c}{r} \left(\fint_{2B} |u - u_{2B}|^p \, d\mu \right)^{1/p} \\ &\leqslant c \left(\fint_{2\lambda B} g_u^{p_0} \, d\mu \right)^{1/p_0}. \end{aligned} \tag{3.2}$$

Assume thus that $2\lambda B \setminus \Omega \neq \emptyset$. Choose a Lipschitz cut-off function η such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on B, $\operatorname{supp} \eta \subset 2B$ and $g_\eta \leqslant c/r$. Then $\eta(u - w) \in N^{1,p}_0(2B \cap \Omega)$ and we may use it as a test function in (2.3). Hence

$$\int_{2B \cap \Omega} g_u^p \, d\mu \leqslant K \int_{2B \cap \Omega} g_v^p \, d\mu,$$

where $v = u + \eta(w - u)$ and $g_v \leqslant (1 - \eta) g_u + \eta g_w + |u - w| g_\eta + g_w$ μ-a.e. Therefore we obtain

$$\int_{B \cap \Omega} g_u^p \, d\mu \leqslant c \int_{(2B \setminus B) \cap \Omega} (1 - \eta)^p g_u^p \, d\mu + c \int_{2B \cap \Omega} |u - w|^p g_\eta^p \, d\mu + c \int_{2B \cap \Omega} \eta^p g_w^p \, d\mu.$$

Adding $c \int_{B \cap \Omega} g_u^p \, d\mu$ to the both sides of the inequality and dividing by $(1 + c)$ implies that

$$\int_{B \cap \Omega} g_u^p \, d\mu \leqslant \theta \int_{2B \cap \Omega} g_u^p \, d\mu + \frac{\theta}{r^p} \int_{2B \cap \Omega} |u - w|^p \, d\mu + \theta \int_{2B \cap \Omega} g_w^p \, d\mu,$$

where $\theta = c/(1+c) < 1$. Applying a standard technical iteration lemma (see [6, Lemma 3.1, Chapter V]) we obtain

$$\int_{B\cap\Omega} g_u^p \, d\mu \leqslant \frac{c}{r^p} \int_{2B\cap\Omega} |u-w|^p \, d\mu + c \int_{2B\cap\Omega} g_w^p \, d\mu. \tag{3.3}$$

We will consider the integrals on the right-hand side on the larger ball $4B$. We estimate the first integral on the right-hand side using Proposition 2.3 with $q = p_0$. This gives

$$\begin{aligned}\left(\frac{c}{r^p} \fint_{4B} |u-w|^p \, d\mu\right)^{1/p} &\leqslant \frac{c}{r} \left(\frac{1}{\operatorname{cap}_{p_0}(S,4B)} \int_{4\lambda B} g_{u-w}^{p_0} \, d\mu\right)^{1/p_0} \\ &\leqslant c \left(\frac{\mu(2B) r^{-p_0}}{\operatorname{cap}_{p_0}(S,4B)} \fint_{4\lambda B} g_{u-w}^{p_0} \, d\mu\right)^{1/p_0}\end{aligned}$$

by the doubling condition. Here the set $S = \{x \in 2B \colon u(x) = w(x)\}$. Since $u = w$ p-q.e. (and thus p_0-q.e.) in $X \setminus \Omega$ and the set $X \setminus \Omega$ is uniformly p_0-fat, we have

$$\operatorname{cap}_{p_0}(S,4B) \geqslant \operatorname{cap}_{p_0}(2B \setminus \Omega; 4B) \geqslant c_f \operatorname{cap}_{p_0}(2B;4B) \geqslant c\mu(2B) r^{-p_0}.$$

