Regularity for some quasilinear parabolic equations in non-divergence form

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Motivation

The regularity theory for uniformly parabolic fully nonlinear equations with small oscillation condition on the coefficients is well advanced since the works of Krylov, Safonov, Caffarelli et al, Wang...

It is not the case for quasilinear degenerate or singular parabolic equations in non divergence form.

In the divergence case, the situation is better understood and our equations will be modelled on the p-Laplacian.

The usual p-Laplacian

$$u_t - \mathrm{div}(|Du|^{p-2}Du) = f$$

Hölder regularity, Harnack estimates were provided by the works of DiBenedetto, Gianazza, Kuusi, Vespri...

Higher regularity: The solutions are also known to be locally $C^{1,\alpha}$ in space for some $\alpha > 0$. This was proved by DiBenedetto and Friedman '85 and Wiegner '86 (some extra conditions are needed for the singular case to ensure the boundedness of u).

Non-divergence equations modeled on the p-Laplacian

Expand the formula of the p-Laplacian:

$$\begin{aligned}\Delta_p u &= \operatorname{div}(|Du|^{p-2}Du)\\ &= |Du|^{p-2}\left[\Delta u + (p-2)\frac{D^2uDu, Du}{|Du|^2}\right]\\ &:= |Du|^{p-2}\left[\Delta u + (p-2)\Delta_\infty^N u\right]\\ &:= |Du|^{p-2}\Delta_p^N u\end{aligned}$$

The normalized p-Laplacian

$$u_t - \Delta_p^N u = f$$

- Homogeneous of degree 1. In contrast, the p-Laplace operator is homogeneous of degree $p-1$.
- Studied by: Banerjee, Garofalo, Jin, Silvestre, Does, Ubostad, Liu, Schikorra, A. Björn, J. Björn, Parviainen, Nyström,...
- Existence and uniqueness proved including game-theoretic arguments and approximation methods.

Connection with tug-of-war games

- The story started in the stationary case: this connection was discovered in the works of Peres, Schramm, Sheffield and Wilson '05, '08, '09 via a relation to a two players zero sum game (link via the Dynamic Programming Principle and mean value property).

- In the parabolic case, Manfredi-Parviainen-Rossi '10 showed that solutions to the normalized p-Laplacian can be obtained as limits of value functions of tug-of-war games with noise (number of rounds is bounded) when the parameter that controls the length of steps goes to zero.

The limiting cases $p = \infty$

Multiplying the r.h.s by p and $p \to \infty$, the equation converges to

$$u_t - \Delta_\infty^N u = 0.$$

Studied by Juutinen and Kawohl '06 (provided existence and uniqueness results for both Dirichlet and Cauchy problems, establish interior and boundary Lipschitz estimates and a Harnack inequality).

Studied also by Wu, Akagi, Barron-Evans-Jensen, Aronsson.

Applications in image processing (Does '11, Caselles-Morel-Sbert '98...).

Stationary case In the stationary setting solutions to

$$\Delta_\infty u := (D^2 u Du, Du) = 0$$

are known to be $C^{1,\alpha}$ in 2D (Evans and Savin '08) and pointwise differentiable in arbitrary dimension (Evans and Smart '10).

The parabolic case is still open.

Limiting case $p = 1$

$$u_t - |Du| \operatorname{div}\left(\frac{Du}{|Du|}\right) = u_t - \Delta u + \Delta_\infty^N u = f$$

the evolution equation for the function u whose level sets follow a mean curvature flow.

Studied by a number of authors like Chen-Giga-Goto, Evans-Spruck, Evans-Soner-Souganidis, Ishii-Souganidis, Oberman, Minicozzi-Colding, etc...

A deterministic game related to the motion by mean curvature was introduced by Spencer '77 and studied by Kohn and Serfaty '06.

- The operator is uniformly parabolic

$$a_{ij}(Du) := \delta_{ij} + (p-2)\frac{\partial_i u \partial_j u}{|Du|^2}$$

$$\min(1, p-1)I \leq a_{ij} \leq \max(1, p-1)I$$

- The operator is singular at $\{Du = 0\}$ but the singularity is bounded.

