Stability analysis of asymptotic profiles for fast diffusion

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Workshop on Nonlinear Parabolic PDEs

INSTITUT MITTAG-LEFFLER

Plan of this talk

§1. Introduction: Asymptotic profiles for fast diffusion

- Previous results
- Problem, motivation

§2. Stability of asymptotic profiles [A-Kajikiya'13]

- Definition of (in)stability for asymptotic profiles
- Stability criteria for isolated profiles

§3. Stability analysis of non-isolated asymptotic profiles [A'16]

§4. Instability of positive radial profiles in thin annular domains [A'16] (cf. [A-Kajikiya'14])

§5. Exponential stability of nondegenerate profiles of least energy

⋮

1. Asymptotic profiles for fast diffusion

Fast Diffusion equation (FD)

Consider the Cauchy-Dirichlet problem for the Fast Diffusion Equation (FD),

$$\begin{aligned} &(1) \qquad \partial_t \left(|u|^{m-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty), \\ &(2) \qquad u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ &(3) \qquad u(\cdot,0) = u_0 \quad \text{in } \Omega, \end{aligned}$$

where $m > 2$ and Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$.

Background: singular diffusion of plasma ($m = 3$ by Okuda-Dawson '73).

Throughout (the most of) this talk, we assume that

$$m < 2^* := \frac{2N}{(N-2)_+} \quad \text{and} \quad u_0 \in H^1_0(\Omega).$$

Behavior of solutions: finite-time extinction

Extinction in finite time

$$\exists t_* = t_*(u_0) \geq 0 \quad \text{s.t.} \quad u(\cdot, t) \equiv 0 \quad \text{in } \Omega \quad \forall t \geq t_*.$$

- Singular diffusion Setting $w = |u|^{m-2}u$, one can rewrite (FD) as

$$\partial_t w = \Delta\left(|w|^{m'-2}w\right) = \nabla \cdot \Big(\underbrace{(m'-1)|w|^{m'-2}}_{\text{diffusion coefficient}}\nabla w\Big),$$

where $m' := m/(m-1) \in (1,2)$.

- Separable solution Put $u(x,t) = \rho(t)\psi(x)$, where $\rho(t) \geq 0$. Then

$$\spadesuit \quad \frac{\mathrm{d}}{\mathrm{d}t}\rho(t)^{m-1} = -\lambda\rho(t) \ \text{ for } \ t > 0, \quad \rho(0) = 1,$$

$$\clubsuit \quad -\Delta\psi(x) = \lambda|\psi|^{m-2}\psi(x) \ \text{ for } \ x \in \Omega, \quad \psi|_{\partial\Omega} = 0.$$

$$\Rightarrow \quad \spadesuit \quad \rho(t) = t_*^{-\frac{1}{m-2}}(t_* - t)_+^{\frac{1}{m-2}} \quad \text{with} \quad t_* := \frac{1}{\lambda}\cdot\frac{m-1}{m-2}.$$

Asymptotic profiles of vanishing solutions

Berryman-Holland ('80) proved that

$$\forall u_0 \in H^1_0(\Omega) \setminus \{0\},\ \exists t_* = t_*(u_0) > 0\ ;\ \|u(t)\|_{H^1_0} \asymp (t_* - t)_+^{\frac{1}{m-2}}.$$

Then there exists an asymptotic profile of the vanishing solution u, i.e.,

(4) $\quad {}^\exists\phi(x) := \lim_{t_n \nearrow t_*} (t_* - t_n)^{-\frac{1}{m-2}} u(x, t_n)$ in $H^1_0(\Omega)$ for ${}^\exists t_n \nearrow t_*$,

and moreover, ϕ solves the Emden-Fowler equation (EF),

(5) $\quad -\Delta\phi = \lambda_m |\phi|^{m-2}\phi$ in Ω, $\quad \phi = 0$ on $\partial\Omega$

with $\lambda_m = \frac{m-1}{m-2} > 0$ (cf. see also ♣).

(cf. Y.-C. Kwong ('88), DiBenedetto-Kwong-Vespri ('91), Savaré-Vespri ('94), Feireisl-Simondon ('00), Bonforte-Grillo-Vazquez ('12)).

Rescaled Problem (RP)

Apply the transformations (then $t \nearrow t_* \Leftrightarrow s \nearrow \infty$),

(7) $v(x,s) := (t_* - t)^{-1/(m-2)} u(x,t)$ and $s := \log(t_*/(t_* - t))$.

Then, $\phi = \lim_{s_n \to \infty} v(s_n)$ with $s_n := \log(t_*/(t_* - t_n))$.

Moreover, rewrite (FD) as Rescaled Problem (RP):

$$
\begin{aligned}
&(8) \qquad \partial_s \left(|v|^{m-2} v\right) = \Delta v + \lambda_m |v|^{m-2} v && \text{in } \Omega \times (0,\infty),\\
&(9) \qquad v = 0 && \text{on } \partial\Omega \times (0,\infty),\\
&(10) \qquad v(\cdot,0) = v_0 && \text{in } \Omega,
\end{aligned}
$$

where $v_0 = t_*(u_0)^{-1/(m-2)} u_0$ and $\lambda_m = \frac{m-1}{m-2} > 0$. Here the function

$$s \mapsto J(v(s)) := \frac{1}{2}\|\nabla v(s)\|_{L^2}^2 - \frac{\lambda_m}{m}\|v(s)\|_{L^m}^m \quad \text{is non-increasing.}$$

Then, $\mathcal{S} :=$ {asymptotic profiles for (FD)} = {nontrivial sol. of (EF)} = {nontrivial stationary sol. of (RP)} = {nontrivial critical points of $J(\cdot)$}.

Asymptotic profiles are stable ?

If (EF) has a unique positive solution ϕ, then all nonnegative solutions of (FD) have the same profile ϕ, i.e., ϕ is "globally stable" (e.g., Berryman-Holland '80).

But, what happens if we take account of sign-changing solutions or of the case that (EF) has multiple positive solutions ?

Q Let ϕ be an asymptotic profile for (FD).

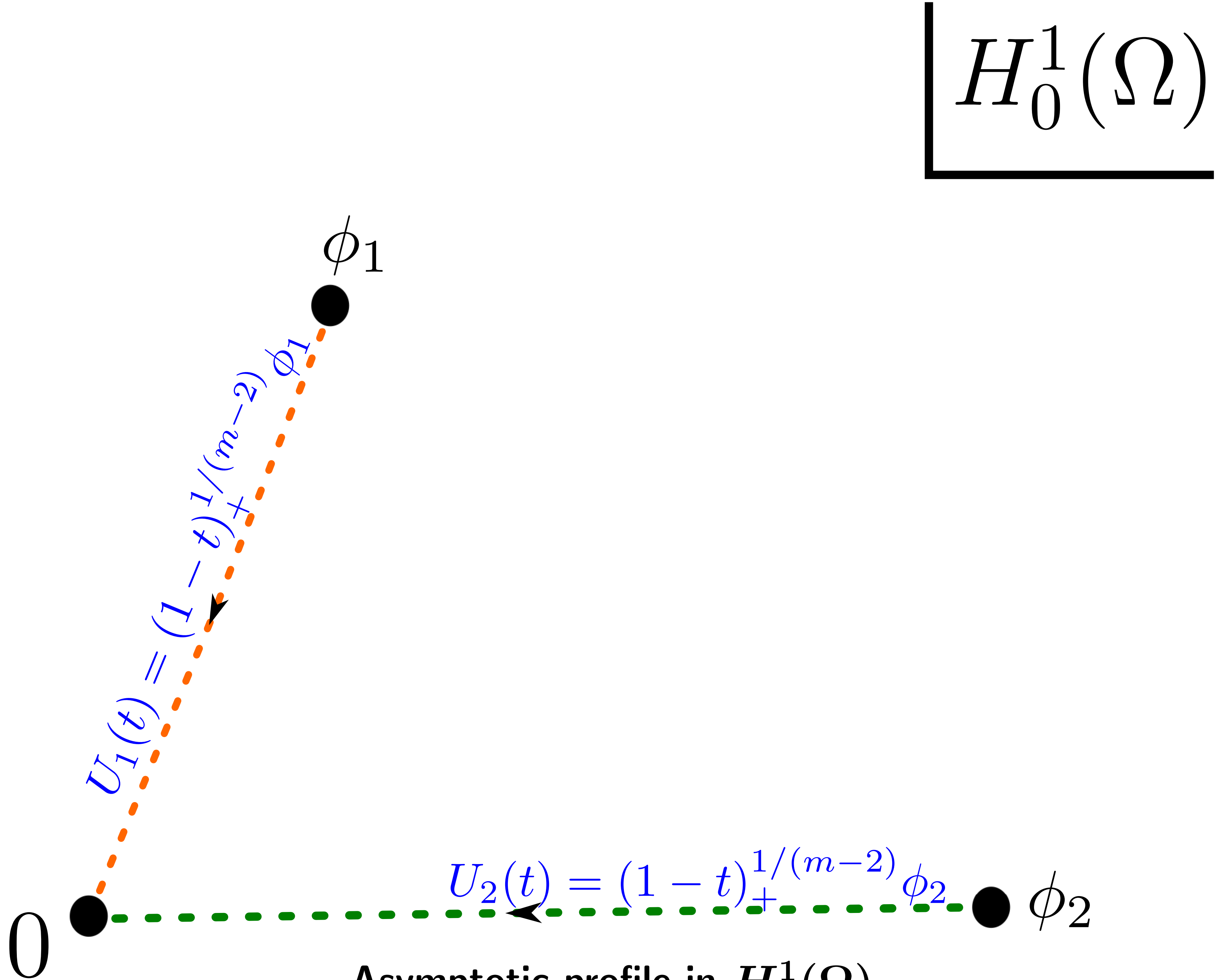
If $u_0 \in H_0^1(\Omega)$ is sufficiently close to ϕ, does the asymptotic profile (of the solution $u = u(x,t)$ of (FD)) for u_0 also coincide with ϕ or not ?

"Stability of asymptotic profile" for vanishing solutions of (FD)

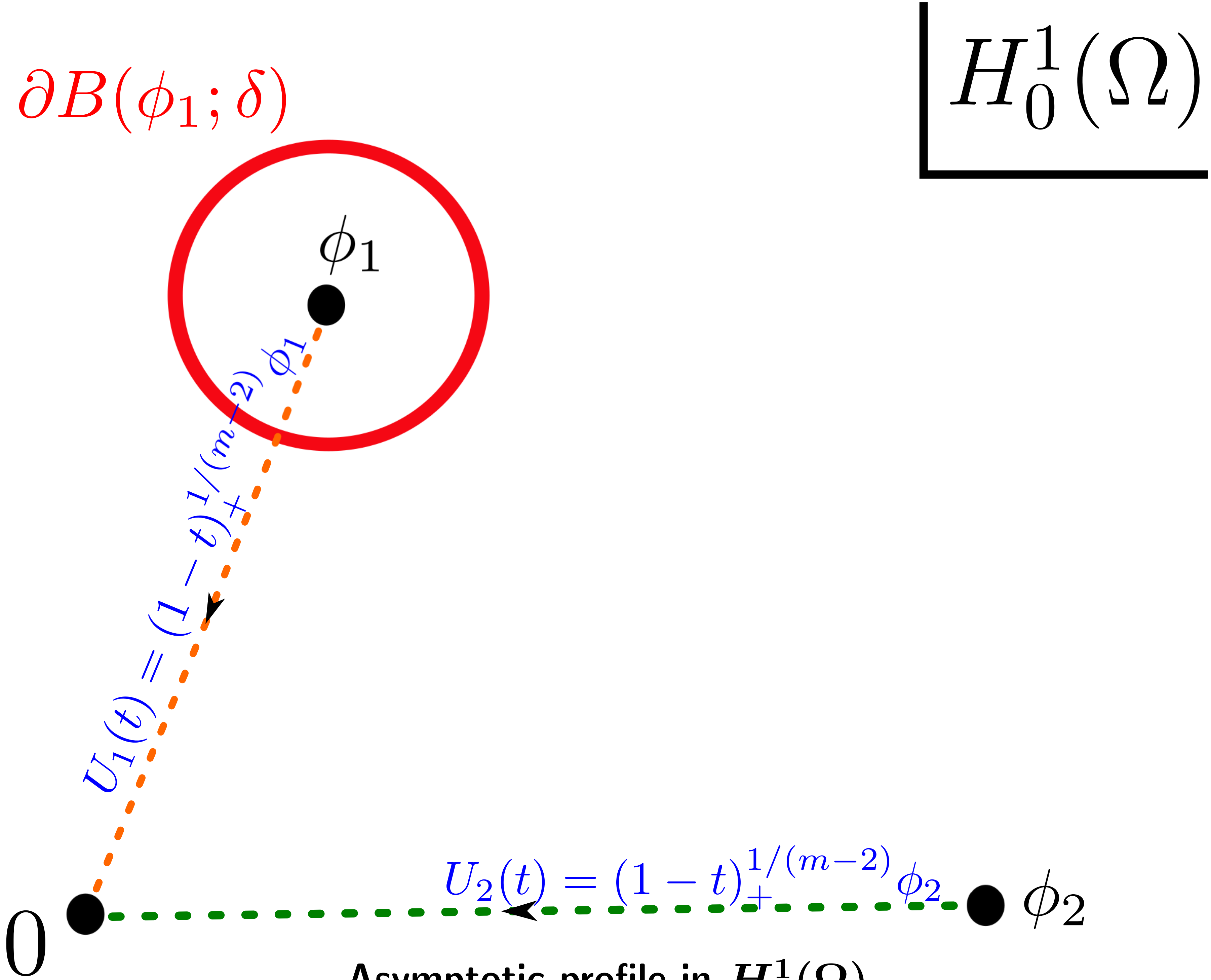
[AK13] G. Akagi, R. Kajikiya, Manuscr. Math. **141** (2013), 559–587.

- Notions of stability and instability of asymptotic profiles for FDE.
- Stability criteria for isolated asymptotic profiles.