Hence,

$$\begin{aligned}\left(\frac{c}{r^p} \fint_{4B} |u-w|^p \, d\mu\right)^{1/p} &\leqslant c \left(\fint_{4\lambda B} g_{u-w}^{p_0} \, d\mu\right)^{1/p_0} \\ &= c \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B\cap\Omega} g_{u-w}^{p_0} \, d\mu\right)^{1/p_0},\end{aligned}$$

because $u - w = 0$ p-q.e. and thus μ-a.e. in $X \setminus \Omega$ and therefore $g_{u-w} = 0$ μ-a.e. in $X \setminus \Omega$. A simple estimation now gives

$$\left(\frac{c}{r^p} \fint_{4B} |u-w|^p \, d\mu\right)^{1/p} \leqslant c \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B\cap\Omega} g_u^{p_0} \, d\mu\right)^{1/p_0} + c \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B\cap\Omega} g_w^{p_0} \, d\mu\right)^{1/p_0}. \tag{3.4}$$

By the Hölder inequality, we have

$$\begin{aligned}\left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B\cap\Omega} g_w^{p_0} \, d\mu\right)^{1/p_0} &= \left(\fint_{4\lambda B} g_w^{p_0} \chi_{4\lambda B\cap\Omega} \, d\mu\right)^{1/p_0} \\ &\leqslant \left(\fint_{4\lambda B} g_w^{p} \chi_{4\lambda B\cap\Omega} \, d\mu\right)^{1/p} \\ &= \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B\cap\Omega} g_w^{p} \, d\mu\right)^{1/p},\end{aligned} \tag{3.5}$$

so that combining (3.3)–(3.5) and using the doubling property, we obtain the inequality

$$\left(\frac{1}{\mu(B)} \int_{B\cap\Omega} g_u^p \, d\mu\right)^{1/p} \leqslant b \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B\cap\Omega} g_u^{p_0} \, d\mu\right)^{1/p_0} + c \left(\frac{1}{\mu(4\lambda B)} \int_{4\lambda B\cap\Omega} g_w^{p} \, d\mu\right)^{1/p}. \tag{3.6}$$

Here the constants b and c depend only on p, Ω and the constants associated to the structure of the space.

Set now

$$g(x) = \begin{cases} g_u^{p_0} & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(x) = \begin{cases} g_w^{p_0} & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

and $s = p/p_0$. The inequalities (3.2) and (3.6) imply that whenever $4\lambda B \subset 2B_0$, the reverse Hölder inequality

$$\fint_B g^s \, d\mu \leqslant b \left(\fint_{4\lambda B} g \, d\mu \right)^s + c \fint_{4\lambda B} f^s \, d\mu$$

holds for $s > 1$ (p is strictly greater than p_0) and with $b = b(p)$. Applying now the Gehring lemma we obtain better integrability of g and the inequality

$$\left(\fint_B g^{\tilde{s}} \, d\mu \right)^{1/\tilde{s}} \leqslant c \left[\left(\fint_{4\lambda B} g^s \, d\mu \right)^{1/s} + \left(\fint_{4\lambda B} f^{\tilde{s}} \, d\mu \right)^{1/\tilde{s}} \right] \tag{3.7}$$

for $c = c(b, c_d, \lambda)$ and $\tilde{s} \in [s, s + \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_0(b, c_d, \lambda)$. Since the diameter of Ω is finite we may choose a finite number of balls $B(x_j, r_j)$, $j = 1, 2, \ldots, N$, such that

$$B(x_j, 2\lambda r_j) \subset B_0 \quad \text{and} \quad \Omega \subset \bigcup_{j=1}^{N} B(x_j, r_j)$$

with fixed λ. The statement now follows if we multiply (3.7) by $\mu(4\lambda B)^{1/\tilde{s}}$ and sum over $B(x_j, r_j)$. This may require changing the constant c a bit, but the change will depend only on the doubling constant c_d and on the domain Ω. Note that with λ and c_d fixed, the constant c will depend essentially on p. □

4. *Proof of the stability result*

Since the proof of the main theorem is rather long, we have divided it into three parts. Before the concluding third part we present two Lemmas, 4.2 and 4.4. The former makes use of the metric version of Rellich–Kondrachov theorem; see [**9**].

