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Abstract: *We introduce a property of Banach spaces, called uniform convex-transitivity, which falls between almost transitivity and convex-transitivity. We will provide examples of uniformly convex-transitive spaces. This property behaves nicely in connection with some vector-valued function spaces. As a consequence, we obtain some new examples of convex-transitive Banach spaces.*

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1 Introduction

In this paper we study the symmetries of some well-known, in fact, almost classical Banach spaces. We denote the closed unit ball of a Banach space X by $\mathbf{B}_{\mathrm{X}}$ and the unit sphere of X by $\mathbf{S}_{\mathrm{X}}$. A Banach space X is called *transitive* if for each $x \in \mathbf{S}_{\mathrm{X}}$ the orbit $\mathcal{G}_{\mathrm{X}}(x) \doteq \{T(x) |\ T\colon \mathrm{X} \to \mathrm{X} \text{ is an isometric automorphism}\} = \mathbf{S}_{\mathrm{X}}$. If $\overline{\mathcal{G}_{\mathrm{X}}(x)} = \mathbf{S}_{\mathrm{X}}$ (resp. $\overline{\mathrm{conv}}(\mathcal{G}_{\mathrm{X}}(x)) = \mathbf{B}_{\mathrm{X}}$) for all $x \in \mathbf{S}_{\mathrm{X}}$, then X is called *almost transitive* (resp. *convex-transitive*). These concepts are motivated by the *Banach-Mazur rotation problem* appearing in [2, p.242], which remains unsolved. We refer to [5] and [8] for a survey and discussion on the matter.

The known concrete examples of convex-transitive spaces are scarce, and the ultimate aim of this paper is to provide more examples by establishing the convex-transitivity of some vector-valued function spaces and other natural spaces. It was first reported by Pelczynski and Rolewicz [18] in 1962 that the space L^p is almost transitive for $p \in [1, \infty)$ and convex-transitive for $p = \infty$ (see also [20]). Later, Wood [23] characterized the spaces $C_0^{\mathbb{R}}(L)$ whose norm is convex-transitive (see Preliminaries). Greim, Jamison and Kaminska [14] proved that if X is almost transitive and $1 \leq p < \infty$, then the Lebesgue-Bochner space $L^p(\mathrm{X})$ is also almost transitive. Recently, an analogous study of the spaces $C_0(L, \mathrm{X})$ was done by Aizpuru and Rambla [1], and some related spaces were studied by Talponen [22]. For some other relevant contemporary results, see [7], [16] and [19].

We will extend these investigations into the vector-valued convex-transitive setting, which differs considerably in many respects from the scalar-valued almost transitive one. For this purpose we will introduce a new concept which is (formally) stronger than convex-transitivity and weaker than almost transitivity, called *uniform convex-transitivity.* With the aid of this class of Banach spaces we produce new natural examples of convex-transitive vector-valued function spaces. The main results of this paper are the following:

- Characterization of locally compact Hausdorff spaces L such that $C_0^{\mathbb{R}}(L)$ is uniformly convex-transitive.
- If X is a uniformly convex-transitive Banach space, then so is $L^\infty_{\mathbb{K}}(\mathrm{X})$.
- If X and $C_0^{\mathbb{R}}(L)$ are uniformly convex-transitive, then so is $C_0^{\mathbb{K}}(L, \mathrm{X})$.

Preliminaries

The scalar field of a Banach space X is denoted by $\mathbb{K}$ and whenever there are several Banach spaces under discussion, then $\mathbb{K}$ is the scalar field of the space denoted by X. The open unit ball of X is denoted by $\mathbf{U}_{\mathrm{X}}$. The group of rotations $\mathcal{G}_{\mathrm{X}}$ of X consists of isometric automorphisms $T\colon \mathrm{X} \to \mathrm{X}$, the group operation being the composition of the maps and the neutral element being the identity map $\mathbf{I}\colon \mathrm{X} \to \mathrm{X}$. We will always consider $\mathcal{G}_{\mathrm{X}}$ equipped with the strong operator topology (SOT). An element $x \in \mathbf{S}_{\mathrm{X}}$ is called a big point if

$\overline{\text{conv}}\mathcal{G}(x) = \mathbf{B}_{\mathrm{X}}$. Thus X is convex-transitive if and only if each $x \in \mathbf{S}_{\mathrm{X}}$ is a big point.

Recall that a topological space is totally disconnected if each connected component of the space is a singleton. In what follows L is a locally compact Hausdorff space and K is a compact Hausdorff space, unless otherwise stated. In [23] Wood characterized convex-transitive $C_0^{\mathbb{R}}(L)$ spaces. Namely, $C_0^{\mathbb{R}}(L)$ is convex-transitive if and only if L is totally disconnected and for every regular probability measure μ on L and $t \in L$ there exists a net $\{\gamma_\alpha\}_\alpha$ of homeomorphisms on L such that the net $\{\mu \circ \gamma_\alpha\}_\alpha$ is ω^*-convergent to the Dirac measure δ_t. The above mapping $\mu \circ \gamma_\alpha$ is given by $\mu \circ \gamma_\alpha(A) = \mu(\gamma_\alpha(A))$ for Borel sets $A \subset L$.

We refer to [17] for background information on measure algebras and isometries of L^p-spaces and to [11] for a suitable source to Banach spaces in general. In what follows Σ is the completed σ-algebra of Lebesgue measurable sets on $[0,1]$ and we denote by $m\colon \Sigma \to \mathbb{R}$ the Lebesgue measure. Define an equivalence relation $\overset{m}{\sim}$ on Σ by setting $A \overset{m}{\sim} B$ if $m((A \cup B) \setminus (A \cap B)) = 0$.

Recall that a rotation R on the space $C_0^{\mathbb{K}}(L, \mathrm{X})$ is said to be of the *Banach-Stone type*, if R can be written as

$$R(f)(t) = \sigma(t)(f \circ \phi(t)), \quad f \in C_0^{\mathbb{K}}(L, \mathrm{X}),$$

where $\phi\colon L \to L$ is a homeomorphism and $\sigma\colon L \to \mathcal{G}_{\mathrm{X}}$ is a continuous map. A Banach space Y is said to be contained *almost isometrically* in a Banach space X if for each $\varepsilon > 0$ there is a linear map $\psi\colon \mathrm{Y} \to \mathrm{X}$ such that

$$||y||_{\mathrm{Y}} \leq ||\psi(y)||_{\mathrm{X}} \leq (1+\varepsilon)||y||_{\mathrm{Y}} \quad \text{for } y \in \mathrm{Y}.$$

2 Uniform convex-transitivity

Provided that the space X under discussion is understood, we denote

$$C_n(x) = \left\{ \sum_{i=1}^{n} a_i T_i(x) | \ T_1, \ldots, T_n \in \mathcal{G}_{\mathrm{X}},\ a_1, \ldots, a_n \in [0,1],\ \sum_{i=1}^{n} a_i = 1 \right\}$$

for $n \in \mathbb{N}$ and $x \in \mathbf{S}_{\mathrm{X}}$. We call a Banach space X *uniformly convex-transitive* if for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ satisfying the following condition: For all $x \in \mathbf{S}_{\mathrm{X}}$ and $y \in \mathbf{B}_{\mathrm{X}}$ it holds that $\text{dist}(y, C_n(x)) \leq \varepsilon$, that is

$$\lim_{n\to\infty} \sup_{x \in \mathbf{S}_{\mathrm{X}}, y \in \mathbf{B}_{\mathrm{X}}} \text{dist}(y, C_n(x)) = 0.$$