2. Stability of asymptotic profiles



Asymptotic profile in $H_0^1(\Omega)$



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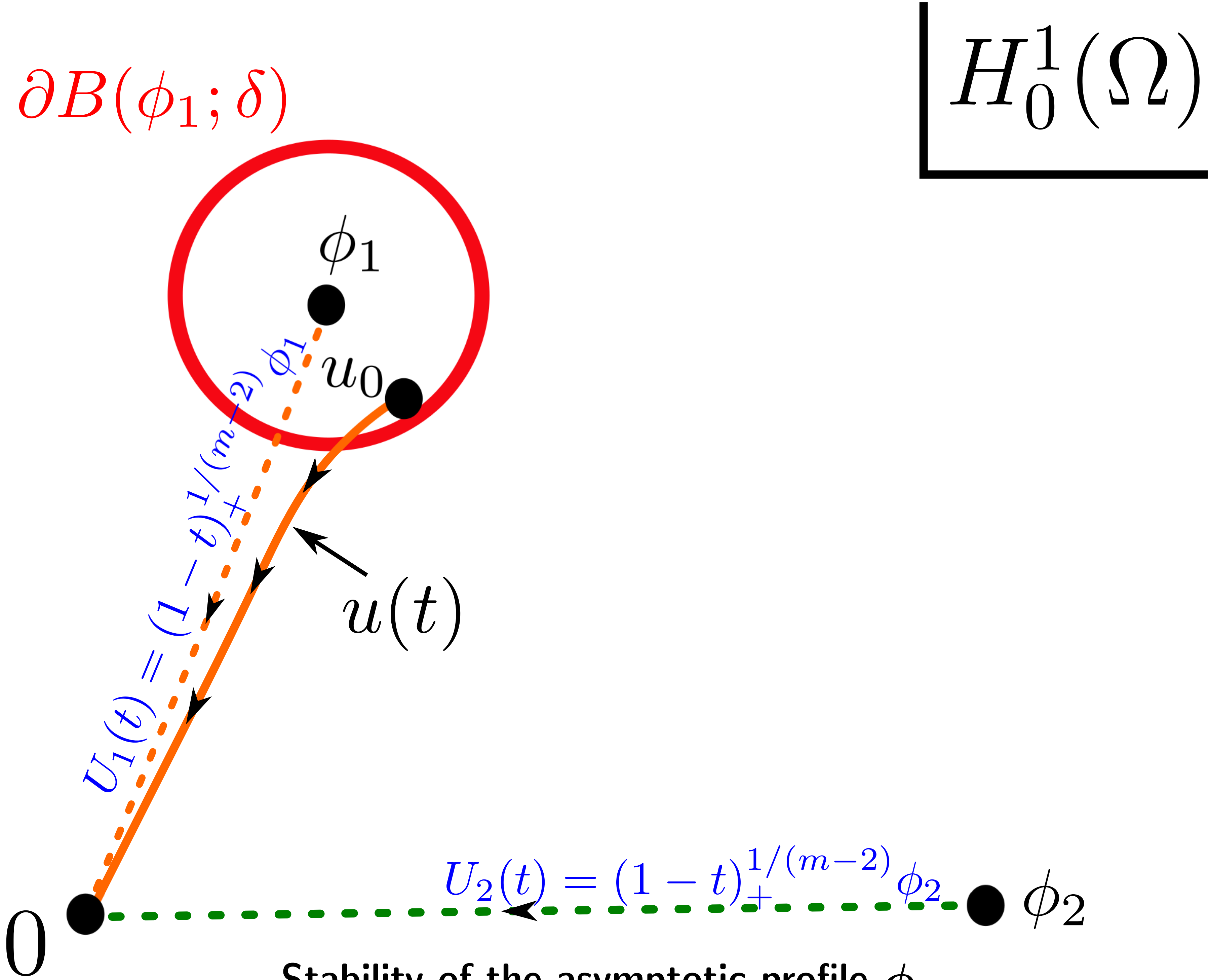
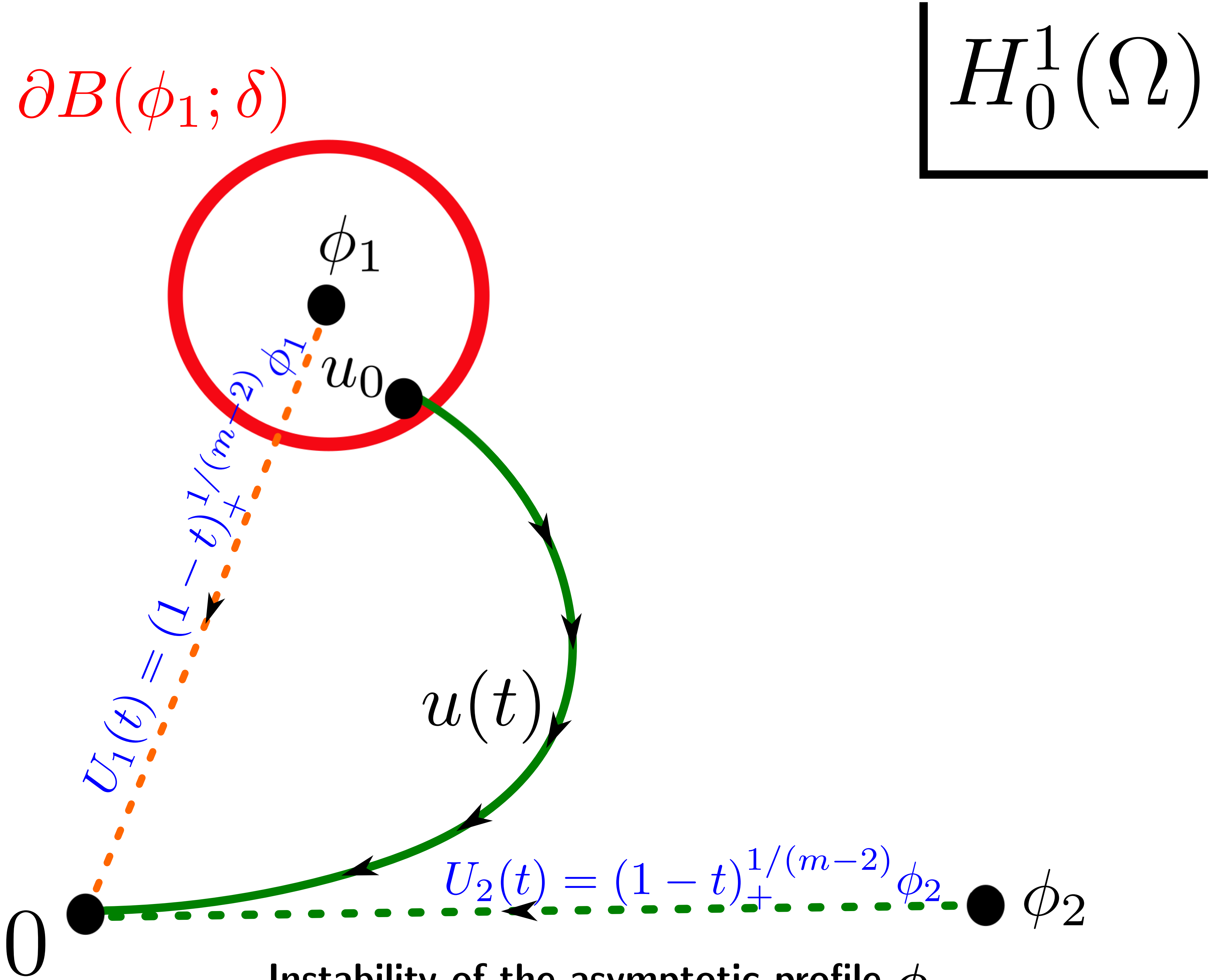


Stability of the asymptotic profile ϕ_1



Instability of the asymptotic profile ϕ_1

Stability/instability of asymptotic profiles

Let us recall the transformation,

$$v(x,s) = (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{and} \quad s = \log(t_*/(t_* - t)) \geq 0.$$

In particular, note the relation $v_0 = t_*(u_0)^{-1/(m-2)} u_0$.

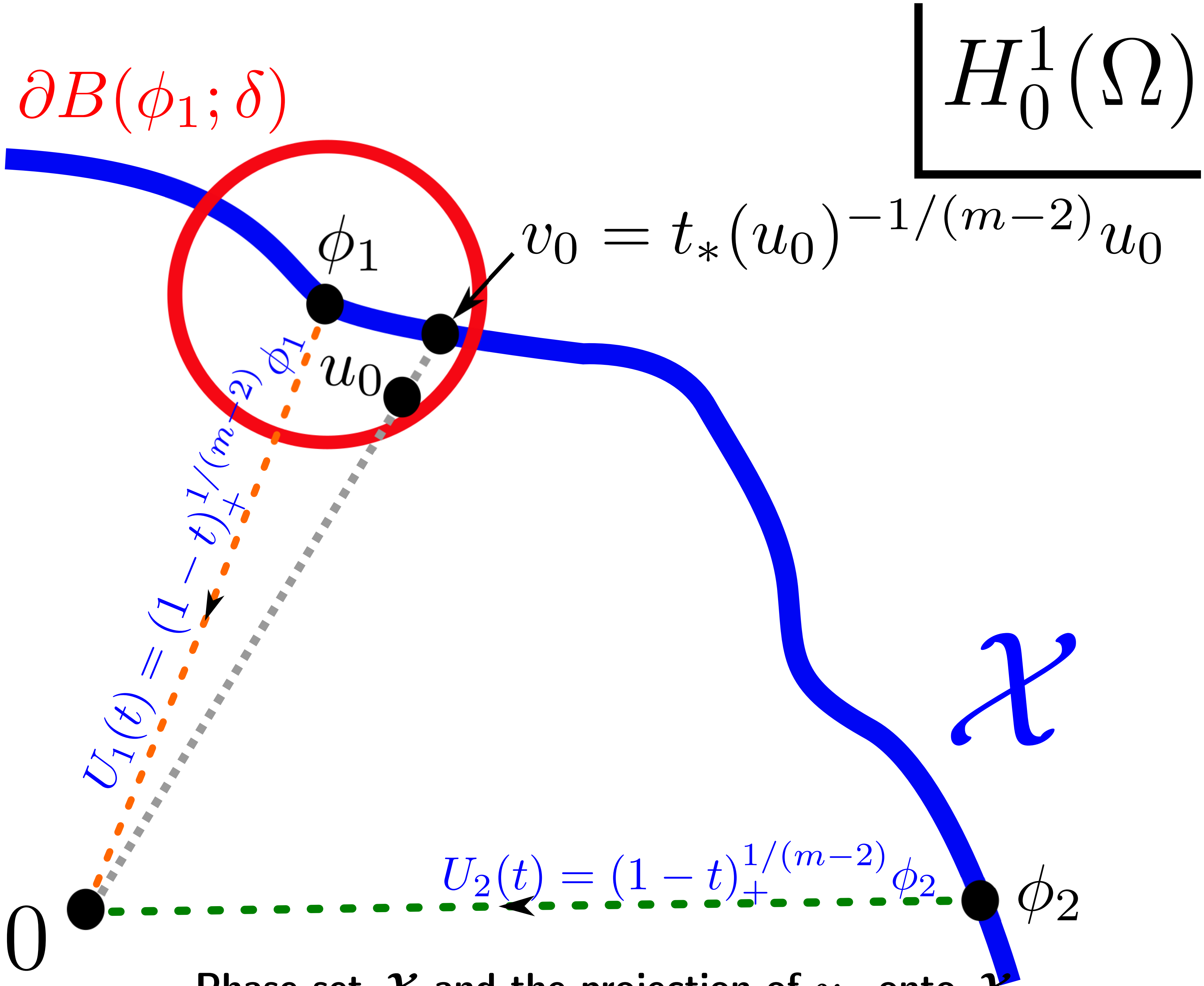
Define the set of initial data for (RP) by

$$\begin{aligned} \mathcal{X} &:= \left\{ t_*(u_0)^{-1/(m-2)} u_0 \colon u_0 \in H_0^1(\Omega) \setminus \{0\} \right\} \\ &= \left\{ v_0 \in H_0^1(\Omega) \colon t_*(v_0) = 1 \right\} \quad (\text{by } \; t_*(\mu u_0) = \mu^{m-2} t_*(u_0)), \end{aligned}$$

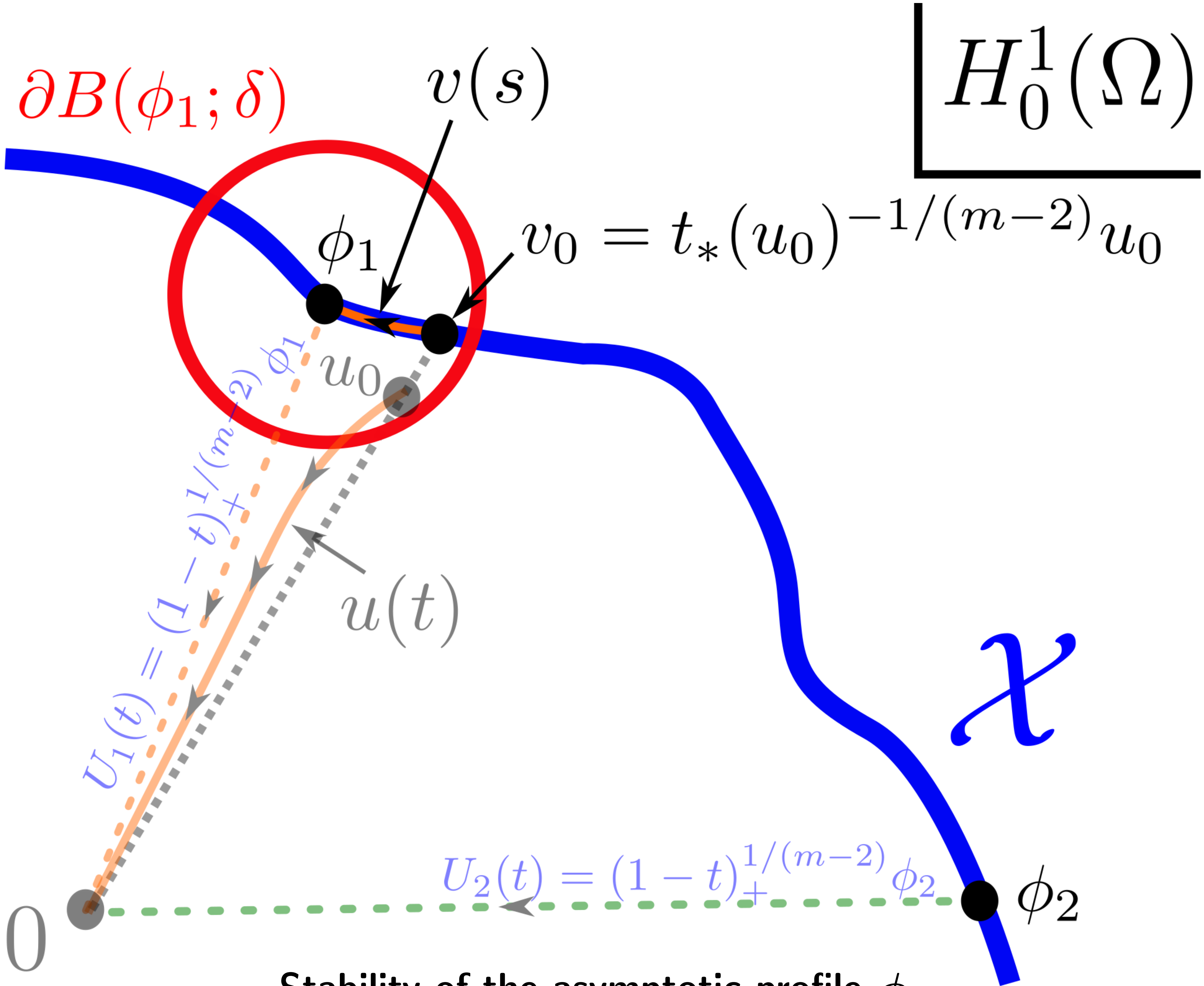
and then, we observe that $\quad u_0 \in H_0^1(\Omega) \setminus \{0\} \quad \Leftrightarrow \quad v_0 \in \mathcal{X}$.