THEOREM 4.1 (Rellich–Kondrachov). *Let (X, d, μ) be a metric space, where μ is doubling. Suppose that all the pairs $\{u_i, g_i\}_{i=1}^{\infty}$ satisfy a weak $(1,p)$-Poincaré inequality. Fix a ball B and assume that the sequence $\|u_i\|_{L^1(B)} + \|g_i\|_{L^p(5\tau B)}$ is bounded. Then there is a subsequence of $(u_i)_{i=1}^{\infty}$ that converges in $L^q(B)$ for each $1 \leqslant q < pQ/(Q-p)$, when $p < Q$ and for each $q \geqslant 1$, when $p \geqslant Q$. Here $Q = \log c_d$ and c_d is the doubling constant of μ.*

By a remark in Paragraph 2.1.7 we can assume that $X \setminus \Omega$ is uniformly p_0-fat. Since $p = \lim_{i\to\infty} p_i$ we can also assume that $p_i \in (q, q^*)$.

Functions u_i are supposed to be not equal to the boundary data w; that is, we assume that there is a set of positive measures where $u_i \neq w$ μ-a.e., otherwise the result is trivial.

We start with a lemma concerning uniform higher integrability of u_i and u.

LEMMA 4.2. *Let u_i and u be as in Theorem 1.1. Then there exists an $\varepsilon_0 > 0$ such that*

$$u_i, u \in L^{p+\varepsilon_0}(\Omega),$$
$$g_{u_i}, g_u \in L^{p+\varepsilon_0}(\Omega),$$

and there is a subsequence such that

$$u_i \longrightarrow u \quad \text{in } L^{p+\varepsilon_0}(\Omega),$$
$$g_{u_i} \longrightarrow g \quad \text{weakly in } L^{p+\varepsilon_0}(\Omega),$$

where g is a weak upper gradient of u.

It is not known whether the sequence of minimal upper gradients converges weakly to the minimal upper gradient of u, or not. Nevertheless, the lemma implies that

$$\|g_u\|_{L^p(\Omega)} \leqslant \liminf_{i\to\infty} \|g_{u_i}\|_{L^p(\Omega)},$$

which is extremely useful in applications.

Proof of Lemma 4.2. By Theorem 1.2, for every p_i there exists a $\delta_i = \delta_i(p_i)$ such that the minimal p_i-weak upper gradient g_{u_i} belongs to the space $L^{p_i+\delta_i}(\Omega)$ and

$$\left(\int_\Omega g_{u_i}^{p_i+\delta_i}\,d\mu\right)^{1/(p_i+\delta_i)} \leqslant c_i \left(\int_\Omega g_{u_i}^{p_i}\,d\mu\right)^{1/p_i} + c_i \left(\int_\Omega g_w^{p_i+\delta_i}\,d\mu\right)^{1/(p_i+\delta_i)}. \tag{4.1}$$

Since u_i is a quasiminimizer of the p_i-energy functional in Ω with boundary data w and thus $u_i - w \in N_0^{1,p_i}(\Omega)$, we have

$$\begin{aligned}\int_\Omega g_{u_i}^{p_i}\,d\mu &\leqslant K \int_\Omega g_w^{p_i}\,d\mu \\ &\leqslant K(\mu(\Omega))^{\delta_i/(p_i+\delta_i)} \left(\int_\Omega g_w^{p_i+\delta_i}\,d\mu\right)^{p_i/(p_i+\delta_i)},\end{aligned}$$

and therefore

$$\left(\int_\Omega g_{u_i}^{p_i+\delta_i}\,d\mu\right)^{1/(p_i+\delta_i)} \leqslant c_i \left(\int_\Omega g_w^{p_i+\delta_i}\,d\mu\right)^{1/(p_i+\delta_i)}. \tag{4.2}$$

Now note that when $p_i \in (q, q^*)$ and $p_i \to p$ we have

$$\delta_i \geqslant \delta_0 = \delta_0(p) \quad \text{and} \quad c_i \leqslant c = c(p).$$