We denote by K_ε the least integer n, which satisfies the above inequality involving ε and such K_ε is called *the constant of uniform convex transitivity of* X *associated to* ε. We call $x \in \mathbf{S}_{\mathrm{X}}$ a *uniformly big point* if

$$\lim_{n\to\infty} \sup_{y \in \mathbf{B}_{\mathrm{X}}} \text{dist}(y, C_n(x)) = 0.$$

Clearly almost transitive spaces are uniformly convex-transitive, and uniformly convex-transitive spaces are convex-transitive. It is well-known that $C^{\mathbb{C}}(S^1)$ is a convex-transitive, non-almost transitive space, and it is easy to see (see e.g. the subsequent Theorem 2.4) that it is even uniformly convex-transitive. Unfortunately, we have not been able so far to find an example of a convex-transitive space which is not uniformly convex-transitive. However, we suspect that such examples exist and we note that the absence of such a complicated space would make some proofs regarding convex-transitive spaces much more simple. Observe that the canonical unit vectors $e_k \in \ell^1$ are far from being uniformly big points:

$$\lim_{n\to\infty} \sup_{y\in \mathbf{B}_{\ell^1}} \mathrm{dist}(y, C_n(e_k)) = 1,$$

even though they are big points, i.e. $\overline{\mathrm{conv}}(\mathcal{G}_{\ell^1}(e_k)) = \mathbf{B}_{\ell^1}$ for $k \in \mathbb{N}$. In any case, we will provide examples of uniformly convex-transitive spaces, most of which are not previously known to be even convex-transitive.

We note that if X is convex-transitive and there exists a uniformly big point $x \in \mathbf{S}_{\mathrm{X}}$, then each $y \in \mathbf{S}_{\mathrm{X}}$ is a uniformly big point. This does not mean, a priori, that X should be uniformly convex-transitive. Next we give an equivalent condition to uniform convex transitivity, which is more applicable in calculations than the condition introduced above.

Proposition 2.1. *Let* X *be a Banach space. The following condition of* X *is equivalent to* X *being uniformly convex-transitive: For each* $\varepsilon > 0$ *there is* $N_\varepsilon \in \mathbb{N}$ *such that for each* $x \in \mathbf{S}_{\mathrm{X}}$ *and* $y \in \mathbf{B}_{\mathrm{X}}$ *there are* $T_1, \ldots, T_{N_\varepsilon} \in \mathcal{G}_{\mathrm{X}}$ *such that*

$$\left|\left| y - \tfrac{1}{N_\varepsilon} \textstyle\sum_{i=1}^{N_\varepsilon} T_i(x) \right|\right| \leq \varepsilon. \tag{1}$$

Proof. It is clear that (1) implies uniform convex transitivity, even for the value $K_\varepsilon = N_\varepsilon$ for each $\varepsilon > 0$. Towards the other direction, let X be a uniformly convex-transitive Banach space, $\varepsilon > 0$ and $x \in \mathbf{S}_{\mathrm{X}},\ y \in \mathbf{B}_{\mathrm{X}}$. Let K be the constant of uniform convex-transitivity of X associated to $\frac{\varepsilon}{4}$. Then there are $a_1, \ldots, a_K \in [0,1],\ \sum_i a_i = 1$ and $T_1, \ldots, T_K \in \mathcal{G}_{\mathrm{X}}$ such that

$$\left|\left| y - \textstyle\sum_{i=1}^{K} a_i T_i(x) \right|\right| \leq \tfrac{\varepsilon}{4}.$$

Put $m = \lceil \frac{4K}{\varepsilon} \rceil \in \mathbb{N}$, so that $K \cdot \frac{1}{m} \leq \frac{\varepsilon}{4}$. Next we define an m-uple $(S_1, \ldots, S_m) \subset \mathcal{G}_{\mathrm{X}}$ as follows: For each $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, K\}$ we put $S_j = T_i$ if $\lceil m \sum_{n<i} a_n \rceil < j \leq \lfloor m \sum_{n\leq i} a_n \rfloor$. (Here $\sum_{\emptyset} a_n = 0$.) By applying the triangle inequality several times, we obtain that

$$\left|\left| y - \tfrac{1}{m} \textstyle\sum_{j=1}^{m} S_j(x) \right|\right| \leq \varepsilon.$$

Hence it suffices to put $N_\varepsilon = m = \lceil \frac{4K}{\varepsilon} \rceil$, where K depends only on the Banach space X and the value of ε. □

In what follows, we will apply the constant N_ε freely without explicit reference to the previous proposition, and if there is no danger of confusion, also without mentioning explicitly X and ε, either.

The following condition on the locally compact space L turns out to be closely related to the uniform convex-transitivity of $C_0^{\mathbb{K}}(L)$:

($*$) For each $\varepsilon > 0$ there is $M_\varepsilon \in \mathbb{N}$ such that for every non-empty open subset $U \subset L$ and compact $K \subset L$ there are homeomorphisms $\phi_1, \ldots, \phi_{M_\varepsilon} : L \to L$ with

$$\frac{1}{M_\varepsilon} \sum_{i=1}^{M_\varepsilon} \chi_{\phi_i^{-1}(U)}(t) \geq 1 - \varepsilon \quad \text{for } t \in K.$$

This condition should be compared with the conditions found by Cabello (see [7, p.110-113], especially condition (g)), which characterize the convex transitivity of $C_0(L)$. Next we will give this characterization the uniformly convex-transitive counterpart. If L is a locally compact Hausdorff space, by αL we denote its one-point compactification and if L is noncompact, we denote such point by ∞. Prior to the theorem we need the following two lemmas.

Lemma 2.2. *([19, Thm. 3.1]) Let T be a normal topological space with* $\dim T \leq 1$. *If $F \subseteq T$ is a closed subset and $f : F \to S_{\mathbb{C}}$ is a continuous map, then f admits a continuous extension $g : T \to S_{\mathbb{C}}$.*

Lemma 2.3. *Let L be a locally compact, Hausdorff,* 0*-dimensional space. Then for every $g \in \mathbf{B}_{C_0^{\mathbb{R}}(L)}$ and $k \in \mathbb{N}$ there exist disjoint clopen sets $C_1, C_2, \ldots, C_{2k-1}$ such that the function $h \in \mathbf{B}_{C_0^{\mathbb{R}}(L)}$ defined by $h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}$ satisfies $\|h - g\| \leq \frac{3}{2k}$.*

Proof. We regard g as defined in αL. Consider $i \in \{-k, -k+1, \ldots, k-1\}$ and let $K_i = g^{-1}[\frac{i}{k}, \frac{i+1}{k}]$. Every $x \in K_i$ has a clopen neighbourhood A_x such that $g(A_x) \subseteq [\frac{2i-1}{2k}, \frac{2i+3}{2k}]$. By compactness there exist $x_1, \ldots, x_n$ such that $K_i \subseteq \bigcup_{j=1}^n A_{x_j} \doteq B_i$. Finally, define $C_0 = B_0$, $C_1 = B_{-k} \setminus C_0$, $\ldots$, $C_k = B_{-1} \setminus (C_0 \cup \cdots \cup C_{k-1})$, $C_{k+1} = B_1 \setminus (C_0 \cup \cdots \cup C_k)$, $\ldots$, $C_{2k-1} = B_{k-1} \setminus (C_0 \cup \cdots \cup C_{2k-2})$. Note that the C_i's are a partition of αL.