(i) $\mathcal{X}$ is homeomorphic to a sphere in $H_0^1(\Omega)$. Moreover, $\mathcal{S} \subset \mathcal{X}$.

(ii) $v_0 \in \mathcal{X} \;\Rightarrow\; v(s) \in \mathcal{X} \;\; \forall s \geq 0$.



Phase set $\mathcal{X}$ and the projection of u_0 onto $\mathcal{X}$



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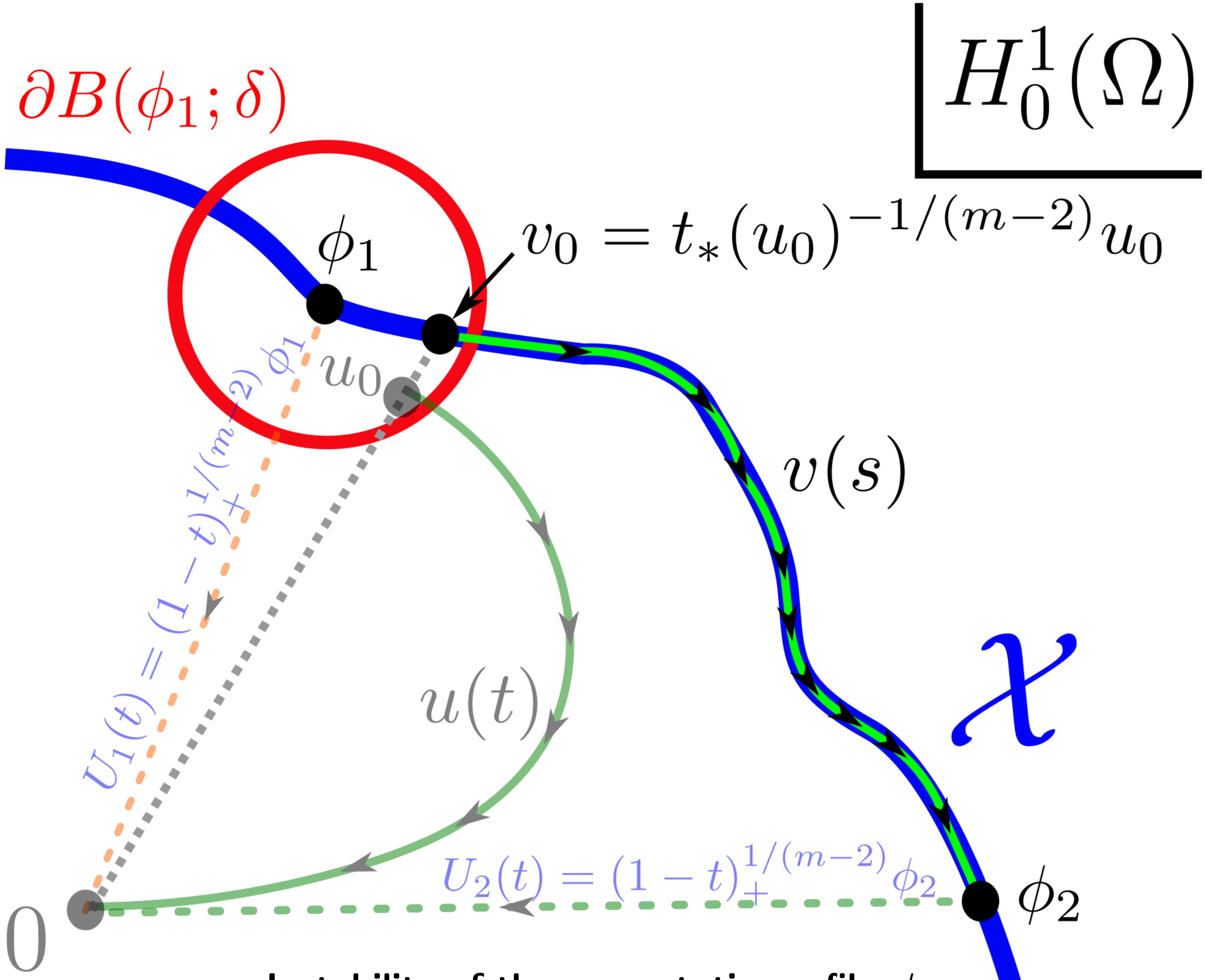


Instability of the asymptotic profile ϕ_1

Definition of the stability/instability of profiles

Definition 1 (Stability of asymptotic profiles [AK13])

Let $\phi \in H_0^1(\Omega)$ be an asymptotic profile of vanishing solutions for (FD).

(i) ϕ is said to be <u>stable</u>, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any solution v of (RP) satisfies

$$v(0) \in \mathcal{X} \cap B_{H_0^1}(\phi; \delta) \quad \Rightarrow \quad \sup_{s \in [0,\infty)} \|v(s) - \phi\|_{H_0^1} < \varepsilon.$$

(ii) ϕ is said to be <u>unstable</u>, if ϕ is not stable.

(iii) ϕ is said to be <u>asymptotically stable</u>, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of (RP) satisfies

$$v(0) \in \mathcal{X} \cap B_{H_0^1}(\phi; \delta_0) \quad \Rightarrow \quad \lim_{s \nearrow \infty} \|v(s) - \phi\|_{H_0^1} = 0.$$

= Stability in Lyapunov's sense of stationary points for (RP) on $\mathcal{X}$.

Stability of asymptotic profiles

Def. Let d_1 be the least energy of J over nontrivial solutions, i.e.,

$$d_1 := \inf_{v \in \mathcal{S}} J(v), \quad \mathcal{S} = \{\ \underline{\text{nontrivial}}\ \text{solutions of (EF)}\}.$$

A least energy solution ϕ of (EF) means $\phi \in \mathcal{S}$ satisfying $J(\phi) = d_1$.

Remark. Every least energy solution of (EF) is sign-definite.

Theorem 2 (Stability of profiles [AK13])

Let $\overline{\phi}$ be a least energy solution of (EF). Then

(i) $\overline{\phi}$ is a stable profile, if $\overline{\phi}$ is isolated in $H_0^1(\Omega)$ from the other least energy solutions.

(ii) $\overline{\phi}$ is an asymptotically stable profile, if $\overline{\phi}$ is isolated in $H_0^1(\Omega)$ from the other sign-definite solutions.

Instability of asymptotic profiles

Theorem 3 (Instability of profiles [AK13])

Let ψ be a sign-changing solution of (EF). Then

(i) ψ is NOT an asymptotically stable profile.

(ii) ψ is an unstable profile, if ψ is isolated in $H^1_0(\Omega)$ from the set $\{w \in \mathcal{S} \colon J(w) < J(\psi)\}$.

Roughly speaking,

- least energy solutions of (EF) are asymptotically stable profiles;
- sign-changing solutions of (EF) are unstable profiles

under appropriate assumptions on the isolation of profiles.

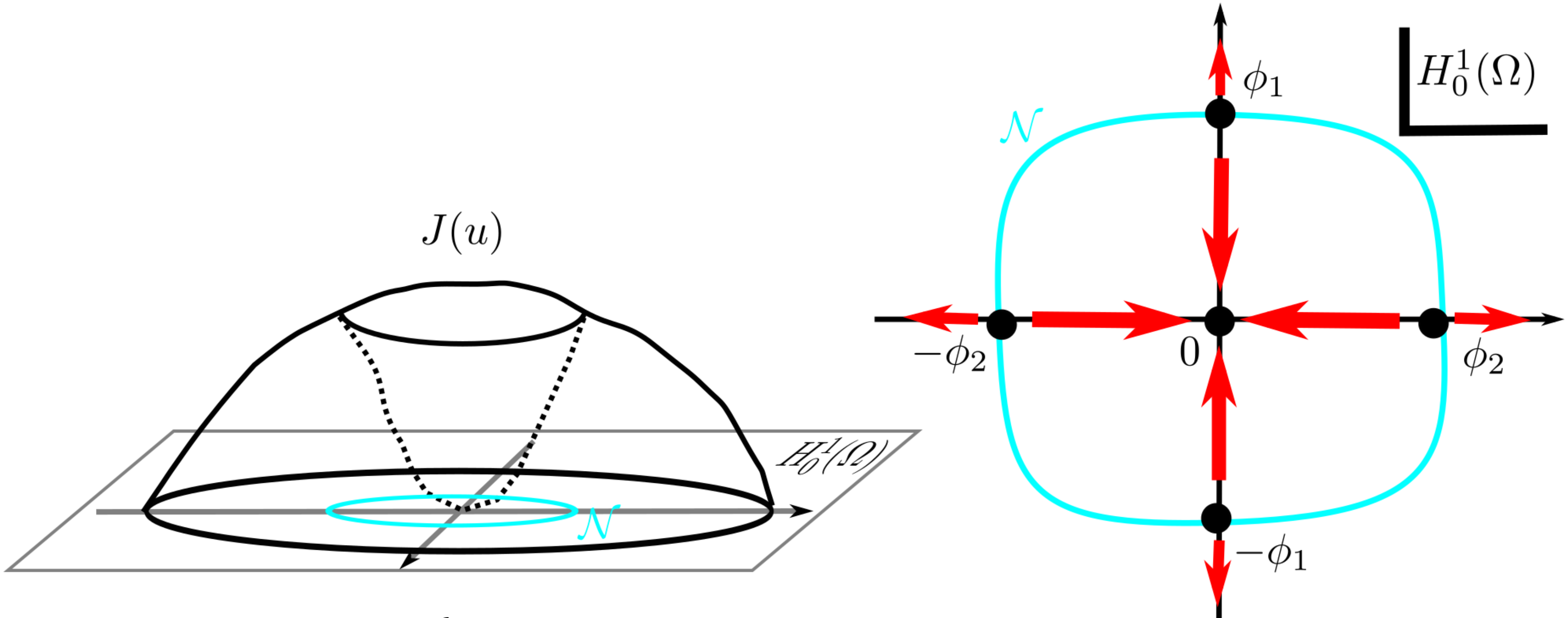
Variational view of the stability criteria

(RP) can be expressed as a (generalized) gradient flow,

$$\partial_s \left(|v|^{m-2}v\right)(s) = -J'(v(s)) \ \text{ in } H^{-1}(\Omega), \quad s > 0.$$

Moreover, the energy functional $J(\cdot)$ has a mountain pass structure.

$$J(w) = \frac{1}{2}\|\nabla w\|_{L^2}^2 - \frac{\lambda_m}{m}\|w\|_{L^m}^m, \quad w \in H_0^1(\Omega), \quad m > 2.$$



(cf. Nehari manifold $\mathcal{N} = \{w \in H_0^1(\Omega) \setminus \{0\} : \langle J'(w), w\rangle = 0\}$)

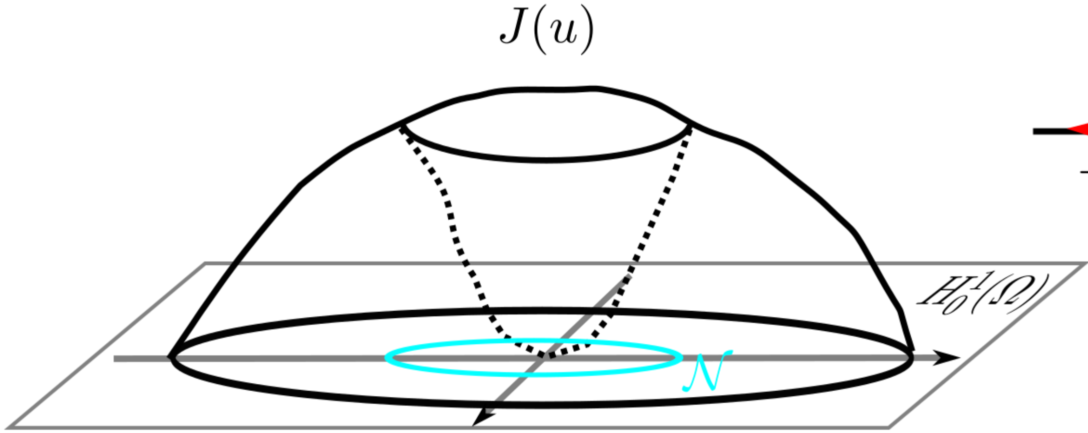
Variational view of the stability criteria

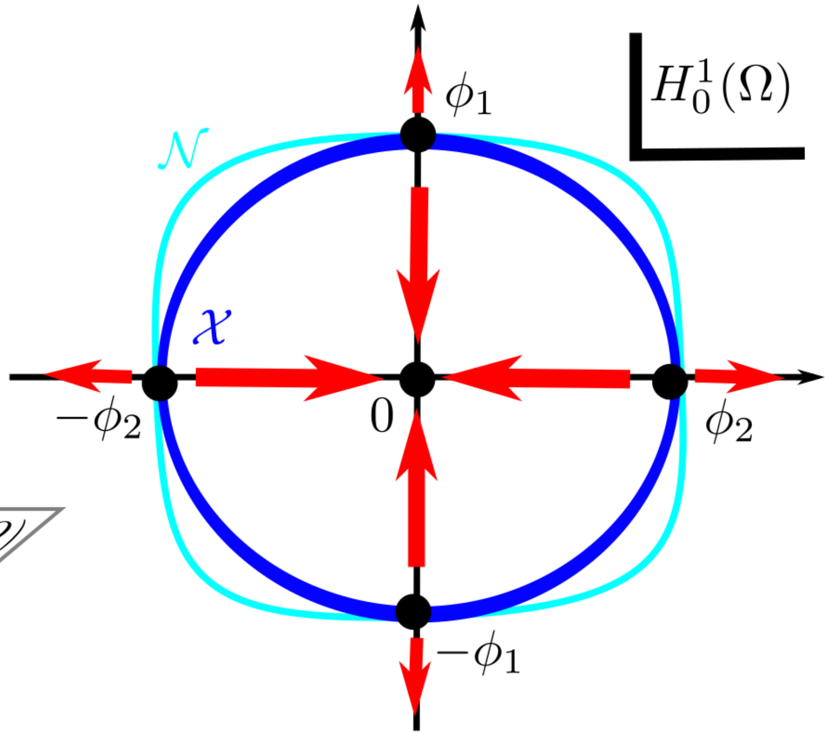
Key properties of the set $\mathcal{X} = \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) = 1\}$

(i) $v_0 \in \mathcal{X} \;\Rightarrow\; \forall s_n \to \infty,\ \exists (n') \subset (n),\ \exists \phi \in \mathcal{S},\ v(s_{n'}) \to \phi.$

(ii) $\mathcal{X}$ is (sequentially) weakly closed in $H_0^1(\Omega)$.