Indeed, in order to prove (4.1) we first show that a reverse Hölder inequality

$$\left(\frac{1}{\mu(B)}\int_{B\cap\Omega} g_{u_i}^{p_i}\,d\mu\right)^{1/p_i} \leqslant b_i \left(\frac{1}{\mu(\sigma B)}\int_{\sigma B\cap\Omega} g_{u_i}^{p_0}\,d\mu\right)^{1/p_0} + c_i \left(\frac{1}{\mu(\sigma B)}\int_{\sigma B\cap\Omega} g_w^{p_i}\,d\mu\right)^{1/p_i} \tag{4.3}$$

holds for some $\sigma > 1$, and then we apply the Gehring lemma. The constant b_i in (4.3) depends on p_i. However, when $p_i \in (q, q^*)$, it may be chosen independently on p_i; that is, $b_i \leqslant b$ for some $b = b(p)$. The bound will depend on p due to the fact that we apply the $(1, p_0)$-Poincaré inequality and p_0 is chosen to be sufficiently close to p. The assertion follows because of the fact that in (4.1) δ_i is inversely proportional to b_i and c_i is comparable to b_i (see, for example, [**25**]).

For i sufficiently large we may assume that

$$p + \varepsilon_0 \leqslant p_i + \delta_0 \leqslant p_i + \delta_i \leqslant s,$$

where $\varepsilon_0 = \delta_0/2$. By this assumption and the uniform bound for c_i, applying the Hölder inequality and (4.2) we find that

$$\begin{aligned}\left(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,d\mu\right)^{1/p+\varepsilon_0} &\leqslant c\left(\int_\Omega g_{u_i}^{p_i+\delta_i}\,d\mu\right)^{1/(p_i+\delta_i)} \\ &\leqslant c\left(\int_\Omega g_w^{s}\,d\mu\right)^{1/s} < \infty.\end{aligned}$$

Since

$$\begin{aligned}\left(\int_\Omega g_{u_i-w}^{p+\varepsilon_0}\,d\mu\right)^{1/p+\varepsilon_0} &\leqslant \left(\int_\Omega g_{u_i}^{p+\varepsilon_0}\,d\mu\right)^{1/p+\varepsilon_0} + \left(\int_\Omega g_w^{p+\varepsilon_0}\,d\mu\right)^{1/p+\varepsilon_0} \\ &\leqslant c\left(\int_\Omega g_w^{s}\,d\mu\right)^{1/s},\end{aligned}$$

it follows that

$$\sup_i \|g_{u_i-w}\|_{L^{p+\varepsilon_0}(\Omega)} < \infty. \tag{4.4}$$

Using Proposition 2.3 we are able to find a uniform $L^{p+\varepsilon_0}$-bound for the sequence $(u_i - w)$ as well. Observe that, decreasing ε_0 if necessary, we may additionally assume that $p+\varepsilon_0 < q^*$. Therefore choose $t = p+\varepsilon_0$, $q = p+\varepsilon_0$ in Proposition 2.3 and fix $B_0 = B(x_0, r_0)$ such that $\Omega \subset B_0$. We note again that the minimal p_i-weak upper gradient of $u_i - w$ satisfies $g_{u_i-w} = 0$ μ-a.e. on the set $S = \{x \in B_0 \colon u(x) = w(x)\}$. On the other hand $u_i - w$ is zero p_i-quasi everywhere on $X \setminus \Omega$ and thus μ-almost everywhere on $X \setminus \Omega$. In addition, observe that p-fatness always implies $p+\varepsilon_0$-fatness, so that $\operatorname{cap}_{p+\varepsilon_0}(S, 2B_0) \geqslant c\mu(B_0)/r^{p+\varepsilon_0}$. It follows that

$$\begin{aligned}
\left(\int_\Omega |u_i - w|^{p+\varepsilon_0}\, d\mu\right)^{1/p+\varepsilon_0} &\leqslant \left(\mu(2B_0) \fint_{2B_0} |u_i - w|^{p+\varepsilon_0}\, d\mu\right)^{1/p+\varepsilon_0} \\
&\leqslant \left(\frac{c\mu(B_0)}{\operatorname{cap}_{p+\varepsilon_0}(S, 2B_0)} \int_{2\tau' B_0} g_{u_i-w}^{p+\varepsilon_0}\, d\mu\right)^{1/p+\varepsilon_0} \\
&\leqslant c r_0 \left(\int_\Omega g_{u_i-w}^{p+\varepsilon_0}\, d\mu\right)^{1/p+\varepsilon_0} \\
&\leqslant c \left(\int_\Omega g_w^s\, d\mu\right)^{1/s},
\end{aligned}$$