Now take $h : L \to \mathbb{R}$ given by $h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}$. It is easy to check that $\|h - g\| \leq \frac{3}{2k}$ and $h \in \mathbf{B}_{C_0^{\mathbb{R}}(L)}$. □

Theorem 2.4. *Let L be a locally compact Hausdorff space. The space $C_0^{\mathbb{R}}(L)$ is uniformly convex-transitive if and only if L is totally disconnected and satisfies ($*$). If the space $C_0^{\mathbb{C}}(L)$ is uniformly convex-transitive, then L satisfies ($*$). Moreover, if $\dim(\alpha L) \leq 1$, then also the converse implication holds.*

Before the proof we comment on the above assumptions.

Remark 2.5. *The spaces $C^{\mathbb{R}}(S^1, \mathbb{R}^2)$ and $C^{\mathbb{C}}(S^1, \mathbb{C})$ are uniformly convex-transitive, their rotations are of the Banach-Stone type, and clearly S^1, $\mathcal{G}_{\mathbb{R}^2}$ and $\mathcal{G}_{\mathbb{C}}$ are not totally disconnected.*

Proof of Theorem 2.4. Let us first consider the *only if* directions. Since uniformly convex-transitive spaces are convex-transitive, we may apply Wood's characterization for convex-transitive $C_0^{\mathbb{R}}(L)$ spaces, and thus we obtain that L must be totally disconnected. Let $C_0^{\mathbb{K}}(L)$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, be uniformly convex-transitive. Next we aim to check that L satisfies $(*)$, so let $U \subset L$ be a non-empty open subset and $K \subset L$ a compact subset. Fix $x_0 \in U$. Since αL is normal, there exist continuous functions $f, g : \alpha L \to [-1, 1]$ satisfying $f(\alpha L \setminus U) = \{0\}$, $f(x_0) = 1$, $g(K) = \{1\}$ and $g(\infty) = 0$. Since both functions vanish at infinity, we can consider that $f, g \in \mathbf{S}_{C_0^{\mathbb{K}}(L)}$.

Fix $\varepsilon > 0$ appearing in condition $(*)$. Let N_ε be the associated constant provided by the uniform convex-transitivity and condition (1). Then by the definition of N_ε and the Banach-Stone characterization of rotations of $C_0^{\mathbb{K}}(L)$ we obtain that there are continuous functions $\sigma_1, \ldots, \sigma_{N_\varepsilon} \colon L \to \mathbb{K}$ and homeomorphisms $\phi_1, \ldots, \phi_{N_\varepsilon} \colon L \to L$ such that

$$\left|\left|g - \tfrac{1}{N_\varepsilon} \textstyle\sum_{i=1}^{N_\varepsilon} \sigma_i (f \circ \phi_i)\right|\right| \leq \varepsilon. \tag{2}$$

In particular, this yields for each $t \in K$ that

$$\begin{aligned} \varepsilon &\geq |1 - \tfrac{1}{N_\varepsilon} \textstyle\sum_{i=1}^{N_\varepsilon} \sigma_i f(\phi_i(t))| = |\tfrac{1}{N_\varepsilon} \textstyle\sum_{i=1}^{N_\varepsilon} 1 - \sigma_i f(\phi_i(t))| \\ &\geq \tfrac{1}{N_\varepsilon} \textstyle\sum_{i=1}^{N_\varepsilon} \chi_{L \setminus \phi_i^{-1}(U)}(t), \end{aligned}$$

where we applied the fact that f vanishes outside U. This justifies $(*)$ for $M_\varepsilon = N_\varepsilon$.

Let us see the *if* direction for $C_0^{\mathbb{R}}(L)$. Let $k \in \mathbb{N}, f \in \mathbf{S}_{C_0^{\mathbb{R}}(L)}$ and $g \in \mathbf{B}_{C_0^{\mathbb{R}}(L)}$. We may assume $\max f = 1$. Take h as in Lemma 2.3, i.e. $h = \sum_{i=1}^{2k-1} \frac{i-k}{k} \chi_{C_i}$ with each C_i clopen and $\|h - g\| \leq \frac{3}{2k}$.

Note that $K \doteq \bigcup_{i=1}^{2k-1} C_i$ is compact and apply $(*)$ to this K, the subset $U \doteq \{t \in L : f(t) > 1 - k^{-1}\}$ and $\varepsilon = \frac{1}{k}$. Write $M \doteq M_\varepsilon$. There exist homeomorphisms $\phi_1, \ldots, \phi_M$ such that if $t \in K$ then $\frac{1}{M} \sum_{i=1}^{M} \chi_{\phi_i^{-1}(U)}(t) \geq 1 - k^{-1}$. For each $j \in \{1, \ldots, 2k-1\}$, define $B_j = \bigcup_{s=j}^{2k-1} C_s$ and let T_j be the rotation on $C_0^{\mathbb{R}}(L)$ given by $T_j x = (\chi_{B_j} - \chi_{B_j^c}) \cdot x$ if $j \leq k$ and $T_j x = (\chi_{B_j} - \chi_{B_j^c} + 2\chi_{L \setminus K}) \cdot x$ if $j > k$. Now only a few calculations are needed to see that

$$\left|\left|g - \tfrac{1}{M(2k-1)} \textstyle\sum_{j=1}^{2k-1} \sum_{i=1}^{M} T_j (f \circ \phi_i)\right|\right| \leq 6k^{-1}$$

and thus $C_0^{\mathbb{R}}(L)$ is uniformly convex transitive.

In order to justify the last sentence in the theorem it is required to verify that if L satisfies $\dim(\alpha L) \leq 1$ and $(*)$, then $C_0^{\mathbb{C}}(L)$ is uniformly convex-transitive. Let $k \in \mathbb{N}$ and let M_k be the corresponding constant in condition $(*)$ associated to value k^{-1}. Fix $f \in \mathbf{S}_{C_0^{\mathbb{C}}(L)}$ and $g \in \mathbf{B}_{C_0^{\mathbb{C}}(L)}$. We may assume without loss of generality, possibly by multiplying f with a suitable complex number of modulus 1, that $f(t_0) = 1$ for a suitable $t_0 \in L$. Let $U \doteq \{t \in L : |1 - f(t)| < k^{-1}\}$ and $K = \{t \in L : \ |g(t)| \geq k^{-1}\}$. Let $\phi_1, \ldots, \phi_{M_k} \colon L \to L$

be homeomorphisms such that $\frac{1}{M_k}\sum_{i=1}^{M_k}\chi_{\phi_i^{-1}(U)}(t)\geq 1-k^{-1}$ for $t\in K$. This means that the average

$$F\doteq \tfrac{1}{M_k}\textstyle\sum_{i=1}^{M_k} f\circ\phi_i\in \mathbf{B}_{C_0^{\mathbb{C}}(L)} \tag{3}$$

satisfies $|1-F(t)|\leq 3k^{-1}$ for each $t\in K$.