(iii)





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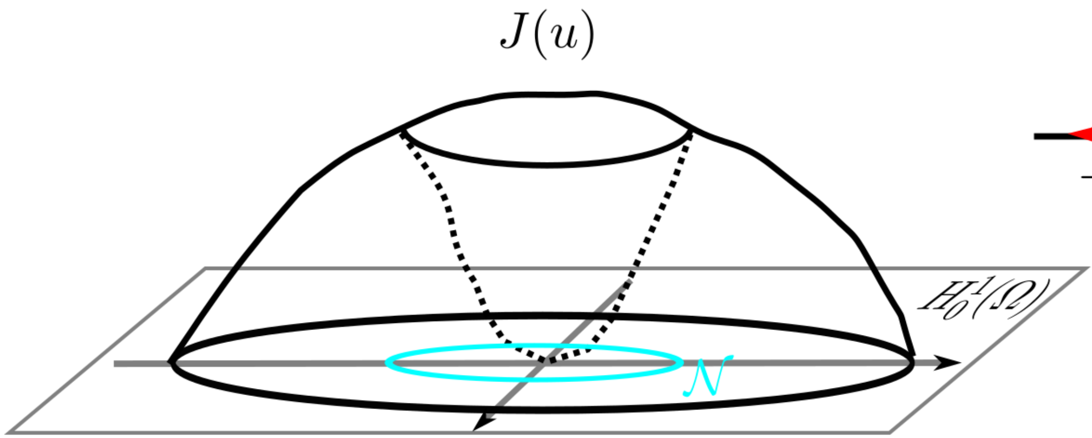
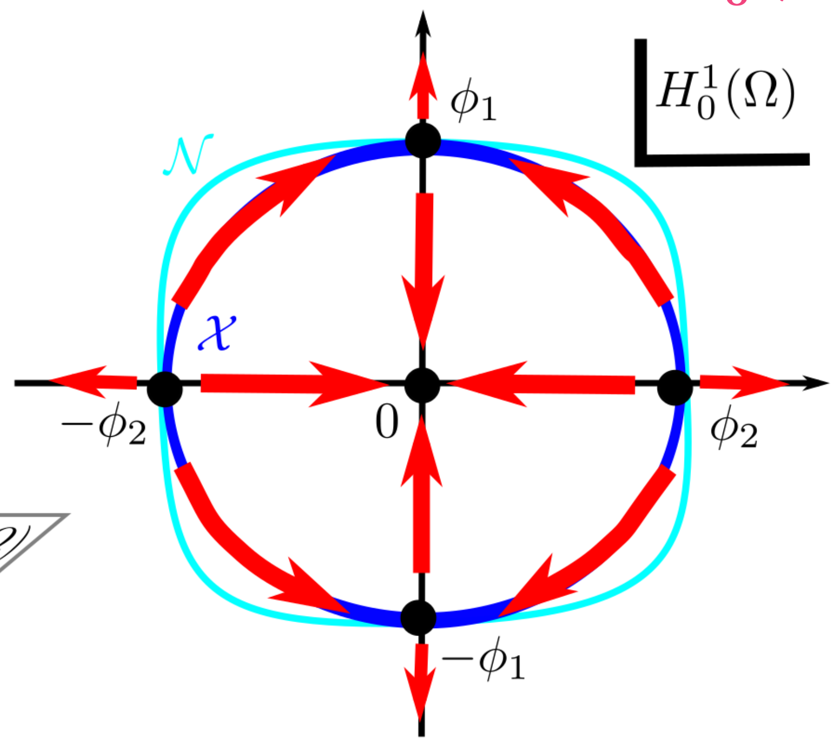
(ii) $\boldsymbol{\mathcal{X}}$ is (sequentially) weakly closed in $H^1_0(\Omega)$.

(iii) $\boldsymbol{\mathcal{X}}$ is a separatrix between stable and unstable sets for (RP) in $H^1_0(\Omega)$.





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Characterization of $\mathcal{X}$

Global dynamics of solutions to (RP) can be completely clarified, i.e.,

$$H_0^1(\Omega) = \mathcal{X}^+ \cup \mathcal{X} \cup \mathcal{X}^-$$

Proposition 4 (Characterization of $\mathcal{X}$)

Let $v(s)$ be a solution of (RP) with $v(0) = v_0$.

(i) If $v_0 \in \mathcal{X} = \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) = 1\}$,then

$$v(s_n) \to \phi \in \mathcal{S} \quad \text{strongly in } H_0^1(\Omega) \text{ as } s_n \to \infty.$$

(ii) If $v_0 \in \mathcal{X}^+ := \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) > 1\}$, then $v(s)$ diverges as $s \to \infty$. Hence $\mathcal{X}^+$ is an unstable set.

(iii) If $v_0 \in \mathcal{X}^- := \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) < 1\}$, then $v(s)$ vanishes in finite time. Hence $\mathcal{X}^-$ is a stable set.

3. Stability of non-isolated asymptotic profiles

Beyond the criteria: annulus case

Let us consider the annular domain,

$$\Omega = A_N(a,b) := \{x \in \mathbb{R}^N : a < |x| < b\}, \quad 0 < a < b.$$

If $(b-a)/a \ll 1$, then least energy solutions of (EF) are not radially symmetric (see [Coffman '84] and also [Y.Y. Li '90], [Byeon '97]).

Then least energy solutions of (EF) form a one-parameter family in $H_0^1(\Omega)$.
So this case is out of the criteria given by Theorem 2.

Non-isolated profiles of least energy

The solitary assumption of asymptotic profiles is essentially needed to verify their asymptotic stability. But, how about the stability ?
Namely, we shall discuss the following question:

Q Are non-isolated asymptotic profiles of least energy always stable or not ?

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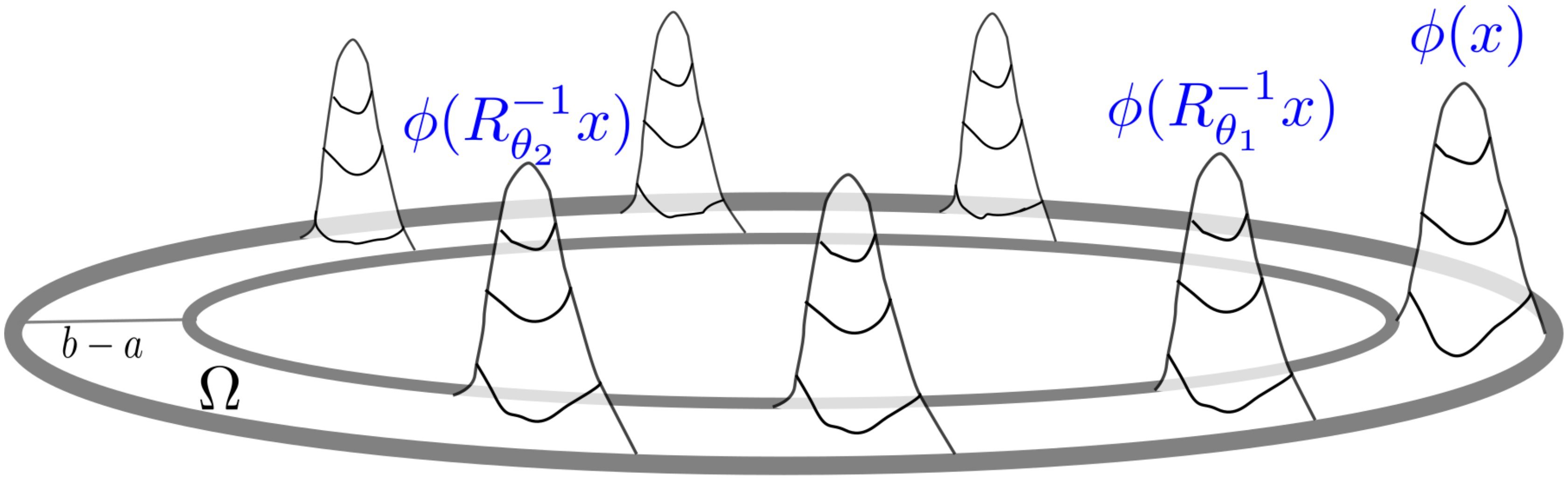

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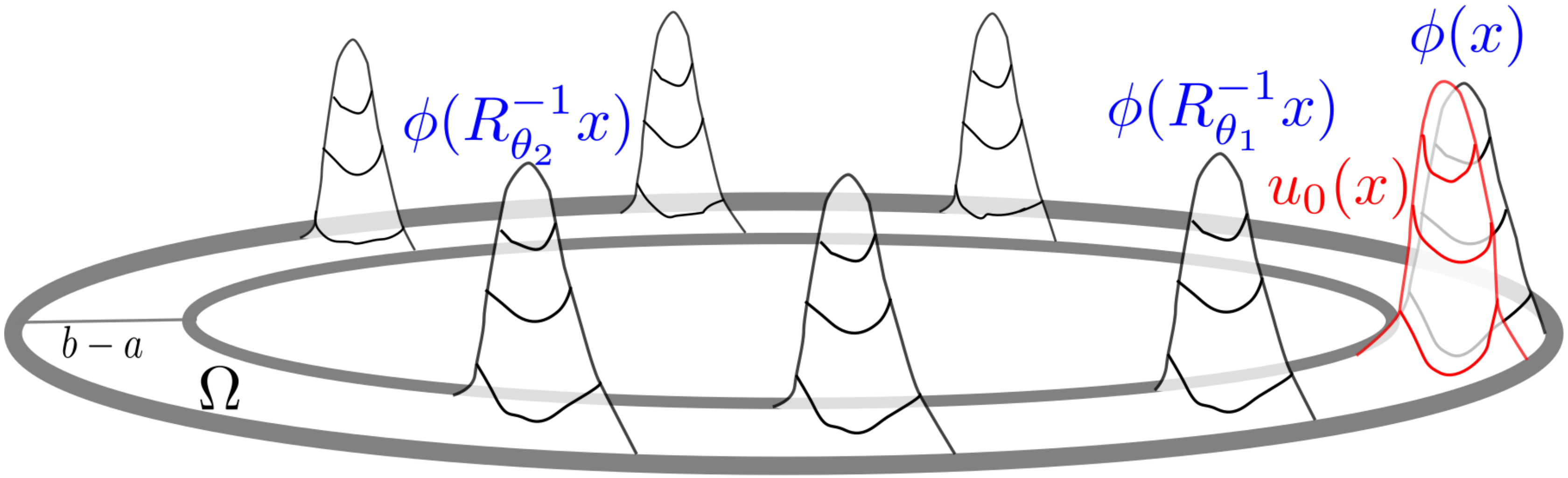

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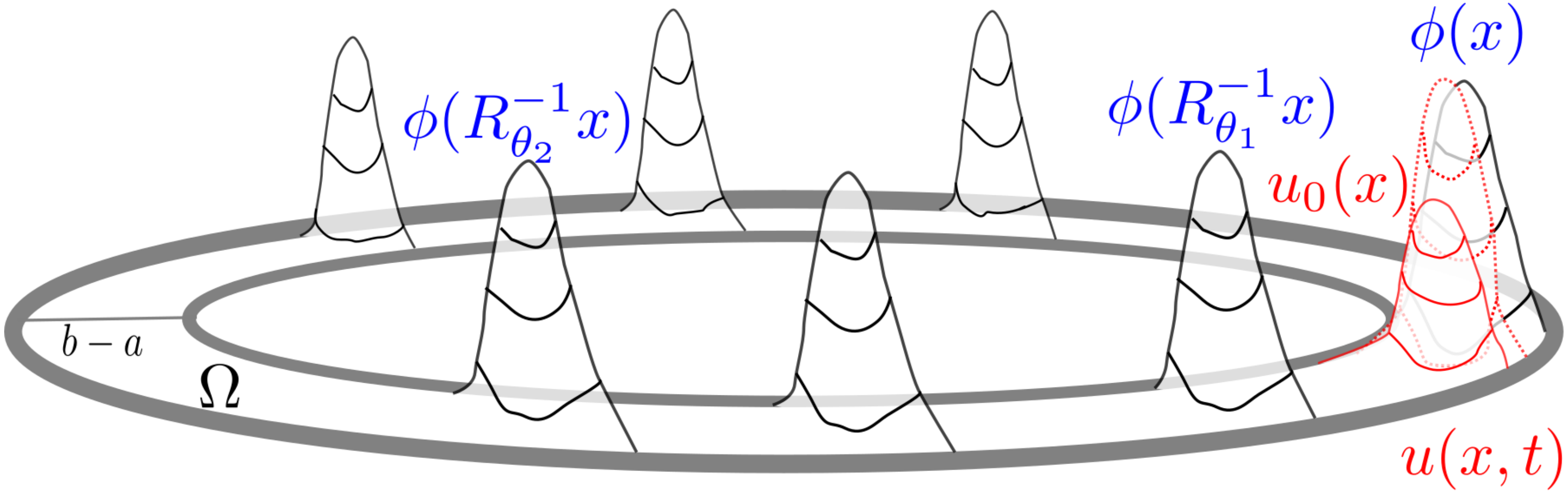

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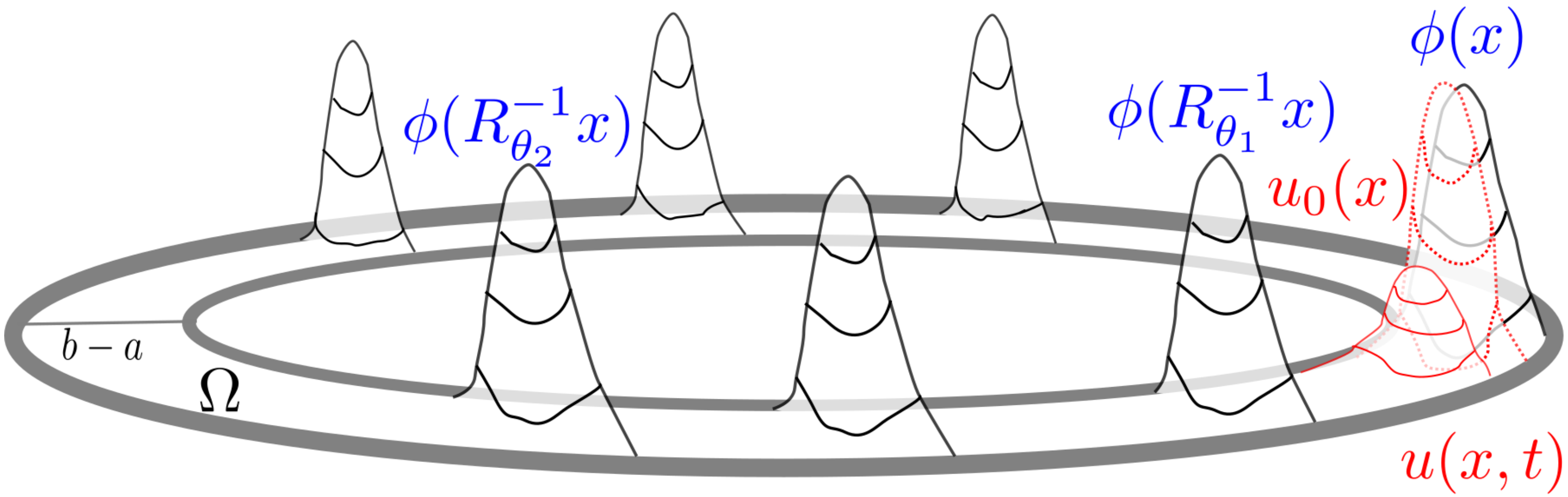