by the Hölder inequality. Together with (4.4) this implies that

$$\sup_i \|u_i - w\|_{N^{1,p+\varepsilon_0}(\Omega)} < \infty.$$

Hence the sequence $(u_i - w)$ is uniformly bounded in $N^{1,p+\varepsilon_0}(\Omega)$. In particular, for i large enough, the pairs $\{u_i, g_{u_i}\}$ satisfy the $(1, p+\varepsilon_0)$-Poincaré inequality with fixed constants c and τ.

Choose a ball B_0 such that $\Omega \subset B_0$, and extend $u_i - w$ to be zero in $X \setminus \Omega$. The sequence

$$\|u_i - w\|_{L^1(B_0)} + \|g_{u_i-w}\|_{L^{p+\varepsilon_0}(5\tau B_0)}$$

is bounded. By the Rellich–Kondrachov theorem, Theorem 4.1, there exist a $\tilde{u} \in L^{p+\varepsilon_0}(B_0)$ and a subsequence such that

$$u_{i_k} - w \to \tilde{u} - w \quad \text{in } L^{p+\varepsilon_0}(B_0).$$

Since $u_i \to u$ μ-a.e. in Ω, it follows that $\tilde{u} = u$ μ-a.e. in Ω and $u \in L^{p+\varepsilon_0}(\Omega)$. As the norms of $g_{u_{i_k}}$ are uniformly bounded in $L^{p+\varepsilon_0}(\Omega)$, there exist $g \in L^{p+\varepsilon_0}(\Omega)$ and a subsequence, also denoted by $g_{u_{i_k}}$ for brevity, such that

$$g_{u_{i_k}} \to g \quad \text{weakly in } L^{p+\varepsilon_0}(\Omega).$$

By [**24**, Lemma 3.6] we see that g is a weak upper gradient of u and $u \in N^{p+\varepsilon_0}(\Omega)$. □

REMARK 4.3. If desired, the statement of Theorem 1.1 could be reformulated, since the proof of Lemma 4.2 shows that it is not necessary to suppose the existence of a limit u *a priori*. This is exceptional for quasiminimizers, and follows from the $N^{1,p+\varepsilon_0}$-boundedness of $(u_i - w)$. Hence, the Rellich–Kondrachov theorem applies. The boundedness is a strong property and holds in this case due to the fact that all quasiminimizers have the same $p+\varepsilon_0$-integrable boundary data.

Let $D \subset \Omega$ be a compact set and for $t > 0$ write

$$D(t) = \{x \in \Omega \,:\, \operatorname{dist}(x, D) < t\}.$$

Then $D(t) \subset\subset \Omega$ for $t \in (0, t_0)$, where $t_0 = \operatorname{dist}(D, X \setminus \Omega)$. We reformulate a lemma by Kinnunen and Martio [15] so that it corresponds to the present case.

LEMMA 4.4. *Let u_i, u be as in Theorem 1.1. Then for almost every $t \in (0, t_0)$ we have*

$$\limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i}\, d\mu \leqslant c \int_{D(t)} g_u^p\, d\mu,$$

where the constant c depends only on K and p.