$\Rightarrow$ definition of viscosity solutions tacking the semicontinuous extensions

u is a viscosity solution of the parabolic normalized p-Laplacian if for each smooth function ϕ

- if $u - \phi$ has a local maximum at (x_0, t_0) then

$$\phi_t(x_0,t_0) - \Delta_p^N \phi(x_0,t_0) \leq f(x_0,t_0) \quad \text{if} \quad D\phi(x_0,t_0) \neq 0$$
$$\phi_t(x_0,t_0) - \Delta\phi(x_0,t_0) - (p-2)\lambda_{\max}(D^2\phi(x_0,t_0) \leq f(x_0,t_0) \text{ if } D\phi(x_0,t_0) = 0$$

- if $u - \phi$ has a local minimum at (x_0, t_0) then

$$\phi_t(x_0,t_0) - \Delta_p^N \phi(x_0,t_0) \geq f(x_0,t_0) \quad \text{if} \quad D\phi(x_0,t_0) \neq 0$$
$$\phi_t(x_0,t_0) - \Delta\phi(x_0,t_0) - (p-2)\lambda_{\min}(D^2\phi(x_0,t_0) \geq f(x_0,t_0) \text{ if } D\phi(x_0,t_0) = 0$$

For $1 \leq p \leq 2$, one needs to exchange $\lambda_{\min}$ and $\lambda_{\max}$ to obtain a similar definition.

U.P $\Rightarrow$ Hölder regularity and Harnack inequality follow from classical theory of fully nonlinear uniformly parabolic equations.

The Lipschitz regularity can be shown by different methods:

- using Bernstein method (study the equation satisfied by $|Du|^p$ and use comparison principle) by Does '12.
- using Ishii-Lions method (based on a contradiction argument, auxiliary function and the theorem of sums) Jin-Silvestre '17.

The connection with tug-of-war games provided new methods to prove the Hölder regularity and the Lipschitz regularity using adapted strategies for the players.

Luiro-Parviainen-Saksman'13 (elliptic), Manfredi-Parviainen-Rossi '10 , Parviainen-Ruosteenoja '16 (includes also the varying case $p(x, t) \geq 2$).

Regularity of the gradient for the normalized p-Laplacian

Theorem (Jin and Silvestre '16)

Let $1 < p < \infty$ and let u be a viscosity solution to $u_t - \Delta_p^N u = 0$ in Q_1. Then there exists $\alpha = \alpha(p, n) > 0$ such that Du is well defined and α-Hölder continuous in $Q_{1/2}$. Moreover the Hölder norm depends only on p, n and $||u||_{L^\infty(Q_1)}$.

Theorem (A. and Parviainen '18)

Let $1 < p < \infty$, $f \in C(Q_1) \cap L^\infty(Q_1)$ and let u be a viscosity solution to $u_t - \Delta_p^N u = f$ in Q_1. Then there exists $\alpha = \alpha(p, n) > 0$ such that

$$||u||_{C^{1+\alpha, \frac{1+\alpha}{2}}(Q_{1/2})} \leq C(p, n)(||u||_{L^\infty(Q_1)} + ||f||_{L^\infty(Q_1)}).$$

Strategies of the proofs I: $f = 0$

Step 1: regularize the equation
Step 2: get uniform Lipschitz estimates
Step 3: show using an induction argument that the oscillation of the gradient is reduced in a shrinking sequence of parabolic cylinders. The iterative step is reduced to a dichotomy between two cases: either the value of the gradient Du stays close to a fixed vector e for most points (x, t) (in measure), or it does not.
One then has to patch these two alternatives together

Resolving the two cases: The good case

Improvement of oscillation: Let e be any unit vector. Assume that

$$(H1) \qquad |\{(x,t) \in Q_1 : e \cdot Du(x,t) \leq 1 - c_0\}| \geq \mu.$$

Then there exist r and δ depending on c_0 and μ such that

$$e \cdot Du(x,t) \leq 1 - \delta c_0 \quad \text{in} \quad Q_r := B_r \times (-r^2, 0]$$

Idea: The function $\max(e \cdot Du, 1 - c_0)$ is a subsolution of some parabolic equation and then use an improvement of oscillation since the equation is uniformly parabolic (need to differentiate the equation).