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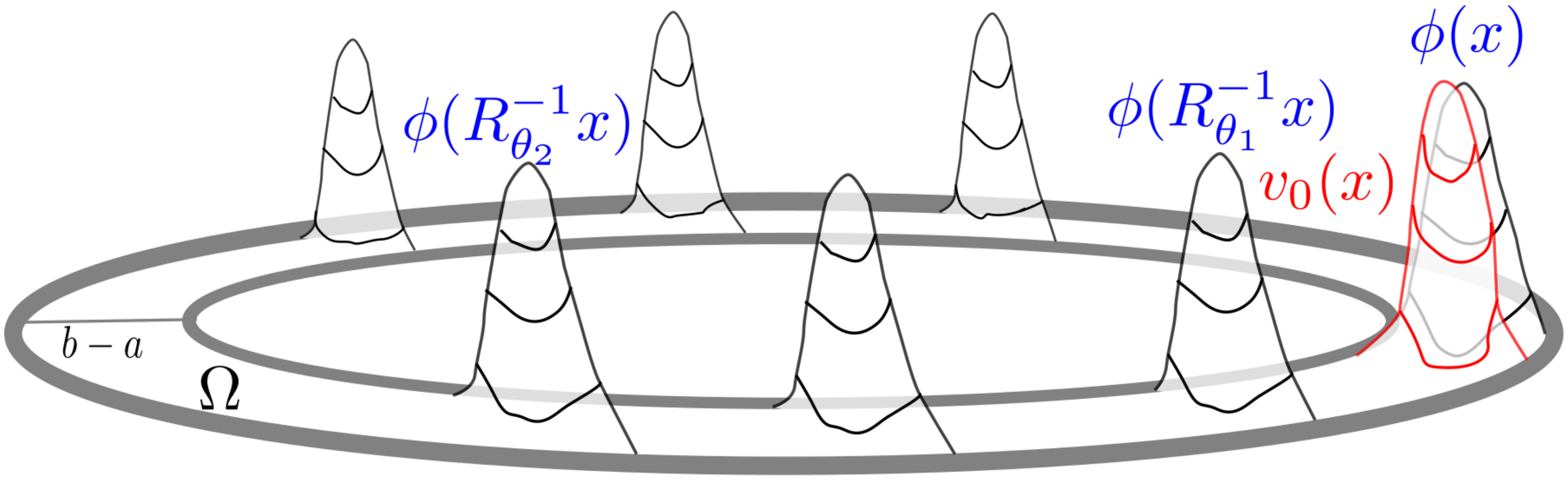

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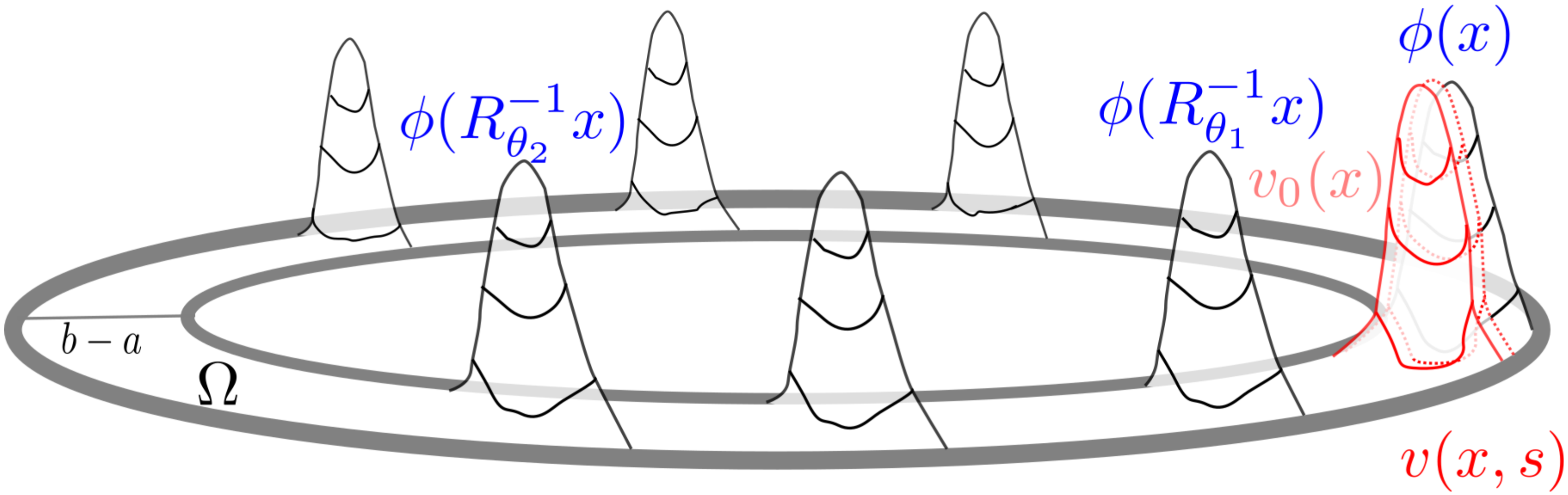

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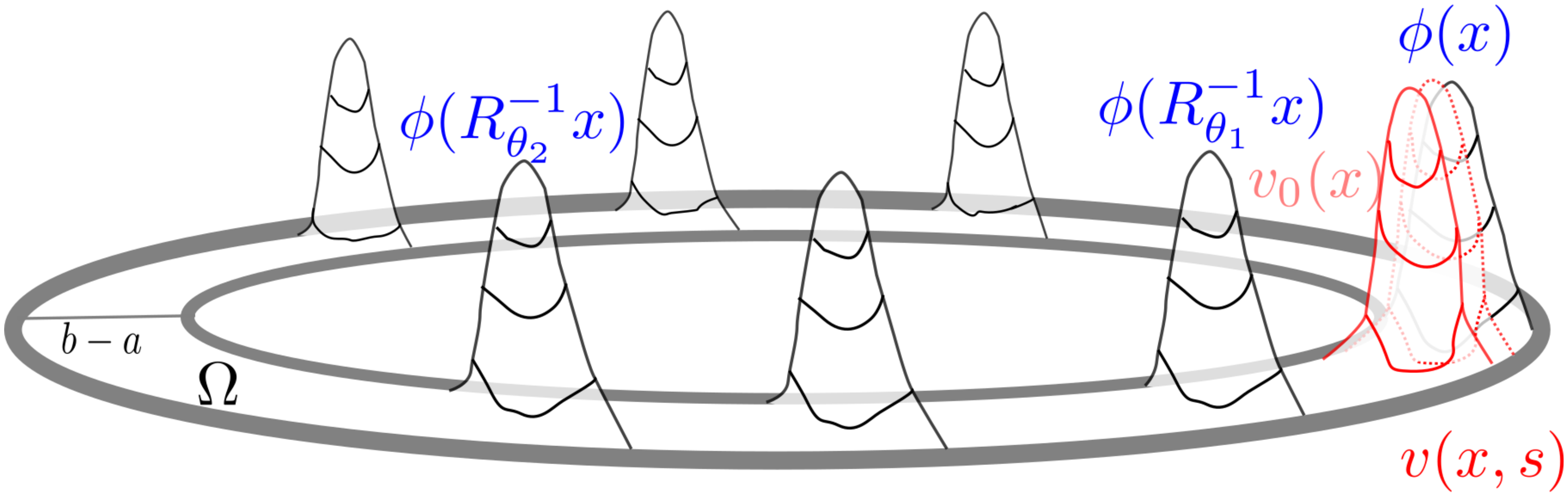

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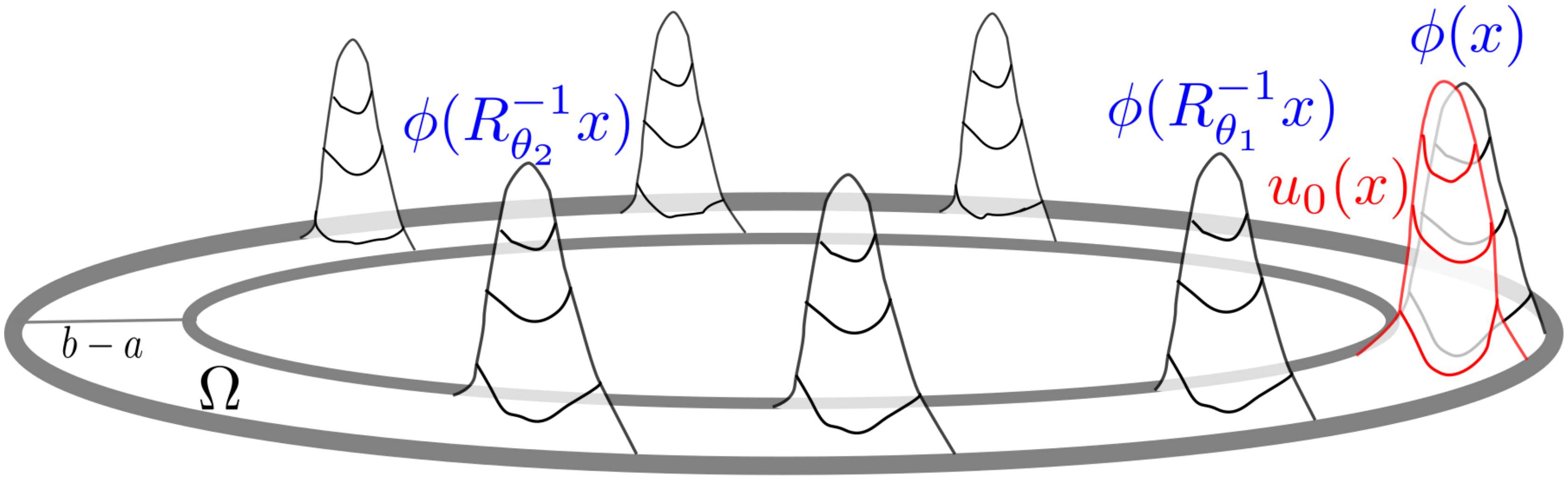

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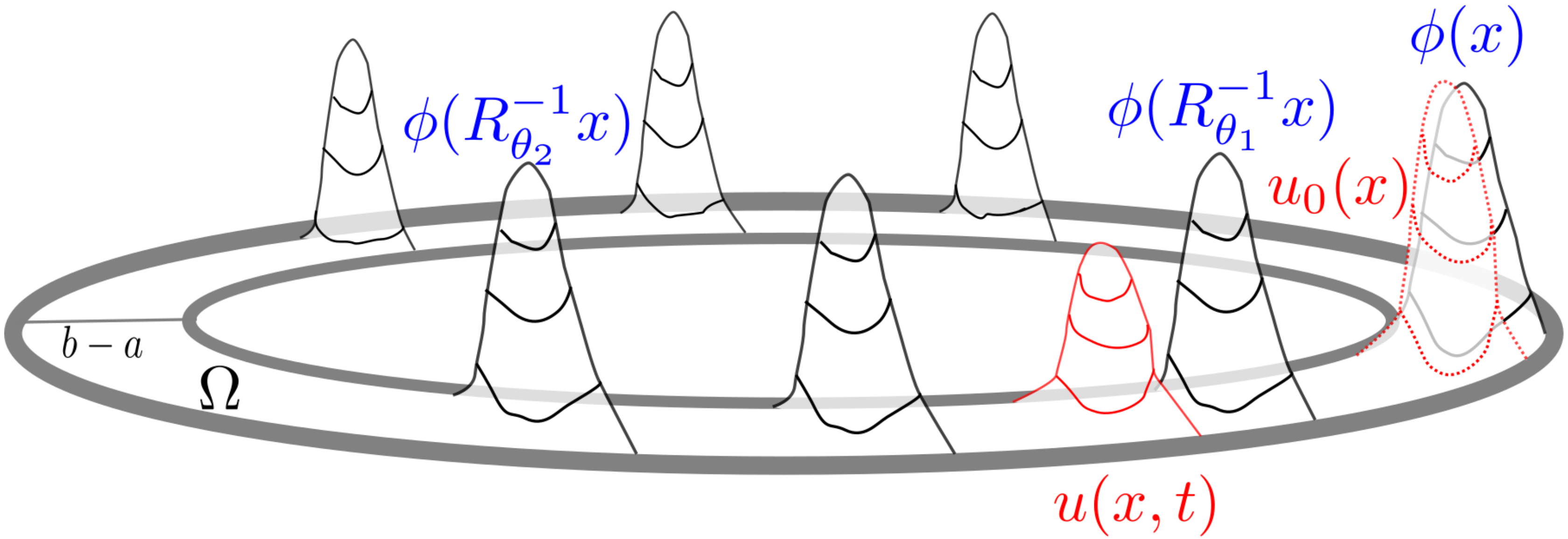

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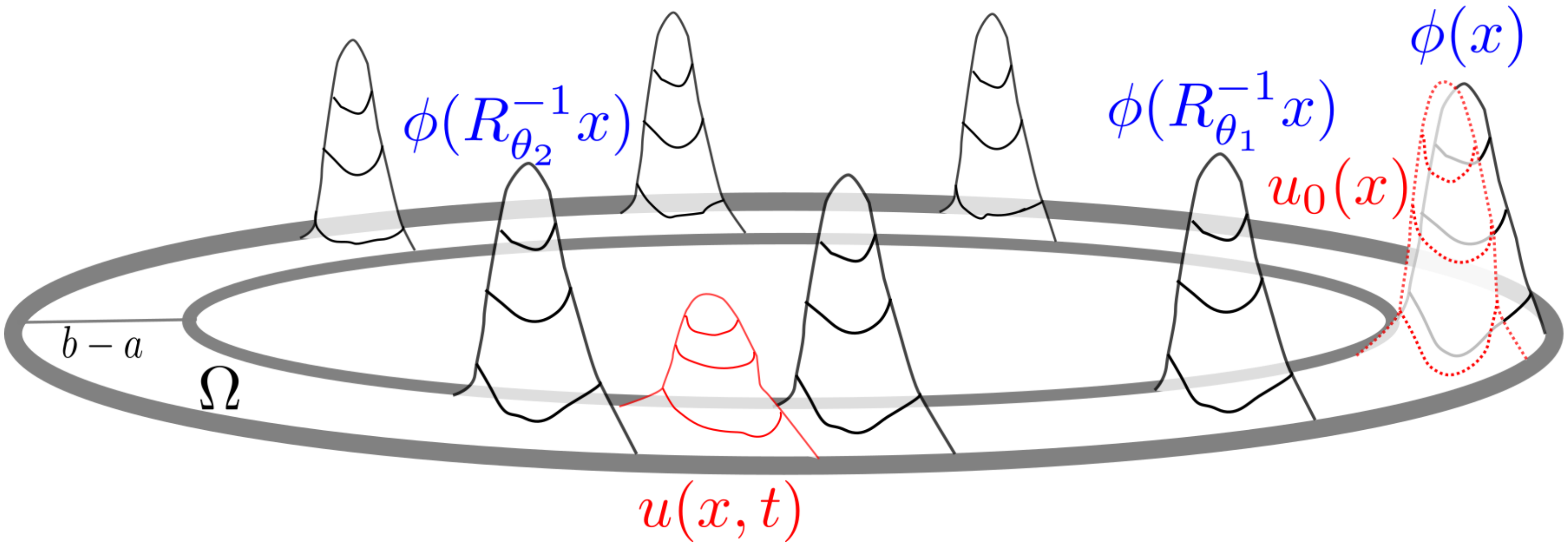