Proof. Let $0 < t' < t < t_0$. Choose a Lipschitz cut-off function η such that $0 \leqslant \eta \leqslant 1$ and

$$\eta = 1 \quad \text{on } D(t'),$$
$$\eta = 0 \quad \text{on } \Omega \setminus D(t).$$

Define a function

$$\phi_i = \eta(u - u_i).$$

For i large enough, $p_i < p + \varepsilon_0$. Then, since u_i and u belong to $N^{1,p+\varepsilon_0}(\Omega)$ it follows that $\phi_i \in N_0^{1,p_i}(D(t))$. Therefore by the quasiminimizing property of u_i we have

$$\int_{D(t')} g_{u_i}^{p_i}\, d\mu \leqslant \int_{D(t)} g_{u_i}^{p_i}\, d\mu \leqslant K \int_{D(t)} g_{u_i+\phi_i}^{p_i}\, d\mu.$$

Lemma 2.1 implies that

$$g_{u_i+\phi_i} \leqslant (1-\eta) g_{u_i} + g_\eta |u - u_i| + \eta g_u$$

μ-a.e., and hence

$$\int_{D(t')} g_{u_i}^{p_i}\, d\mu \leqslant c\biggl(\int_{D(t)} (1-\eta)^{p_i} g_{u_i}^{p_i}\, d\mu + \int_{D(t)} g_\eta^{p_i} |u - u_i|^{p_i}\, d\mu + \int_{D(t)} \eta^{p_i} g_u^{p_i}\, d\mu \biggr)$$

with c depending only on D and p_i. Observing that $\eta \equiv 1$ on $D(t')$ we add $c \int_{D(t')} g_{u_i}^{p_i}\, d\mu$ to both sides of the inequality and obtain

$$(1+c) \int_{D(t')} g_{u_i}^{p_i}\, d\mu \leqslant c \left(\int_{D(t)} g_{u_i}^{p_i}\, d\mu + \int_{D(t)} g_\eta^{p_i} |u - u_i|^{p_i}\, d\mu + \int_{D(t)} \eta^{p_i} g_u^{p_i}\, d\mu \right).$$

Define now, on $(0, t_0)$, a function

$$\Psi(t) = \limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i}\, d\mu.$$

By definition, Ψ is a non-decreasing function of t and by the uniform higher integrability of u_i it is finite for every $t \in (0, t_0)$. Therefore its set of points of discontinuity is at most countable. Let t be a point of continuity of Ψ. Taking limes superior on both sides of the last inequality we obtain

$$(1+c)\Psi(t') \leqslant c\Psi(t) + c \limsup_{i\to\infty} \int_{D(t)} |u - u_i|^{p_i}\, d\mu + c \int_{D(t)} g_u^p\, d\mu.$$

The second term on the right-hand side tends to zero. To see this, apply first the Hölder inequality and then Lemma 4.2. Hence, since t is a point of continuity of Ψ we obtain

$$(1+c)\Psi(t) \leqslant c\Psi(t) + c \int_{D(t)} g_u^p\, d\mu,$$

and furthermore

$$\Psi(t) \leqslant c \int_{D(t)} g_u^p\, d\mu. \qquad \square$$

Proof of Theorem 1.1. In order to show that u is a quasiminimizer of the p-energy integral with boundary data w, we need to show first that $u - w \in N_0^{1,p}(\Omega)$. This does not follow immediately from the compactness argument used to extract the convergent subsequence.

We proceed as follows. For every $\varepsilon > 0$ and for i sufficiently large, $p_i > p - \varepsilon$ such that $u_i - w \in N_0^{1,p-\varepsilon}(\Omega)$. By the Sobolev inequality (Lemma 2.4) we get

$$\begin{aligned} \|u_i - w\|_{N_0^{1,p-\varepsilon}(\Omega)} &\leqslant c\|g_{u_i-w}\|_{L^{p-\varepsilon}(\Omega)} \\ &\leqslant c\|g_{u_i-w}\|_{L^p(\Omega)}; \end{aligned}$$

that is, the norms of $(u_i - w)$ are uniformly bounded in $N_0^{1,p-\varepsilon}(\Omega)$.