Next we will define some auxiliary mappings. Put $\alpha\colon \mathbf{S}_{\mathbb{C}}\times[0,1]\to\mathbf{S}_{\mathbb{C}}$; $\alpha(z,s)=-i^{2s}z$. Note that this is a continuous map, and $\alpha(z,0)=-z$, $\alpha(z,1)=z$ for $z\in\mathbf{S}_{\mathbb{C}}$. Taking into account Lemma 2.2 with $T=\alpha L$, let $\beta_g\colon L\to\mathbf{S}_{\mathbb{C}}$ be a continuous extension of the function $\frac{g(\cdot)}{|g(\cdot)|}$ defined on K.

For $j\in\{1,\ldots,k\}$ we define rotations on $C_0^{\mathbb{C}}(L)$ by putting $e_{ja}(x)(t)=\beta_g(t)\cdot x(t)$ and $e_{jb}(x)(t)=\alpha(\beta_g(t),\min(1,\max(0,k|g(t)|-j)))\cdot x(t)$. The main point above is that $(e_{ja}+e_{jb})(F)(t)=0$ for $(j,t)\in\{1,\ldots,k\}\times L$ such that $|g(t)|\leq\frac{j}{k}$ and $(e_{ja}+e_{jb})(F)(t)=2F(t)\beta_g(t)$ for $(j,t)\in\{1,\ldots,k\}\times L$ such that $g(t)\geq\frac{j+1}{k}$. Thus, by using (3) we obtain that

$$\left|\left|\ |g(t)|\beta_g(t)-\tfrac{1}{2k}\textstyle\sum_{j=1}^{k}(e_{ja}+e_{jb})(F)(t)\right|\right|\leq 2k^{-1}\quad\text{for } t\in L.$$

Here $||g(\cdot)-|g(\cdot)|\beta_g(\cdot)||\leq k^{-1}$, so that $C_0^{\mathbb{C}}(L)$ is uniformly convex-transitive.
□

Note that Theorem 2.4 yields the fact that if $C_0^{\mathbb{R}}(L)$ is uniformly convex-transitive, then so is $C_0^{\mathbb{C}}(L)$. By the above reasoning one can also see that if $C_0^{\mathbb{K}}(L)$ is convex-transitive and $|L|>1$, then L contains no isolated points and thus it follows that each non-empty open subset of L is uncountable (see also [3, Thm. 1]). Cabello pointed out [7, Cor. 1] that locally compact spaces L having a basis of clopen sets C such that $L\setminus C$ is homeomorphic to C, have the property that $C_0^{\mathbb{R}}(L)$ is convex-transitive. Consequently, this provides a route to the fact that the spaces L^∞, ℓ^∞/c_0 and $C(\Delta)$ over $\mathbb{R}$, where Δ is the Cantor set, are convex-transitive. By applying Theorem 2.4 and following Cabello's argument with slight modifications, one arrives at the conclusion that these spaces are in fact uniformly convex-transitive. When studying [7] it is helpful to observe that each occurence of 'basically disconnected' in the paper must be read as *totally disconnected*, ([6]).

It is quite easy to verify that if $L_1,\ldots,L_n$, where $n\in\mathbb{N}$, are totally disconnected locally compact Hausdorff spaces satisfying $(*)$, then so is the product $L_1\times\cdots\times L_n$. It follows that the space $C_0^{\mathbb{R}}(L_1\times\cdots\times L_n)$ (also known as the injective tensor product $C_0^{\mathbb{R}}(L_1)\hat{\otimes}_\varepsilon\ldots\hat{\otimes}_\varepsilon C_0^{\mathbb{R}}(L_n)$, up to isometry) is uniformly convex-transitive.

3 Uniform convex-transitivity of Banach-valued function spaces

With a proof similar to that of lemma 2.3, we obtain the following:

Lemma 3.1. *Let L be a locally compact, Hausdorff, 0-dimensional space and X a Banach space over $\mathbb{K}$. Given $g \in \mathbf{B}_{C_0^{\mathbb{K}}(L,X)}$ and $j \in \mathbb{N}$, there exist nonzero $x_1, \ldots, x_n \in \mathbf{B}_{\mathrm{X}}$ and disjoint clopen sets $C_1, C_2, \ldots, C_n \subset L$ such that the function $h \in \mathbf{B}_{C_0^{\mathbb{K}}(L,\mathrm{X})}$ defined by $h(t) = \sum_{i=1}^{n} \chi_{C_i}(t) x_i$ satisfies $\|h - g\| < \frac{1}{j}$.*

Theorem 3.2. *Let L be a locally compact Hausdorff space and X a Banach space over $\mathbb{K}$. Consider the following conditions:*

(1) L is totally disconnected and satisfies $(*)$, i.e. $C_0^{\mathbb{R}}(L)$ is uniformly convex-transitive.

(2) X is uniformly convex-transitive.

(3) $C_0^{\mathbb{K}}(L, \mathrm{X})$ is uniformly convex-transitive.

We have the implication $(1) + (2) \implies (3)$. If the rotations of $C_0^{\mathbb{K}}(L,\mathrm{X})$ are of the Banach-Stone type and $\dim_{\mathbb{K}}(\mathrm{X}) \geq 1$, then $(3) \implies (*) + (2)$. If additionally $\mathbb{K} = \mathbb{R}$ and $\mathcal{G}_{\mathrm{X}}$ is totally disconnected, then $(3) \implies (1) + (2)$.

Recall Remark 2.5 related to the last claim above.

Proof of Theorem 3.2. We begin by proving the implication $(1) + (2) \implies (3)$. Fix $k \in \mathbb{N}$. Then condition $(*)$ provides us with an integer N_k associated to $\frac{1}{4k}$. Let $f \in \mathbf{S}_{C_0^{\mathbb{K}}(L,\mathrm{X})}$ and $g \in \mathbf{B}_{C_0^{\mathbb{K}}(L,\mathrm{X})}$. Take h and $C_1, \ldots, C_n \subset L$ as in Lemma 3.1 with $j = 2k$.