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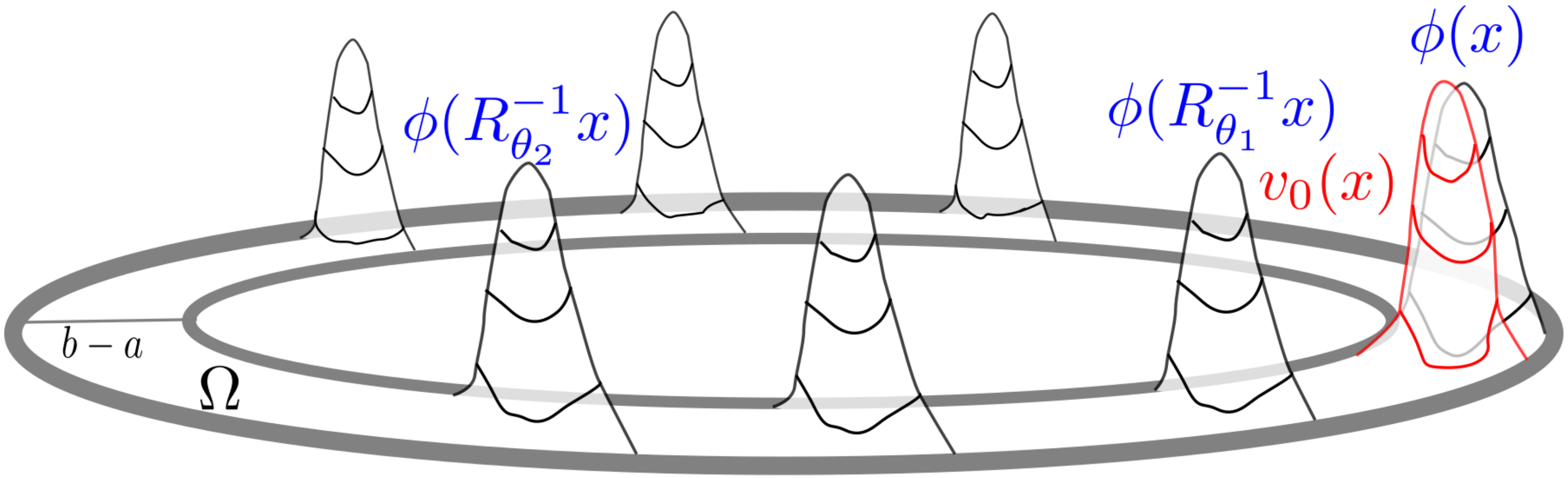

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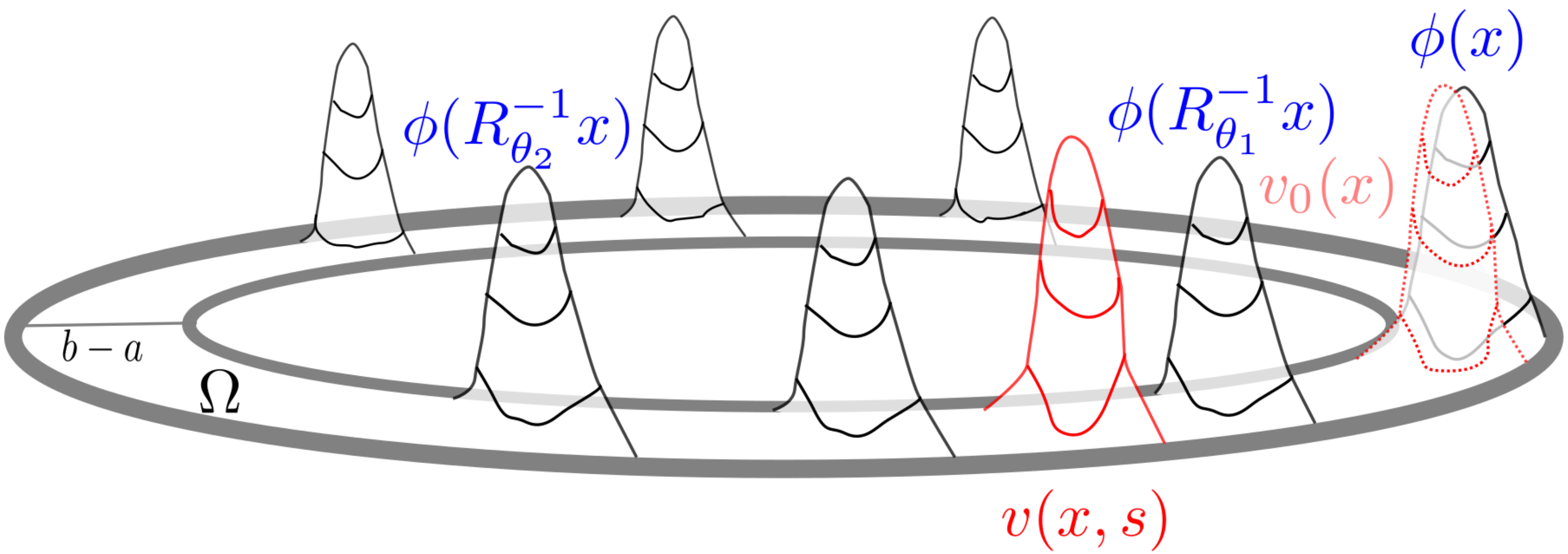

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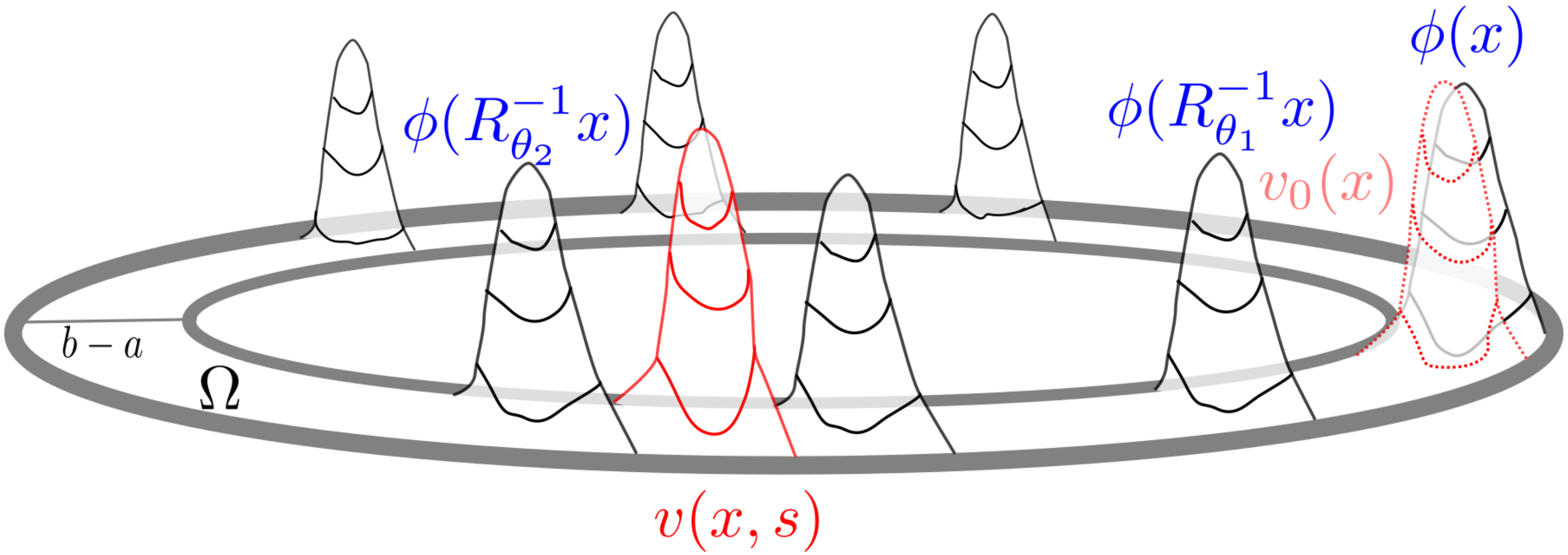


(ϕ is unstable)

Stability of all least energy profiles

Our result reads,

Theorem 5 (Stability of non-isolated profiles [A16])

Let $\phi > 0$ be any least energy solution of (EF).
Then ϕ is stable in the sense of Definition 1 (for possibly sign-changing data).

[A16] G. Akagi, Comm. Math. Phys. **345** (2016), 077-100.

Idea of proof

Goal

$$v_0 \sim \phi \ \text{ on } \mathcal{X} \quad \Rightarrow \quad \forall s \geq 0, \quad v(s) \sim \phi \ \text{ on } \mathcal{X}$$

Key claim:

$$v_0 \sim \phi \ \text{ on } \mathcal{X} \quad \Rightarrow \quad \sup_{s \geq 0} \|v(s) - v_0\|_{H^{-1}(\Omega)} \ll 1.$$

Remark: It is not true, if we do not restrict the phase space onto $\mathcal{X}$. Indeed, ϕ is a saddle point of $J(\cdot)$ over $H_0^1(\Omega)$.

$\Rightarrow$ The set $\mathcal{X}$ plays a crucial role !!

In particular, in the current setting, one may expect the existence of a "center manifold" on $\mathcal{X}$, since ϕ belongs to a one-parameter family of stationary points.

Łojasiewicz-Simon inequality for $J(\cdot)$

Let $\phi \geq 0$ be a least energy solution of (EF). Then by maximum principle,

$$0 < \phi(x) \leq {}^{\exists}L_{\phi} \text{ in } \Omega, \quad \partial_{\nu}\phi(x) < 0 \text{ on } \partial\Omega.$$

To prove the main result, we shall employ (see [Feireisl-Simondon'00]):

Proposition 6 (Łojasiewicz-Simon inequality for $J(\cdot)$)

$\forall L > L_{\phi}$, $\exists \theta \in (0, 1/2]$, $\exists \omega > 0$, $\exists \delta > 0$ s.t.

(ŁS) $$|J(w) - J(\phi)|^{1-\theta} \leq \omega \, \|J'(w)\|_{H^{-1}(\Omega)},$$

for all $w \in H_0^1(\Omega)$, $|w(\cdot)| \leq L$ a.e. in Ω, $\|w - \phi\|_{H_0^1(\Omega)} < \delta$.

ŁS for Lyapunov stability

Test (RP): $\partial_s(|v|^{m-2}v) = -J'(v)$ by $\partial_s v(s)$ to see that

$$\Big(\partial_s(|v|^{m-2}v), \partial_s v\Big) = -\frac{\mathrm{d}}{\mathrm{d}s} J(v(s)).$$

Suppose that $v(s)$ is uniformly bounded for $s \geq 0$. Then

$$C_2 \left\| \partial_s \left(|v|^{m-2} v \right) (s) \right\|_{H^{-1}(\Omega)}^2 \leq -\frac{\mathrm{d}}{\mathrm{d}s} J(v(s))$$

for some $C_2 > 0$ (depending on $L := \sup_{s \geq 0} \|v(s)\|_{L^\infty}$). Note by the Ł-S inequality that

$$\begin{aligned} \left\| \partial_s \left(|v|^{m-2} v \right) (s) \right\|_{H^{-1}(\Omega)} &\overset{\text{(RP)}}{=} \|J'(v(s))\|_{H^{-1}(\Omega)} \\ &\overset{\text{(ŁS)}}{\geq} \omega^{-1} \Big(J(v(s)) - J(\phi) \Big)^{1-\theta}. \end{aligned}$$

ŁS for Lyapunov stability

We obtain

$$C_2\omega^{-1}\Big(J(v(s)) - J(\phi)\Big)^{1-\theta}\left\|\partial_s\left(|v|^{m-2}v\right)(s)\right\|_{H^{-1}(\Omega)} \leq -\frac{\mathrm{d}}{\mathrm{d}s}\Big(J(v(s)) - J(\phi)\Big)$$

In case $J(v(s)) - J(\phi) > 0$, it follows that

$$\left\|\partial_s\left(|v|^{m-2}v\right)(s)\right\|_{H^{-1}(\Omega)} \leq -C_3\frac{\mathrm{d}}{\mathrm{d}s}\underbrace{\Big(J(v(s)) - J(\phi)\Big)^{\theta}}_{=:\,\mathcal{H}(s)}.$$

Integrate both sides over $(0, s)$.

ŁS for Lyapunov stability

Then

$$\int_0^s \left\| \partial_s \left(|v|^{m-2} v \right)(s) \right\|_{H^{-1}(\Omega)} \mathrm{d}s \le -C_3 \int_0^s \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{H}(s) \, \mathrm{d}s$$
$$= -C_3 \mathcal{H}(s) + C_3 \mathcal{H}(0)$$
$$\le C_3 \mathcal{H}(0)$$
$$= C_3 \Big(J(v(0)) - J(\phi) \Big)^\theta,$$

which implies

$$\left\| |v|^{m-2} v(s) - |v|^{m-2} v(0) \right\|_{H^{-1}(\Omega)} \le C_3 \Big(J(v(0)) - J(\phi) \Big)^\theta$$
$$\ll 1 \quad \text{if } \; v(0) \sim \phi \; \text{ on } \mathcal{X}.$$

$\Rightarrow$ desired conclusion (by fundamental inequalities).

A uniform extinction estimate for (FD)

Lemma 7 (Uniform estimate for rescaled solutions)

$\exists C = C(N, m) > 0$; $\forall s_0 \in (0, \log 2)$, $\forall v_0 \in \mathcal{X}$,

$$\|v(s)\|_{L^\infty(\Omega)} \leq C\,(e^{s_0} - 1)^{-\frac{N}{\kappa}} R(v_0)^{\frac{4m}{\kappa(m-2)}} \quad \text{for all } s \geq s_0.$$

with $\kappa := 2N - Nm + 2m > 0$ (by $m < 2^*$).