Resolving the two cases: The bad case

If the condition $(H1)$ is not satisfied at some step, that is, for some unit vector e,

$$|\{(x,t) \in Q_1 : e \cdot Du(x,t) \leq 1 - c_0\}| < \mu.$$

1) Show that in this case $u(x,t) - e \cdot x$ has a small oscillation in $Q_{1/2}$ (relying on the uniform ellipticity of the equation and barrier arguments).

2) Apply the result of Yu Wang:
If u is a solution to a parabolic equation of the form $u_t = F(D^2u, Du)$ in Q_1 where F is smooth and uniformly elliptic in a neighborhood of $(D^2\phi, D\phi)$ for some smooth solution ϕ and if $||u - \phi||_\infty$ is sufficiently small, then u is smooth.

Strategies of the proofs II: f bounded

For $f \neq 0$: first by rescaling we can assume that f is small enough then, use another characterization of $C^{1,\alpha}$ functions:
rate of approximation by linear maps,

$$\underset{B_r \times (-r^2, 0]}{osc} (u(x,t) - q \cdot x) \leq r^{(1+\alpha)}.$$

Proof proceeds by induction.
Main arguments: compactness arguments and improvement of flatness.

We would like to show that $\exists \rho > 0$ and q_k such that

$$\operatorname{osc}_{Q_{r_k}}(u - q_k \cdot x) \leq r_k^{1+\alpha}$$

where $Q_{r_k} := B_{r_k} \times (-r_k^2, 0]$ and $r_k = \rho^k$ with ρ depending only on p, n.

In order to find the next vector q_{k+1} it suffices to show an improvement of flatness for

$$w(x,t) := \frac{u(r_k, r_k^2 t) - q_k \cdot x r_k}{r_k^{1+\alpha}}.$$

We improve our approximation of u in a smaller cylinder by finding a linear approximation for w.

Difficulty: w satisfies an equation of the type

$$w_t = \Delta w + (p-2)\frac{(D^2u(Dw+\eta),(Dw+\eta))}{|Dw+\eta|^2} + f$$

where $\eta := \frac{q_k}{r_k^\alpha}$.

Providing a linear approximation for w independently of the value of η is based on a contradiction argument which requires a uniform Hölder estimate with respect to η for the associated homogeneous equation.

Degenerate or singular parabolic equations in non-divergence form

$$u_t - |Du|^\gamma \Delta_p^N u = f$$

where $-1 < \gamma < \infty$, $1 < p < \infty$ and f is a bounded and continuous function.

Generalizes both the standard parabolic p-Laplace equation $(\gamma = p - 2)$ and the normalized version $(\gamma = 0)$.

These equations maybe degenerate or very singular. The definition for $\gamma < 0$ is a non-trivial issue. These equations fall into the general framework studied by Ohnuma and Sato '97 (for $f = 0$).

Scaling:

- $u(\lambda x, \lambda^{\gamma+2} t)$ is still a solution but λu does not solve a similar equation (an-isotropic diffusion).
- This requires careful geometric techniques, so-called intrinsic scaling, to resolve the nonhomogeneity.

- Existence and uniqueness : Demengel '11 (relying on Perron's method), Imbert-Jin-Silvestre'17 (approximation arguments).

- For $\gamma = p - 2$ equivalence between weak and viscosity solutions (Juutinen-Lindqvist-Manfredi '01).

- Hölder regularity and Lipschitz estimates using Ishii-Lions method (Demengel '11, Imbert-Jin-Silvestre'17).

- Using the Lipschitz continuity in space and a simple comparison argument, show that the solution is Hölder continuous in time.