Let $B = \bigcup_{i=1}^{n} C_i$ and $K = \{t \in L : \|g(t)\| \geq k^{-1}\}$. Note that B is a compact clopen set and $K \subset B$. There are $y \in \mathbf{S}_{\mathrm{X}}$ and $T_1, \ldots, T_{N_k} \in \mathcal{G}_{C_0^{\mathbb{K}}(L,\mathrm{X})}$ such that

$$\left\| y - \left(\tfrac{1}{N_k} \textstyle\sum_{i=1}^{N_k} T_i f \right)(t) \right\| < \tfrac{1}{k}, \quad \text{for } t \in B. \tag{4}$$

Indeed, observe that the continuous map $L \to \mathbb{R};\ t \mapsto ||f(t)||$ attains its supremum, the value 1. Thus, let $t_0 \in L$ be such that $||f(t_0)|| = 1$ and let $y = f(t_0) \in \mathbf{S}_{\mathrm{X}}$. Write $V = \{t \in L : \ ||f(t) - y|| < \frac{1}{2k}\}$. By using $(*)$ there are homeomorphisms $\sigma_1, \ldots, \sigma_{N_k} : L \to L$ such that

$$\tfrac{1}{N_k} \textstyle\sum_{i=1}^{N_k} \chi_V(\sigma_i(t)) \geq 1 - \tfrac{1}{4k}, \quad \text{for } t \in B. \tag{5}$$

Let $T_i \in \mathcal{G}_{C_0^{\mathbb{K}}(L,\mathrm{X})}$ be given by $(T_i F)(t) = F(\sigma_i(t))$ for $1 \leq i \leq N_k$ and $F \in C_0^{\mathbb{K}}(L, \mathrm{X})$. Thus, for all $t \in B$ we obtain by (5) and the definition of V that

$$\begin{aligned} & \left\| y - \left(\tfrac{1}{N_k} \textstyle\sum_{i=1}^{N_k} T_i f \right)(t) \right\| \\ = \ & \left\| \tfrac{1}{N_k} \textstyle\sum_{i=1}^{N_k} y - \tfrac{1}{N_k} \sum_{i=1}^{N_k} f(\sigma_i(t)) \right\| \leq \tfrac{1}{N_k} \textstyle\sum_{i=1}^{N_k} ||y - f(\sigma_i(t))|| \\ < \ & (1 - \tfrac{1}{4k}) \cdot \tfrac{1}{2k} + \tfrac{1}{4k} \cdot 2 < \tfrac{1}{k}. \end{aligned}$$

Since X is uniformly convex-transitive, there is an integer $2M = N_\varepsilon$ satisfying (1) for the value $\varepsilon = k^{-1}$. Let $S_1^{(i)}, \ldots, S_{2M}^{(i)} \in \mathcal{G}_{\mathrm{X}}$ for $1 \leq i \leq n$ such that

$$\left|\left|x_i - \tfrac{1}{2M}\textstyle\sum_{l=1}^{2M} S_l^{(i)}(y)\right|\right| < k^{-1} \quad \text{for } 1 \leq i \leq n. \tag{6}$$

Then for each $1 \leq l \leq 2M$ we define a rotation on $C_0^{\mathbb{K}}(L, \mathrm{X})$ by

$$R_l(F)(t) = \chi_{L \setminus B}(t)(-1)^l F(t) + \sum_{i=1}^{n} \chi_{C_i} S_l^{(i)}(F(t)), \quad F \in C_0^{\mathbb{K}}(L, \mathrm{X}),\ t \in L.$$

Indeed, this defines rotations, since the sets $L \setminus B$ and C_i are clopen. It is easy to see by combining (4) and (6) that

$$\left|\left|g - \tfrac{1}{2M}\textstyle\sum_{l=1}^{2M} R_l \tfrac{1}{N_k}\sum_{i=1}^{N_k} T_i f\right|\right| < 2k^{-1}.$$

This verifies the first implication.

Next we will prove the implication (3) $\implies$ $(*)$+(2) under the assumption that the rotations are of the Banach-Stone type. In fact, the verification of claim (3) $\implies$ $(*)$ reduces to the analogous scalar-valued case, which was treated in the proof of Theorem 2.4. Moreover, by using the Banach-Stone representation of rotations and functions of type $f \otimes x, g \otimes y \in \mathbf{S}_{C_0^{\mathbb{K}}(L,\mathrm{X})}$ it is easy to verify that the uniform convex-transitivity of $C_0^{\mathbb{K}}(L, \mathrm{X})$ implies that of X.

Finally, let us prove the total disconnectedness of L in the case when $\mathcal{G}_{\mathrm{X}}$ is totally disconnected and $\mathbb{K} = \mathbb{R}$. Assume to the contrary that L contains a connected subset C, which is not a singleton. Pick $t, s \in C,\ t \neq s$, and $x \in \mathbf{S}_{\mathrm{X}}$. Let $x^* \in \mathbf{S}_{\mathrm{X}^*}$ with $x^*(x) = 1$. Let $f, g \in \mathbf{S}_{C_0^{\mathbb{R}}(L)}$ be functions with disjoint supports and such that $f(t) = g(s) = 1$. Consider $f \otimes x, f \otimes x - g \otimes x \in \mathbf{S}_{C_0^{\mathbb{R}}(L,\mathrm{X})}$. Since $C_0^{\mathbb{R}}(L, \mathrm{X})$ is convex-transitive we obtain that $f \otimes x - g \otimes x \in \overline{\mathrm{conv}}(\mathcal{G}_{C_0^{\mathbb{R}}(L,\mathrm{X})}(f \otimes x))$.

It follows easily by taking into account the Banach-Stone representation of rotations of $C_0^{\mathbb{R}}(L, \mathrm{X})$ and by studying the convex combinations in $\mathrm{conv}(\mathcal{G}_{C_0^{\mathbb{R}}(L,\mathrm{X})}(f \otimes x))$ that there exists a continuous map $\sigma\colon L \to \mathcal{G}_{\mathrm{X}}$ such that

$$x^*(\sigma(t)(x)), x^*(-\sigma(s)(x)) > 0.$$

By using the facts that $\sigma(t) \neq \sigma(s)$ and that $\mathcal{G}_{\mathrm{X}}$ is totally disconnected we obtain that $\sigma(C)$ is not connected. However, we have a contradiction, since $\sigma(C)$ is a continuous image of a connected set. This contradiction shows that L must be totally disconnected. □

By following the argument in the previous proof with slight modifications one obtains an analogous result in the convex-transitive setting.

Theorem 3.3. *If $C_0^{\mathbb{R}}(L)$ is convex-transitive and* X *is a convex-transitive space over $\mathbb{K}$, then $C_0^{\mathbb{K}}(L, \mathrm{X})$ is convex-transitive.*

Proof. The proof of Theorem 3.2 has the convex-transitive counterpart with convex combinations of rotations in place of averages of rotations. Indeed, in the equation (4) one uses the convex-transitivity of $C_0^{\mathbb{R}}(L)$ and the corresponding Banach-Stone type rotations applied on $C_0^{\mathbb{K}}(L,\mathrm{X})$. After equation (4) the argument proceeds similarly. Note that in the convex-transitive setting there does not exist, a priori, an upper bound M depending only on ϵ. □

Recall that the Lebesgue-Bochner space $L^p(\mathrm{X})$ consists of strongly measurable maps $f\colon [0,1]\to \mathrm{X}$ endowed with the norm

$$||f||^p_{L^p(\mathrm{X})} = \int_0^1 ||f(t)||^p_{\mathrm{X}}\ \mathrm{d}t, \quad \text{for } p\in[1,\infty)$$

and $||f||_{L^\infty(\mathrm{X})} = \operatorname*{ess\,sup}_{t\in[0,1]} ||f(t)||_X$. We refer to [10] for precise definitions and background information regarding the Banach-valued function spaces appearing here.