(cf. [DiBenedetto-Kwong-Vespri '91] for $v_0 \geq 0$)

- For $0 < s_0 \ll 1$, one can prove that

$$\sup_{s \in [0, s_0]} \|v(s) - v_0\|_{H_0^1(\Omega)} \ll 1.$$

- By Lemma 7, one can apply the ŁS argument for $v(s)$ on $[s_0, \infty)$:

$$v(s_0) \sim \phi \text{ on } \mathcal{X} \quad \Rightarrow \quad \sup_{s \geq s_0} \|v(s) - v_0\|_{H_0^1(\Omega)} \ll 1.$$

4. Instability of positive radial profiles in annular domains

Positive radial profiles in annular domains

Let us recall the annular domain,

$$\Omega = A_N(a,b) := \{x \in \mathbb{R}^N : a < |x| < b\}, \quad 0 < a < b.$$

Then (EF) admits a unique positive radial solution $\phi > 0$ (cf. [Ni '83]).

If $(b-a)/a \ll 1$, then least energy solutions of (EF) are not radially symmetric.

Thereby, the positive radial profile ϕ may NOT take the least energy and it is NOT sign-changing. Hence ϕ is also beyond the scope of the stability criteria of [AK13].

Instability of positive radial profiles for $N = 2$

[AK14] G. Akagi, R. Kajikiya, AIHP (C) 31 (2014), no.6 1155–1173.

Theorem 8 (Instability of positive radial profiles [AK14])

Let $\Omega = A_N(a, b)$ and assume that

(11) $$\left(\frac{b}{a}\right)^{(N-3)_+} \left(\frac{b-a}{\pi a}\right)^2 < \frac{m-2}{N-1}.$$

Let ϕ be the unique positive radial solution of (EF).

Then ϕ is NOT asymptotically stable in the sense of profile.

In addition, if $(b-a)/a \ll 1$ and $N = 2$, then ϕ is unstable.

Q Can we prove the instability <u>for general N</u> under the <u>quantitative condition (11)</u> ?

Instability of positive radial profiles for general N

Our result reads,

Theorem 9 (Instability of positive radial profiles [A16])

Let $\Omega = A_N(a,b)$ and assume that

(11)
$$\left(\frac{b}{a}\right)^{(N-3)_+}\left(\frac{b-a}{\pi a}\right)^2 < \frac{m-2}{N-1}.$$

Then the positive radial profile ϕ is unstable.

Non-radial perturbation to ϕ ($N = 2$ for simplicity):

$$\phi_\varepsilon(x) = (1+\varepsilon\cos\theta)\phi(r) \quad \text{for} \ \ x = x(r,\theta).$$

Then $\phi_\varepsilon \notin \mathcal{X}$. However, $v_{0,\varepsilon} := t_*(\phi_\varepsilon)^{-1/(m-2)}\phi_\varepsilon \in \mathcal{X}$.

Proof

Under (11), one can (explicitly) construct $v_{0,\varepsilon} \in \mathcal{X}$ such that

$$v_{0,\varepsilon} \to \phi \text{ in } H_0^1(\Omega) \text{ as } \varepsilon \to 0_+ \quad \text{and} \quad J(v_{0,\varepsilon}) < J(\phi) \text{ if } \varepsilon > 0.$$

Therefore there exist $\psi_\varepsilon \in \mathcal{S}$ such that the solution v_ε of (RP) with $v_\varepsilon(0) = v_{0,\varepsilon}$ satisfies

$$v_\varepsilon(s) \to \psi_\varepsilon \text{ in } H_0^1(\Omega), \quad J(\psi_\varepsilon) \le J(v_{0,\varepsilon}) < J(\phi).$$

Claim. ψ_ε does not converge to ϕ as $\varepsilon \to 0_+$.

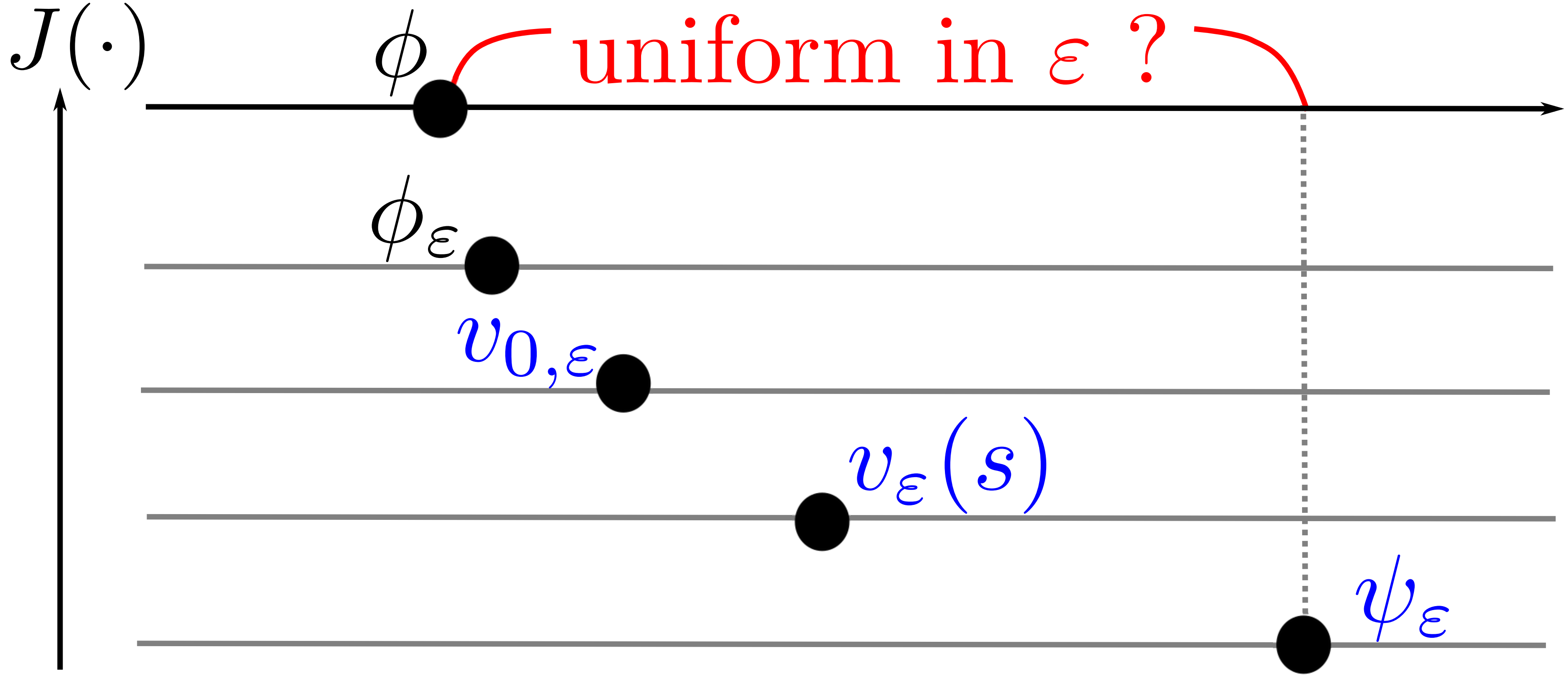
Suppose on the contrary that $\psi_\varepsilon \to \phi$. Then, due to the ŁS inequality (6),

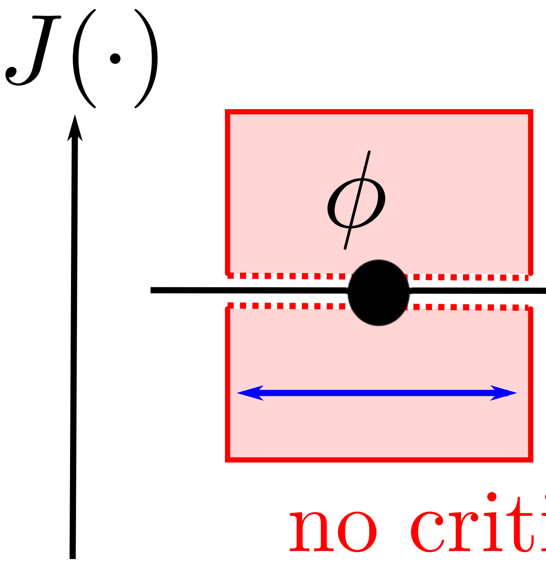
$$J(\psi_\varepsilon) = J(\phi) \quad \text{for } \varepsilon \ll 1,$$

which is a contradiction to the difference of the energy.

Consequently, the solution v_{ε_n} of (RP) with $v_{\varepsilon_n}(0) = v_{0,\varepsilon_n}$ cannot stay within a small neighborhood of ϕ. □

Instability of other profiles





By Łojasiewicz-Simon ineq.:

$$|J(\phi) - J(w)|^{1-\theta} \leq \omega \|J'(w)\|_{H^{-1}(\Omega)}$$

uniform in ε

no critical points with different energies

Remarks

The main results can be extended to local minimizers of $\boldsymbol{J}$ over $\boldsymbol{\mathcal{X}}$, i.e., $\phi \in H_0^1(\Omega) \setminus \{0\}$ satisfying

$$J(\phi) = \inf\{J(w) \colon w \in \mathcal{X} \cap B_{H_0^1(\Omega)}(\phi; r_0)\} \quad \text{for some } r_0 > 0.$$

(1) Theorem 5 is extended as follows:

Theorem 10 (Stability of local minimizers of J over $\mathcal{X}$)

Let ϕ be a local minimizer of J over $\mathcal{X}$. Then ϕ is stable in the sense of Definition 1.

(2) Theorem 9 is extended to

Theorem 11 (Instability of sign-definite profiles)

Let ϕ be a positive or negative profile except for local minimizers of J over $\mathcal{X}$. Then ϕ is unstable.

5. Exponential stability of asymptotic profiles

Hierarchy of stability

Notion of stability

Exponential stability

$\Downarrow$

Asymptotic stability

$\Downarrow$

Stability

Q Can we prove exponential stability for some class of isolated asymptotic profiles of least energy ?

Hierarchy of stability

Notion of stability	$\Leftarrow$	Criteria
Exponential stability		**?**
$\Downarrow$		$\cap$
Asymptotic stability		isolated profiles of LE
$\Downarrow$		$\cap$
Stability		any profiles of LE

LE = least energy

Q Can we prove exponential stability for <u>some class of</u> isolated asymptotic profiles of least energy ?

Exponential stability of non-degenerate LESs

Theorem 12 (Exponential stability of non-degenerate LESs)

Let $\phi > 0$ be a non-degenerate least energy solution of (EF), namely,

$$\mathcal{L}_\phi := -\Delta + \lambda_m (m-1)|\phi|^{m-2} \text{ is invertible.}$$

Then ϕ is exponentially stable, i.e., ϕ is stable, and moreover,

- there exist $C, \mu, \delta_0 > 0$ s.t. any solution $v(x,s)$ of (RP) satisfies

$$\|v(s) - \phi\|_{H_0^1(\Omega)} \leq Ce^{-\mu s} \quad \text{for all} \quad s \geq 0,$$

  provided that $v(0) \in \mathcal{X}$ and $\|v(0) - \phi\|_{H_0^1(\Omega)} < \delta_0$.

In particular, $\mu = \mu(\Omega, N, m, \|\mathcal{L}_\phi^{-1}\|)$.

cf.) Exponential convergence of any nonnegative solution for (FD) with $m < m_\sharp$ for some $m_\sharp \in (2, \infty)$ (Bonforte-Grillo-Vazquez '12).

Outline of proof

Since $J''(\phi) = \mathcal{L}_\phi$ is non-degenerate, one can prove the following gradient inequality:

Proposition 13 (Gradient inequality)

For any $\omega > \|\mathcal{L}_\phi^{-1}\|_{\mathcal{L}(H^{-1}(\Omega);H_0^1(\Omega))}$, there exists $\delta > 0$ such that

$$|J(w) - J(\phi)|^{1/2} \leq \omega \|J'(w)\|_{H^{-1}(\Omega)} \tag{12}$$

for all $w \in H_0^1(\Omega)$ satisfying $\|w - \phi\|_{H_0^1(\Omega)} < \delta$.

Remark: In ŁS inequalities, it could be difficult to identify the exponent θ (indeed, θ might be less than $1/2$). On the other hand, $\theta = 1/2$ will play a crucial role to prove the exponential stability.

Outline of proof

Test (RP): $\partial_s(|v|^{m-2}v) = -J'(v)$ by $\partial_s v(s)$ to see that

$$\underbrace{\Big(\partial_s(|v|^{m-2}v), \partial_s v\Big)}_{(m-1)\int_\Omega |v|^{m-2}|\partial_s v|^2 \, \mathrm{d}x = \frac{4}{mm'}\int_\Omega |\partial_s v^{\frac{m}{2}}|^2 \, \mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}s} J(v(s)).$$

Since ϕ is stable (i.e., $\|v(s)\|_{H_0^1(\Omega)} \approx \|\phi\|_{H_0^1(\Omega)}$), one can derive

$$C \left\|\partial_s \left(|v|^{m-2}v\right)(s)\right\|^2_{H^{-1}(\Omega)} \le -\frac{\mathrm{d}}{\mathrm{d}s} J(v(s))$$

for some $C > 0$ depending on ϕ.

Here, by gradient inequality, we find that

$$\begin{aligned} \left\|\partial_s \left(|v|^{m-2}v\right)(s)\right\|_{H^{-1}(\Omega)} &\overset{\text{(RP)}}{=} \|J'(v(s))\|_{H^{-1}(\Omega)} \\ &\overset{\text{(GI)}}{\ge} \omega^{-1}\Big(J(v(s)) - J(\phi)\Big)^{1/2}. \end{aligned}$$

Outline of proof

We obtain

$$C\omega^{-2}\Big(J(v(s)) - J(\phi)\Big) \le -\frac{\mathrm{d}}{\mathrm{d}s}\Big(J(v(s)) - J(\phi)\Big)$$

It follows that

$$0 \le J(v(s)) - J(\phi) \le (J(v(0)) - J(\phi))\, e^{-\mu s}.$$

On the other hand, as in the proof of stability, one can derive

$$\|\partial_s(|v|^{m-2}v)(s)\|_{H^{-1}(\Omega)} \le -\frac{C}{\theta}\frac{\mathrm{d}}{\mathrm{d}s}\Big(J(v(s)) - J(\phi)\Big)^{\frac{1}{2}}.$$

Integrate this over (s, ∞). Then

$$\int_s^\infty \|\partial_\sigma(|v|^{m-2}v)(\sigma)\|_{H^{-1}(\Omega)}\,\mathrm{d}\sigma \le \frac{C}{\theta}\Big(J(v(s)) - J(\phi)\Big)^{\frac{1}{2}}.$$

Outline of proof

Hence

$$\left\|\phi^{m-1} - |v|^{m-2}v(s)\right\|_{H^{-1}(\Omega)} \leq \int_s^{\infty} \|\partial_\sigma(|v|^{m-2}v)(\sigma)\|_{H^{-1}(\Omega)}\, \mathrm{d}\sigma$$
$$\leq \frac{C}{\theta}\Big(J(v(s)) - J(\phi)\Big)^{\frac{1}{2}} \leq Ce^{-\frac{\mu}{2}s}.$$

Furthermore, we can also derive

- $\|\phi - v(s)\|_{L^m}^m \leq \langle \phi^{m-1} - |v|^{m-2}v(s), \phi - v(s)\rangle_{H_0^1} \leq Ce^{-\frac{\mu}{2}s}$,
- $\frac{1}{2}\Big(\|\nabla v(s)\|_{L^2}^2 - \|\nabla\phi\|_{L^2}^2\Big) \leq$ (diff. of J and $\|\cdot\|_{L^m}^m$) $\leq Ce^{-\frac{\mu}{2m}s}$,

and then, we finally obtain

$$\|v(s) - \phi\|_{H_0^1}^2 = \|\nabla v(s)\|_{H_0^1}^2 - \|\nabla\phi\|_{H_0^1}^2 - 2\left(\nabla\phi, \nabla(v(s) - \phi)\right)_{L^2}$$
$$\leq Ce^{-\frac{\mu}{2m}s}. \quad \square$$

Examples of non-degenerate least energy solutions

- (Dancer ('88)). Let $2 < m < 2^*$ and Ω be a bounded convex domain in $\mathbb{R}^2$, which is symmetric w.r.t. the coordinate axes. Then positive solution is unique and nondegenerate (see also [Pacella '05]).