As $X \setminus \Omega$ is uniformly p_0-fat, it is also uniformly $(p-\varepsilon)$-fat for ε small enough. In addition, $u_i \to u$ μ-a.e., and so by Lemma 2.6

$$u - w \in N_0^{1,p-\varepsilon}(\Omega)$$

for all $\varepsilon > 0$ such that $p_0 < p - \varepsilon$. Hence, by Proposition 2.7, the p-fatness of $X \setminus \Omega$ implies also that

$$u - w \in N_0^{1,p}(\Omega).$$

It remains to show that for every open $\Omega' \subset\subset \Omega$ and every $\phi \in N_0^{1,p}(\Omega')$, we have

$$\int_{\overline{\Omega'}} g_u^p \, d\mu \leqslant K \int_{\overline{\Omega'}} g_{u+\phi}^p \, d\mu. \tag{4.5}$$

Let $\varepsilon > 0$ be arbitrary. For i sufficiently large, $p_i > p - \varepsilon$. Since, by Lemma 4.2, (g_{u_i}) converges weakly to a weak upper gradient g of u, for every μ-measurable subset E of Ω, we have

$$\begin{aligned} \int_E g_u^{p-\varepsilon} \, d\mu \leqslant \int_E g^{p-\varepsilon} \, d\mu &\leqslant \liminf_i \int_E g_{u_i}^{p-\varepsilon} \, d\mu \\ &\leqslant \liminf_i \left(\int_E g_{u_i}^{p_i} \, d\mu \right)^{(p-\varepsilon)/p_i} \mu(E)^{1-(p-\varepsilon)/p_i} \\ &\leqslant \liminf_i \left(\int_E g_{u_i}^{p_i} \, d\mu \right)^{(p-\varepsilon)/p} \mu(E)^{\varepsilon/p}, \end{aligned}$$

where we have used the Hölder inequality. Passing to zero with ε we conclude that

$$\int_E g_u^p \, d\mu \leqslant \liminf_i \int_E g_{u_i}^{p_i} \, d\mu. \tag{4.6}$$

We will first show that the inequality (4.5) holds for every Lipschitz function compactly supported in Ω', that is, for $\phi \in \mathrm{Lip}_C(\Omega')$. Fix $\varepsilon > 0$ and choose open sets Ω'' and Ω_0 such that

$$\Omega' \subset\subset \Omega'' \subset\subset \Omega_0 \subset\subset \Omega$$

and

$$\int_{\Omega_0 \setminus \overline{\Omega'}} g_u^p \, d\mu < \varepsilon.$$

Let η be a Lipschitz cut-off function such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in a neighbourhood of $\overline{\Omega'}$, and $\eta = 0$ in $\Omega \setminus \Omega''$. Define a function ϕ_i as

$$\phi_i = \phi + \eta(u - u_i).$$

Since $\phi \in \mathrm{Lip}_C(\Omega')$ and both $u_i, u \in N^{1,p+\varepsilon_0}(\Omega)$, it follows that $\phi_i \in N_0^{1,p_i}(\Omega'')$ for i large enough. Hence by the quasiminimizing property of u_i we get

$$\begin{aligned}\int_{\Omega''} g_{u_i}^{p_i}\,d\mu &\leqslant K\int_{\Omega''} g_{u_i+\phi_i}^{p_i}\,d\mu \\ &= K\int_{\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,d\mu + K\int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,d\mu. \end{aligned} \tag{4.7}$$

Since $\eta \equiv 1$ in a neighbourhood of $\overline{\Omega'}$ it follows that

$$u_i + \phi_i = u + \phi \quad \text{in a neighbourhood of } \overline{\Omega'}. \tag{4.8}$$

On the other hand, in $\Omega'' \setminus \overline{\Omega'}$ we have $\phi \equiv 0$, and therefore

$$u_i + \phi_i = u_i + \eta(u - u_i).$$

Lemma 2.1 implies that

$$g_{u_i+\phi_i} \leqslant (1-\eta)g_{u_i} + \eta g_u + g_\eta|u - u_i|$$

μ-a.e. in ω. Therefore

$$\int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,d\mu \leqslant c\int_{\Omega''\setminus\overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i}\,d\mu + c\int_{\Omega''\setminus\overline{\Omega'}} \eta^{p_i} g_u^{p_i}\,d\mu + c\int_{\Omega''\setminus\overline{\Omega'}} g_\eta^{p_i}|u-u_i|^{p_i}\,d\mu. \tag{4.9}$$

We estimate the integrals on the right-hand side separately.