Recall that L^∞ is convex-transitive (see [18] and [20]). Greim, Jamison and Kaminska proved that $L^p(\mathrm{X})$ is almost transitive if X is almost transitive and $1\leq p<\infty$, see [14, Thm. 2.1]. We will present the analogous result for uniformly convex-transitive spaces, that is, if X is uniformly convex-transitive, then $L^p(\mathrm{X})$ are also uniformly convex-transitive for $1\leq p\leq\infty$.

Theorem 3.4. *Let* X *be a uniformly convex-transitive space over* $\mathbb{K}$. *Then the Bochner space* $L^p_{\mathbb{K}}(\mathrm{X})$ *is uniformly convex-transitive for* $1\leq p\leq\infty$.

We will make some preparations before giving the proof. Suppose that $(A_n)_{n\in\mathbb{N}}$ is a countable measurable partition of the unit interval and $(x_n)_{n\in\mathbb{N}}\subset \mathrm{X}$. We will use the short-hand notation $F=\sum_n \chi_{A_n}x_n$ for the function $F\in L^\infty(\mathrm{X})$ defined by $F(t)=x_n$ for a.e. $t\in A_n$ for each $n\in\mathbb{N}$. The following two auxiliary observations are obtained immediately from the fact that the countably valued functions are dense in $L^\infty(\mathrm{X})$ and the triangle inequality, respectively.

Fact 3.5. *Consider* $F=\sum_n \chi_{A_n}x_n$, *where* (A_n) *is a measurable partition of* $[0,1]$ *and* $(x_n)\subset \mathbf{B}_{\mathrm{X}}$. *Functions* F *of such type are dense in* $\mathbf{B}_{L^\infty(\mathrm{X})}$.

Fact 3.6. *Let* X *be a Banach space and* $T_1,\ldots,T_n\in\mathcal{G}_{\mathrm{X}},\ n\in\mathbb{N}$. *Assume that* $x,y,z\in \mathrm{X}$ *satisfy* $||y-\frac{1}{n}\sum_i T_i(x)||=\varepsilon\geq 0$ *and* $||x-z||=\delta\geq 0$. *Then* $||y-\frac{1}{n}\sum_i T_i(z)||\leq\varepsilon+\delta$.

Proof of Theorem 3.4. We mainly concentrate on the case $p=\infty$. Fix $k\in\mathbb{N}$, $x\in\mathbf{S}_{\mathrm{X}}$, $(x_n),(y_n)\subset\mathbf{B}_{\mathrm{X}}$ and measurable partitions (A_n) and (B_n) of the unit interval. Let N_k be the integer provided by the uniform convex transitivity of X associated to the value $\varepsilon=\frac{1}{k}$. Write

$$F=\sum_n \chi_{A_n}x_n \text{ and } G=\sum_n \chi_{B_n}y_n.$$

We assume additionally that $||F|| = 1$.

For each $n \in \mathbb{N}$ there are isometries $\{T_i^{(n)}\}_{i \leq N_k} \subset \mathcal{G}_{\mathrm{X}}$ such that

$$\left|\left|\tfrac{1}{N_k}\sum_{i=1}^{N_k} T_i^{(n)}(x) - y_n\right|\right| < \tfrac{1}{k} \quad \text{for } n \in \mathbb{N}. \tag{7}$$

Observe that one obtains rotations on $L^\infty(\mathrm{X})$ by putting

$$R_i(f)(t) = \sum_n \chi_{B_n} T_i^{(n)}(f(t))$$

for a.e. $t \in [0,1]$, where $f \in L^\infty(\mathrm{X})$, $i \leq N_k$, and the above summation is understood in the sense of pointwise convergence almost everywhere. We define a convex combination of elements of $\mathcal{G}_{L^\infty(\mathrm{X})}$ by

$$\mathrm{A}_1(f) = \tfrac{1}{N_k}\sum_{i=1}^{N_k} R_i(f), \quad f \in L^\infty(\mathrm{X}).$$

Condition (7) implies that

$$||G - \mathrm{A}_1(\chi_{[0,1]}x)|| < \frac{1}{k}. \tag{8}$$

By the definition of F one can find $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) > 0$ and

$$||x_{n_0}||_{\mathrm{X}} > 1 - \frac{1}{k}. \tag{9}$$

Put $\Delta_n = [1 - 2^{-n}, 1 - 2^{-(n+1)}]$ for $n \leq k$. By composing suitable bijective transformations one can construct measurable mappings $g_n \colon [0,1] \to [0,1]$ and $\hat{g}_n \colon [0,1] \to [0,1]$ such that

$$g_n(A_{n_0}) \overset{m}{\sim} [0,1] \setminus \Delta_n \text{ and } g_n([0,1] \setminus A_{n_0}) \overset{m}{\sim} \Delta_n, \tag{10}$$

$$\text{the measure } \mu_n(\cdot) \doteq m(g_n(\cdot)) \colon \Sigma \to \mathbb{R} \text{ is equivalent to } m \tag{11}$$

and

$$\hat{g}_n \circ g_n(t) = t \quad \text{for a.e. } t \in [0,1] \tag{12}$$

for each $n \leq k$.

Next we will apply some observations which appear e.g. in [13] and [12]. Denote by $\Sigma\backslash_m$ the quotient σ-algebra of Lebesgue measurable sets on $[0,1]$ formed by identifying the m-null sets with $\emptyset$. Note that (11) gives in particular that the map $\phi_n \colon \Sigma\backslash_m \to \Sigma\backslash_m$ determined by $\phi_n(A) \overset{m}{\sim} g_n(A)$ for $A \in \Sigma$ is a Boolean isomorphism for each $n \leq k$. Observe that $\hat{g}_n(A) \overset{m}{\sim} \phi_n^{-1}(A)$ for $A \in \Sigma$ and $n \leq k$.

By (9) there are rotations $\{T_i\}_{i \leq N_k} \subset \mathcal{G}_{\mathrm{X}}$ such that

$$\left|\left|x - \tfrac{1}{N_k}\sum_{i=1}^{N_k} T_i(x_{n_0})\right|\right|_{\mathrm{X}} < \tfrac{2}{k}. \tag{13}$$

According to (12) we may define mappings $S_i \colon L^\infty(\mathrm{X}) \to L^\infty(\mathrm{X})$ for $n \leq k$ and $i \leq N_k$ by putting

$$S_i^{(n)}(F)(t) = T_i(F(\hat{g}_n(t))) \quad \text{for a.e. } t \in [0,1],\ F \in L^\infty(\mathrm{X}).$$

By (11) we get that $S_i^{(n)} \in \mathcal{G}_{L^\infty(\mathrm{X})}$ (see also [12, p. 467-468]).