- (Lin ('94)). Let $2 < m < 2^*$ and Ω be a bounded convex domain in $\mathbb{R}^2$. Then least energy solution is unique and nondegenerate.

- (Grossi ('00)). Let $N \geq 3$ and $2^* - \delta < m < 2^*$ with a small $\delta > 0$. Let $\Omega \subset \mathbb{R}^N$ be convex in x_i and symmetric w.r.t. $[x_i = 0]$ for each $1 \leq i \leq N$. Then positive solution is unique and nondegenerate.

- (Dancer ('03)). Let $2 < m < 2 + \delta$ with a small $\delta > 0$ and Ω be any bounded smooth domain in $\mathbb{R}^N$. Then positive solution is unique and nondegenerate.

6. Sobolev-critical case

Work in progress

joint work with N. Ikoma (Keio Univ., Japan)

FDEs with the Sobolev critical exponent

Let us consider the Sobolev-critical case (i.e., the case $m = 2^*$),

$$\begin{aligned} \partial_t \left(|u|^{2^*-2}u\right) &= \Delta u + \mu u && \text{in } \Omega \times (0,\infty), \\ u &= 0 && \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where Ω is a b'dd domain of $\mathbb{R}^N$, $N \geq 3$, $\mu < \lambda_1(\Omega)$ and $2^* = \frac{2N}{(N-2)_+}$ (here $\lambda_1(\Omega)$ is the principal eigenvalue of $-\Delta$). Then one can prove that

- for each $u_0 \in H_0^1(\Omega) \setminus \{0\}$ there exists $t_*(u_0) > 0$ such that

  $$c(t_* - t)_+^{1/(2^*-2)} \leq \|u(t)\|_{H_0^1(\Omega)} \leq C(t_* - t)_+^{1/(2^*-2)}$$

  for some $0 < c < C < +\infty$.

FDEs with the Sobolev critical exponent

Set $v(x,s) := (t_* - t)_+^{-1/(2^*-2)} u(x,t)$ with $s = \log(t_*/(t_* - t))$.
Then $v(x,s)$ solves (RP), that is,

$$\begin{aligned} \partial_s \left(|v|^{2^*-2} v\right) &= \Delta v + \mu v + \lambda_* |v|^{2^*-2} v && \text{in } \Omega \times (0,\infty), \\ v &= 0 && \text{on } \partial\Omega \times (0,\infty), \\ v(\cdot,0) &= v_0 && \text{in } \Omega, \end{aligned}$$

where $\lambda_* = \frac{2^*-1}{2^*-2} > 0$ and $v_0 = t_*(u_0)^{-1/(2^*-1)} u_0$. Moreover,

- it holds that $c \le \|v(s)\|_{H_0^1(\Omega)} \le C$ for all $s \ge 0$,
- $J_\mu(v(s)) := \frac{1}{2}\|\nabla v(s)\|_{L^2}^2 + \frac{\mu}{2}\|v(s)\|_{L^2}^2 - \frac{\lambda_*}{2^*}\|v(s)\|_{L^{2^*}}^{2^*}$ is non-increasing in s,
- however, the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is no longer compact.

Sobolev-critical case (contd.)

Even if $m = 2^*$, the following facts are still valid:

- $(v(s_n))$ is a (PS)-sequence for $J(\cdot)$ along some sequence $s_n \to \infty$,
- $J_\mu(w) \geq d_1$ for all $w \in \mathcal{X}$ (proof requires more effort),
- an asymptotic profile $\phi(x)$ of $u(x,t)$ can be defined as a limit of $v(x, s_n)$ (in $H_0^1(\Omega)$) along a seq. $s_n \to +\infty$ and characterized by

  (BN) $$-\Delta\phi - \mu\phi = \lambda_* |\phi|^{2^*-2}\phi \ \text{ in } \Omega, \quad \phi|_{\partial\Omega} = 0.$$

- notions of stability of asymptotic profiles can be also defined in the same manner for (regular) profiles,

On the other hand, it is unclear

does each solution $v(x,s)$ have a (regular) asymptotic profile ?

due to the lack of compactness embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

Sobolev-critical case without lower order term

As for the case $\mu = 0$, i.e.,

$$\begin{aligned} \partial_t\left(|u|^{2^*-2}u\right) &= \Delta u && \text{in } \ \Omega \times (0,\infty), \\ u &= 0 && \text{on } \ \partial\Omega \times (0,\infty), \\ u(\cdot,0) &= u_0 && \text{in } \ \Omega, \end{aligned}$$

Galaktionov and King ('02) If u_0 is positive and radial, then

$$\|u(t)\|_{L^\infty} = C(t_* - t)_+^{\frac{1}{2^*-2}} |\log(t_* - t)|^{\frac{N+2}{2N-4}}(1 + o(1))$$

as $t \nearrow t_*$. It also yields

$$\|v(s)\|_{L^\infty} = C|s - \log t_*|^{\frac{N+2}{2N-4}}(1 + o(1)) \ \text{ as } s \nearrow \infty.$$

Sobolev-critical case without lower order term

If $m = 2^*$ and $\Omega = \mathbb{R}^N$, (FD) and $J(\cdot)$ are invariant under the scaling,

$$v(x, s) \mapsto v_\mu(\xi, s) = \mu^{\frac{N-2}{2}} v(\mu\xi, s).$$

In particular, we remark that

- d_1 is never attained by non-trivial solutions to (EF). Furthermore, it is characterized with a Talenti function $W(x)$ by

$$d_1 := \inf_{w \in \mathcal{S}} J(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla W(x)|^2 dx - \frac{\lambda_m}{m}\int_{\mathbb{R}^N} |W(x)|^m dx.$$

Global compactness result

By applying Struwe's global compactness result (Struwe '84, Bahri-Coron '88),

Proposition 14 (Global compactness)

There exist $k \in \mathbb{N} \cup \{0\}$, sequences (R_n^j) in $(0, +\infty)$ and (x_n^j) in Ω, a solution $\phi \in H_0^1(\Omega)$ of (EF) and nontrivial solutions $\psi^j \in D^{1,2}(\mathbb{R}^N)$ $(j = 1, 2, \dots, k)$ to the limiting problem

$$-\Delta \psi^j = \lambda_* |\psi^j|^{m-2} \psi^j \quad \text{in } \mathbb{R}^N$$

such that, up to a subsequence,

$$R_n^j \to \infty \quad \text{and} \quad \left\| v(s_n) - \phi - \sum_{j=1}^{k} \psi_n^j \right\|_{D^{1,2}(\mathbb{R}^N)} \to 0$$

as $n \to \infty$. Here $\psi_n^j(x) = (R_n^j)^{(N-2)/2} \psi^j (R_n^j (x - x_n^j))$.

Global compactness result

Proposition 15 (Global compactness)

Moreover, we have

$$J(v(s_n)) \to J(\phi) + \sum_{j=0}^{k} J_{\mathbb{R}^N}(\psi^j)$$

with

$$J_{\mathbb{R}^N}(w) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w(x)|^2 dx - \frac{\lambda_m}{m}\int_{\mathbb{R}^N} |w(x)|^m dx.$$

Furthermore, if $i \neq j$, then

$$\frac{R_m^i}{R_m^j} + \frac{R_m^j}{R_m^i} + R_m^i R_m^j |x_m^i - x_m^j|^2 \to +\infty \quad \text{as } m \to +\infty.$$

Star-shaped domain case

Let us consider the case that Ω is strictly star-shaped w.r.t. 0 and $u_0 \geq 0$ (hence $v_0 \geq 0$ and $v(\cdot, s_n) > 0$). Then $\phi \geq 0$ and $\psi^j > 0$.

On the other hand, by a well-known nonexistence result, (EF) admits no positive solution. Hence $\phi \equiv 0$.

Furthermore, we claim that $k \neq 0$. Indeed, if $k = 0$, then

$$v(s_n) \to 0 \quad \text{strongly in } H_0^1(\Omega) \quad \text{and} \quad J(v(s_n)) \to 0.$$

However, since $J(v(s_n)) \geq d_1 > 0$ by $v(s_n) \in \mathcal{X}$, it yields a contradiction.

Star-shaped domain case (contd.)

Moreover, one can observe that $\psi^j = W$, a Talenti function, and

$$J(v(s_n)) \to k\beta \quad \text{with} \quad \beta := J_{\mathbb{R}^N}(W).$$

Remark. k is uniquely determined, for $J(v(\cdot))$ is nonincreasing.

Observation

Let Ω be strictly star-shaped and assume that $u_0 \geq 0$ and $k\beta < J(u_0) \leq (k+1)\beta$. Then $v(\cdot)$ forms at least one and at most k bubbles along a sequence $s_n \to +\infty$, i.e.,

$$v(x, s_n) \sim \sum_{j=1}^{k} \psi_n^j(x) \quad \text{for} \quad n \gg 1,$$

where $\psi_n^j(x) = (R_n^j)^{(N-2)/2} W(R_n^j(x - x_n^j))$, for some $s_n \to +\infty$.

Brezis-Nirenberg result

Proposition 16 (Brezis-Nirenberg '83)

- In case $n \geq 4$, for any $\mu \in (0, \lambda_1(\Omega))$,
- In case $n = 3$, there exists $\mu_* \in [0, \lambda_1(\Omega))$ such that for any $\mu \in (\mu_*, \lambda_1(\Omega))$,

the Dirichlet problem

(BN) $$-\Delta\phi - \mu\phi = \lambda_* |\phi|^{2^*-2}\phi \ \text{ in } \Omega, \quad \phi|_{\partial\Omega} = 0$$

admits a positive solution $\phi > 0$.

Remark In case $n = 3$ and $\Omega = B(0; 1) \subset \mathbb{R}^3$,

- $\mu \leq \mu_* = \lambda_1(\Omega)/4$,
- (BN) has no positive solution for any $\mu \leq \mu_* = \lambda_1(\Omega)/4$.

Lemma 17 (Local compactness [BN '83, Struwe '90])

Any sequence (u_n) in $H_0^1(\Omega)$ satisfying

$$J_\mu(u_n) \to {}^\exists\beta < \frac{1}{N} S_0^{N/2}, \quad J'_\mu(u_n) \to 0 \ \text{ strongly in } H^{-1}(\Omega)$$

is precompact in $H_0^1(\Omega)$. Here S_0 denotes the infimum of the Rayleigh quotient,

$$S_0 := \inf_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^{2/2^*}_{L^{2^*}(\Omega)}}.$$

Remark Under the assumptions of the BN result, one can check the above for the mountain-pass level, that is, $d_{1,\mu} = \inf_{w \in \mathcal{S}} J_\mu < \frac{1}{N} S_0^{N/2}$.

Results

Theorem 18 (Convergence to least energy profiles)

In addition to the assumptions of the Brezis-Nirenberg result, suppose that

$$v_0 \in \mathcal{X}, \quad J(v_0) < \frac{1}{N} S_0^{N/2}.$$

For any $s_n \to +\infty$, there exist a subsequence (n') of (n) and a non-trivial solution ϕ of (BN) such that

$$v(s_{n'}) \to \phi \quad \text{strongly in } H_0^1(\Omega).$$

Moreover, if either $\phi > 0$ or $N = 3, 4,$ then

$$v(s) \to \phi \quad \text{strongly in } H_0^1(\Omega) \ \text{ as } \ s \to +\infty.$$

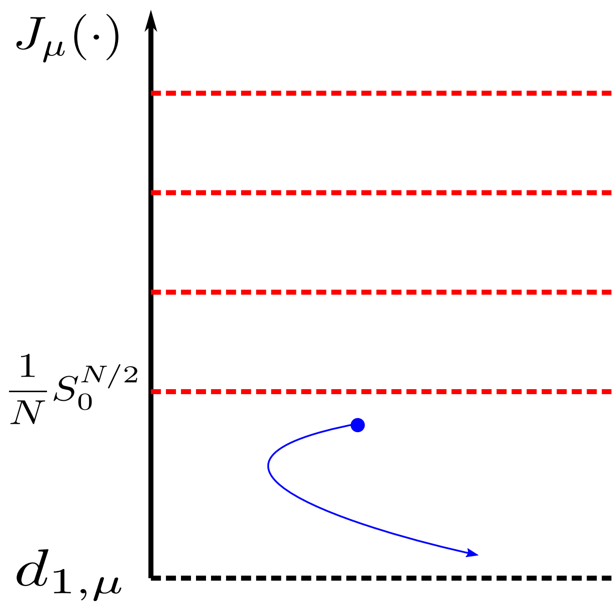
Results

Theorem 19 (Stability of least energy profiles)

Asymptotic profiles of least energy are stable.

Key of proofs

- Lack of compactness $\longrightarrow$ Local compactness result
- Lack of uniform boundedness for $v(x, s)$ $\longrightarrow$ refine arguments to exclude the use of uniform boundedness
  - ŁS inequality for power nonlinearities (with a singularity at the origin) [Feireisl-Simondon '00]: Ł ineq. is applied to a cut-offed function. Remove unif. b'ddness of solutions [A-Schimperna-Segatti, preprint]
  - Energy arguments to handle $\partial_t(|v|^{2^*-2}v)$ Improvable for the critical case



Remark

- “Compactness” is needed to realize a “regular” asymptotic profile.
  - FDE with a lower order term (cf. Brezis-Nirenberg type)
  - “Symmetric” domains with a hole
- “Non-compactness” causes a “singular” asymptotic profile (e.g., $m = 2^*$ and $\mu = 0$).
  - Behavior of such singular solutions along a full sequence
  - How to extend the notion of asymptotic profiles to singular ones ?
  - How to define stability and instability of singular profiles ?
  - How is the stability and instability of each singular profile ?

Thank you for your attention !

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