Since $\eta \equiv 1$ on a neighbourhood of $\overline{\Omega'}$, there exists a compact set $D \subset \overline{\Omega''}$ such that $D \cap \overline{\Omega'} = \emptyset$ and

$$\int_{\Omega''\setminus\overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i}\,d\mu \leqslant \int_D g_{u_i}^{p_i}\,d\mu.$$

For t sufficiently small we have $D(t) \subset \Omega_0 \setminus \overline{\Omega'}$. Therefore we choose t such that we may apply Lemma 4.4; in other words,

$$\limsup_{i\to\infty} \int_{D(t)} g_{u_i}^{p_i}\,d\mu \leqslant c\int_{D(t)} g_u^p\,d\mu.$$

Consequently,

$$\begin{aligned}\limsup_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}} (1-\eta)^{p_i} g_{u_i}^{p_i}\,d\mu &\leqslant \limsup_{i\to\infty}\int_D g_{u_i}^{p_i}\,d\mu \\ &\leqslant \limsup_{i\to\infty}\int_{D(t)} g_{u_i}^{p_i}\,d\mu \\ &\leqslant c\int_{D(t)} g_u^p\,d\mu \\ &\leqslant c\varepsilon, \end{aligned} \tag{4.10}$$

by the choice of Ω_0.

Also, the second integral is arbitrarily small. Again by the choice of Ω_0 we have

$$\limsup_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}} \eta^{p_i} g_u^{p_i}\,d\mu \leqslant \int_{\Omega''\setminus\overline{\Omega'}} g_u^p\,d\mu \leqslant \int_{\Omega_0\setminus\overline{\Omega'}} g_u^p\,d\mu \leqslant \varepsilon. \tag{4.11}$$

Observe that for a Lipschitz function its minimal p-weak upper gradient is bounded by its Lipschitz constant μ-almost everywhere. This allows us to conclude that

$$\begin{aligned}\limsup_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}} g_\eta^{p_i}|u-u_i|^{p_i}\,d\mu &\leqslant c\limsup_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}}|u-u_i|^{p_i}\,d\mu\\ &\leqslant c\limsup_{i\to\infty}\left(\int_{\Omega''\setminus\overline{\Omega'}}|u-u_i|^{p+\varepsilon_0}\,d\mu\right)^{p_i/(p+\varepsilon_0)}\\ &=0,\end{aligned}\tag{4.12}$$

by Lemma 4.2.

By the estimates (4.10)–(4.12) we find from (4.9) that

$$\limsup_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,d\mu\leqslant c\varepsilon.\tag{4.13}$$

Finally by (4.6)–(4.8) and (4.13) we have

$$\begin{aligned}\int_{\overline{\Omega'}} g_u^p\,d\mu &\leqslant \liminf_{i\to\infty}\int_{\Omega''} g_{u_i}^{p_i}\,d\mu\\ &\leqslant K\liminf_{i\to\infty}\int_{\Omega''} g_{u_i+\phi_i}^{p_i}\,d\mu\\ &\leqslant K\liminf_{i\to\infty}\int_{\overline{\Omega'}} g_{u+\phi}^{p_i}\,d\mu + K\liminf_{i\to\infty}\int_{\Omega''\setminus\overline{\Omega'}} g_{u_i+\phi_i}^{p_i}\,d\mu\\ &\leqslant K\int_{\overline{\Omega'}} g_{u+\phi}^{p}\,d\mu + c\varepsilon.\end{aligned}\tag{4.14}$$

Passing to zero with ε we obtain the desired inequality for any $\phi\in \mathrm{Lip}_C(\Omega')$.

The result for $\phi\in N_0^{1,p}(\Omega')$ follows by approximation; that is, if $\phi\in N_0^{1,p}(\Omega')$ then for any $\varepsilon>0$ we may find a function $\phi_\varepsilon\in\mathrm{Lip}_C(\Omega')$ such that

$$\|\phi_\varepsilon-\phi\|_{N^{1,p}(\Omega')}<\varepsilon. \qquad \square$$

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