The function $\chi_{[0,1]}x$ can be approximated by convex combinations as follows:

$$\left|\left|\chi_{[0,1]}x - \tfrac{1}{k}\sum_{n=1}^{k}\tfrac{1}{N_k}\sum_{i=1}^{N_k} S_i^{(n)}(F)\right|\right|_{L^\infty(\mathrm{X})} \leq \tfrac{1}{k}(2+\sum_{i=1}^{k-1} 2k^{-1}). \tag{14}$$

Indeed, for $n \leq k$ and a.e. $t \in [0,1] \setminus \Delta_n$ it holds by (13) that

$$\left|\left|x - \tfrac{1}{N_k}\sum_{i=1}^{N_k} S_i^{(n)}(F)(t)\right|\right|_{\mathrm{X}} = \left|\left|x - \tfrac{1}{N_k}\sum_{i=1}^{N_k} T_i^{(n)}(x_n)\right|\right|_{\mathrm{X}} \leq \tfrac{2}{k}.$$

On the other hand, $||x - \frac{1}{N_k}\sum_{i=1}^{N_k} S_i^{(n)}(F)(t)||_{\mathrm{X}} \leq 2$ for a.e. $t \in \Delta_n$. In (14) we apply the fact that Δ_n are pairwise essentially disjoint.

Denote $\mathrm{A}_2 = \frac{1}{k}\sum_{n=1}^{k}\frac{1}{N_k}\sum_{i=1}^{N_k} S_i^{(n)} \in \mathrm{conv}(\mathcal{G}_{L^\infty(\mathrm{X})})$. By combining the estimates (8) and (14) we obtain by Fact 3.6 that

$$||G - \mathrm{A}_1\mathrm{A}_2(F)|| < \frac{5}{k}.$$

Observe that $\mathrm{A}_1\mathrm{A}_2$ is an average of $N_k N_k$ many rotations on $L^\infty(\mathrm{X})$. We conclude by Fact 3.5 that $L^\infty(\mathrm{X})$ is uniformly convex-transitive.

The case $1 \leq p < \infty$ is a straightforward modification of the proof of [14, Thm. 2.1], where one replaces $U_i x_i$ by suitable averages belonging to $\mathrm{conv}(\mathcal{G}_{\mathrm{X}}(x_i))$ for each i. □

In fact it is not difficult to check the following fact: If the rotations of $L^\infty(\mathrm{X})$ are of the Banach-Stone type, then $L^\infty(\mathrm{X})$ is convex-transitive if and only if each $x \in \mathbf{S}_{\mathrm{X}}$ is a uniformly big point.

We already mentioned that ℓ^∞/c_0 is uniformly convex-transitive as a real space. Next we generalize this result to the vector-valued setting.

Theorem 3.7. *Let* X *be a uniformly convex-transitive Banach space over* $\mathbb{K}$. *Then* $\ell^\infty(\mathrm{X})/c_0(\mathrm{X})$ *(over* $\mathbb{K}$*) is uniformly convex-transitive.*

Proof. Observe that the formula

$$T((x_n)_n) = (S_n x_{\pi(n)})_n, \tag{15}$$

where $\pi\colon \mathbb{N} \to \mathbb{N}$ is a bijection and $S_n \in \mathcal{G}_{\mathrm{X}},\ n \in \mathbb{N}$, defines a rotation on $\ell^\infty(\mathrm{X})$. Also note that such an isometry T restricted to $c_0(\mathrm{X})$ is a member of $\mathcal{G}_{c_0(\mathrm{X})}$.

If $T \in \mathcal{G}_{\ell^\infty(\mathrm{X})}$ is as in (15), then $\widehat{T}\colon x + c_0(\mathrm{X}) \mapsto T(x) + c_0(\mathrm{X})$, for $x \in \ell^\infty(\mathrm{X})$, defines a rotation $\ell^\infty(\mathrm{X})/c_0(\mathrm{X}) \to \ell^\infty(\mathrm{X})/c_0(\mathrm{X})$. Indeed, it is clear that $\widehat{T}\colon \ell^\infty(\mathrm{X})/c_0(\mathrm{X}) \to \ell^\infty(\mathrm{X})/c_0(\mathrm{X})$ is a linear bijection. Moreover,

$$\inf_{z\in c_0(\mathrm{X})} ||x - z|| = \inf_{z\in c_0(\mathrm{X})} ||T(x) - T(z)|| = \inf_{z\in c_0(\mathrm{X})} ||T(x) - z||,$$

so that $\widehat{T}\colon \ell^\infty(\mathrm{X})/c_0(\mathrm{X}) \to \ell^\infty(\mathrm{X})/c_0(\mathrm{X})$ is an isometry.

Fix $u, v \in \mathbf{S}_{\ell^\infty(\mathrm{X})/c_0(\mathrm{X})}$. If $x, y \in \ell^\infty(\mathrm{X})$ are such that $u = x + c_0(\mathrm{X})$ and $v = y + c_0(\mathrm{X})$, then

$$\mathrm{dist}(x, c_0(\mathrm{X})) = \limsup_{n\to\infty} ||x_n|| = 1 = \mathrm{dist}(y, c_0(\mathrm{X})) = \limsup_{n\to\infty} ||y_n||, \tag{16}$$

since $u, v \in \mathbf{S}_{\ell^\infty(\mathrm{X})/c_0(\mathrm{X})}$. Hence we may pick $x, y \in \mathbf{S}_{\ell^\infty(\mathrm{X})}$ such that $u = x + c_0(\mathrm{X})$ and $v = y + c_0(\mathrm{X})$.

Fix $k \in \mathbb{N}$, $e \in \mathbf{S}_\mathrm{X}$ and let $A = \{n \in \mathbb{N} :\ ||x_n|| \geq 1 - \frac{1}{2k}\}$. Observe that A is an infinite set by (16). Since X is uniformly convex-transitive, there exists $N_{(k)} \in \mathbb{N}$ such that for each $n \in A$ there are $T_1^{(n)}, \ldots, T_{N_{(k)}}^{(n)} \in \mathcal{G}_\mathrm{X}$ such that

$$\left|\left|e - \tfrac{1}{N_{(k)}} \textstyle\sum_{l=1}^{N_{(k)}} T_l^{(n)} x_n\right|\right| < \tfrac{1}{k}. \tag{17}$$

Fix $j_{(k)} \in \mathbb{N}$ such that

$$\frac{1}{j_{(k)}}(2 + (j_{(k)} - 1)(\frac{1}{k})) < \frac{2}{k}. \tag{18}$$

Denote by $p_1, \ldots, p_{j_{(k)}} \in \mathbb{N}$ the $j_{(k)}$ first primes. Let $\phi_1, \ldots, \phi_{j_{(k)}} : \mathbb{N} \to \mathbb{N}$ be permutations such that

$$\phi_i(\mathbb{N} \setminus A) \subset \{p_i^m |\ m \in \mathbb{N}\} \quad \text{for } i \in \{1, \ldots, j_{(k)}\}. \tag{19}$$