Theorem (Imbert-Jin-Silvestre '17)

Let $-1 < \gamma < \infty$, $1 < p < \infty$ *and* u *a viscosity solution to*

$$u_t - |Du|^\gamma \Delta_p^N u = 0 \ in \ Q_1.$$

Then there exists $\alpha = \alpha(p, n, \gamma) > 0$ *such that*

$$|Du(x,t) - Du(y,s)| \leq C(p,n,\gamma)||u||_{L^\infty(Q_1)}(|x-y|^\alpha + |t-s|^{\alpha/2}$$

$$|u(x,t) - u(x,s)| \leq C|t-s|^{\frac{1+\alpha}{2-\alpha\gamma}}.$$

Ideas of the proof

Step 1 regularize the equation
Step 2 Uniform Lipschitz estimate in space and Hölder in time
Step 3 First we show that if the projection of Du onto the direction e is away from 1 in a positive portion of Q_1, then $Du \cdot e$ has improved oscillation in a smaller cylinder.

For every $1/2 < l < 1$ and $\mu > 0$, there exist $\tau, \delta, > 0$ depending only on n, p, γ, μ and l such that for arbitrary e if

$$|\{(x,t) \in Q_1;\ Du \cdot e \leq l\}| > \mu |Q_1|,$$

then

$$Du \cdot e \leq (1-\delta) \text{ in } Q_\tau^{1-\delta} := B_\tau \times (-\tau^2 (1-\delta)^{-\gamma}, 0].$$

The intrinsic scaling plays a crucial role in the iteration process.

If such an improvement of oscillation takes place in all directions e and at all scales, it leads to the Hölder continuity of Du at $(0, 0)$ by iteration and scaling.

In case this does not hold some vector e use the small perturbation.

1) Show that in this case $u(x,t) - e \cdot x$ has a small oscillation in $Q_{1/2}$ (relying on barrier arguments to extend the small oscillation for all values of t, this part is harder especially for the singular case).

2) Apply the result of Yu Wang:

- If u is a solution to a parabolic equation of the form $u_t = F(D^2u, Du)$ in Q_1 where F is smooth and uniformly elliptic in a neighborhood of $(D^2\phi, D\phi)$ for some smooth solution ϕ and if $||u-\phi||_\infty$ is sufficiently small, then u is smooth.

Harnack inequalities

Harnack inequalities are a more subtle topic since the behavior of the solutions show clear distinctions depending on the range of p : subcritical $(1 < p < p^*\ )$, supercritical $(p^* < p < 2)$, and degenerate $(p > 2)$ for the critical number $p^* = 2n/(n+1)$. This topic attracted lots of interests: DiBenedetto, Gianazza, Kuusi, Vespri, ...

Theorem ((Parviainen-Vázquez '18)

Let $u \geq 0$ *be a viscosity solution in* Q_1 *of*

$$u_t - |Du|^{\gamma} \Delta_p^N u = 0.$$

Assume that

$$\gamma + 2 > \frac{2d}{d+1} \quad \text{with} \quad d := 1 + (n-1)\frac{\gamma+1}{p-1}.$$

Suppose that $u(x_0, t_0) > 0$*. Then there exist* $\mu = \mu(n, p, \gamma)$ *and* $C = C(n, p, \gamma)$ *such that*

$$u(x_0, t_0) \leq \mu \inf_{Br(x_0)} u(\cdot, t_0 + \theta)$$

where $\theta = \frac{Cr^{\gamma+2}}{u(x_0,t_0)^{\gamma}}$ *and* $B_{4r}(x_0) \times (t_0 - 4\theta, t_0 + 4\theta) \subset Q_1$*.*

They show that the radial solutions to the original problem can be interpreted as solutions to the divergence form $\gamma+2$-parabolic equation, but in a fictitious space dimension d given by

$$d-1=(n-1)\frac{\gamma+1}{p-1}.$$

The equivalence applies only to radially symmetric solutions but this will be enough to find suitable special solutions of Barenblatt type for the expansion of positivity.
Then they combine this with an Oscillation estimate.

Open questions

- Boundary regularity
- Optimal α for the Hölder regularity of the gradient
- Estimates involving lower norm of f (L^q norm for some q)
- Estimates for the Hessian (for $\gamma = 0$ and p close to 2: Høeg and Lindqvist, preprint '18).
- Hölder regularity for the gradient for the degenerate case ($\gamma \neq 0$) and $f \neq 0$, f continuous and bounded (work in progress)
- The limit cases $p = \infty, 1$

Thanks for your attention