For $l \in \{1, \ldots, N_{(k)}\}$ put $S_{i,n,l} = T_l^{(\phi_i^{-1}(n))}$ if $\phi_i^{-1}(n) \in A$ and otherwise put $S_{i,n,l} = \mathbf{I}$. Define a convex combination of rotations on $\ell^\infty(\mathrm{X})$ by letting

$$\mathrm{A}_1(z)|_n = \tfrac{1}{j_{(k)}} \textstyle\sum_{i=1}^{j_{(k)}} \tfrac{1}{N_{(k)}} \sum_{l=1}^{N_{(k)}} S_{i,n,l}(z_{\phi_i^{-1}(n)}),$$

where $(z_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathrm{X})$. Consider $\mathrm{A}_1 \in L(\ell^\infty(\mathrm{X}))$ and $\overline{e} = (e, e, e, \ldots) \in \ell^\infty(\mathrm{X})$. We obtain that

$$||\overline{e} - \mathrm{A}_1((x_n))||_{\ell^\infty(\mathrm{X})} < \frac{2}{k}. \tag{20}$$

Indeed, for each $n \in \mathbb{N}$ it holds for at least $j_{(k)} - 1$ many indices i that

$$\tfrac{1}{N_{(k)}} \textstyle\sum_{l=1}^{N_{(k)}} S_{i,n,l}(x_{\phi_i^{-1}(n)}) = \tfrac{1}{N_{(k)}} \sum_{l=1}^{N_{(k)}} T_l^{(\phi_i^{-1}(n))}(x_{\phi_i^{-1}(n)}),$$

where one uses the definition of $S_{i,n,l}$, (19) and the fact that the sets $\{p_i^m |\ m \in \mathbb{N}\}, \{p_j^m |\ m \in \mathbb{N}\}$ are mutually disjoint for $i \neq j$. Thus (17) and (18) yield that

$$\left|\left|e - \tfrac{1}{j_{(k)}} \textstyle\sum_{i=1}^{j_{(k)}} \tfrac{1}{N_{(k)}} \sum_{l=1}^{N_{(k)}} S_{i,n,l}(x_{\phi_i^{-1}(n)})\right|\right| < \tfrac{2}{k}$$

holds for all $n \in \mathbb{N}$.

Next we will define another convex combination A_2 of rotations on $\ell^\infty(\mathrm{X})$ as follows. By using again the uniform convex transitivity of X we obtain $T_{n,l} \in \mathcal{G}_\mathrm{X},\ 1 \leq l \leq N_{(k)},\ n \in \mathbb{N}$, such that

$$\left|\left|y_n - \tfrac{1}{N_{(k)}} \textstyle\sum_{l=1}^{N_{(k)}} T_{n,l} e\right|\right| < \tfrac{1}{k}$$

holds for $n \in \mathbb{N}$. Define

$$\mathrm{A}_2(z)|_n = \tfrac{1}{N_{(k)}} \sum_{l=1}^{N_{(k)}} T_{n,l} z_n.$$

Combining the convex combinations yields

$$||y - \mathrm{A}_2\mathrm{A}_1 x||_{\ell^\infty(\mathrm{X})} < \frac{3}{k}$$

according to Fact 3.6. Since the applied rotations induce rotations on $\ell^\infty(\mathrm{X})/c_0(\mathrm{X})$, we may consider the corresponding convex combinations in $L(\ell^\infty(\mathrm{X})/c_0(\mathrm{X}))$ and thus

$$||v - \widehat{\mathrm{A}_2\mathrm{A}_1} u||_{\ell^\infty(\mathrm{X})/c_0(\mathrm{X})} < \frac{3}{k}.$$

Tracking the formation of the convex combinations reveals that $\widehat{\mathrm{A}_2\mathrm{A}_1}$ can be written as an average of $N_{(k)} j_{(k)} N_{(k)}$ many rotations on $\ell^\infty(\mathrm{X})/c_0(\mathrm{X})$. □

Since $C(\beta\mathbb{N} \setminus \mathbb{N})$ is linearly isometric to ℓ^∞/c_0, an application of Theorem 3.2 yields that $C(\beta\mathbb{N} \setminus \mathbb{N}, \mathrm{X})$ is uniformly convex-transitive if X is uniformly convex-transitive. However, let us recall that this space is linearly isometric to $\ell^\infty(\mathrm{X})/c_0(\mathrm{X})$ if and only if X is finite-dimensional.

4 Roughness and projections

Let X be a Banach space. For each $x \in \mathbf{S}_\mathrm{X}$ we denote

$$\eta(\mathrm{X}, x) \doteq \limsup_{\|h\| \to 0} \frac{\|x+h\| + \|x-h\| - 2}{\|h\|}.$$

Given $\varepsilon > 0$, the space X is said to be *ε-rough* if $\inf_{x \in \mathbf{S}_\mathrm{X}} \eta(\mathrm{X}, x) \geq \varepsilon$. In addition, 2-rough spaces are usually called *extremely rough*.

We will denote the *coprojection constant* of X by

$$\rho(\mathrm{X}) = \sup_P ||\mathbf{I} - P||,$$

where the supremum is taken over all linear norm-1 projections $P\colon \mathrm{X} \to \mathrm{Y}$.

A Banach space X is called *uniformly non-square* if there exists $a \in (0,1)$ such that if $x, y \in \mathbf{B}_\mathrm{X}$ and $\|x - y\| \geq 2a$ then $\|x + y\| < 2a$. These spaces were introduced in [15] by R. C. James, who also proved that this property lies strictly between uniform convexity and reflexivity. Next we will illustrate how the previous concepts are related.

Theorem 4.1. *Let* X *be a Banach space. Then the following conditions are equivalent:*

(1) X *contains* $\ell^1(2)$ *almost isometrically.*

(2) X *is not uniformly non-square.*

(3) $\rho(\mathrm{X}) = 2$.

Moreover, if $\sup_{x\in \mathbf{S}_{\mathrm{X}}} \eta(\mathrm{X}, x) = 2$, *then* $\rho(\mathrm{X}) = 2$.

We will require some preparations before the proof. Recall that given $x, y \in \mathrm{X}$ the function $t \mapsto \frac{||x+ty||-||x||}{t}$ is monotone in t and thus the limit $\lim_{t\to 0^+} \frac{||x+ty||-||x||}{t}$ exists and is finite.

Lemma 4.2. *Let* X *be a Banach space and* $x, y \in \mathrm{X},\ x \neq 0$. *Then*

$$\begin{aligned} &\lim_{t\to 0^+} \tfrac{||x+t(y+\theta x)||-||x||}{t} = \lim_{t\to 0^+} \tfrac{||x-t(y+\theta x)||-||x||}{t} \\ = \ &\lim_{t\to 0^+} \tfrac{||x+t(y+\theta x)||+||x-t(y+\theta x)||-2||x||}{2t} \end{aligned}$$

for $\theta \doteq \lim_{t\to 0^+} \frac{||x-ty||-||x+ty||}{2t||x||}$.

Proof. Observe that for all maps $a\colon [0,1] \to \mathbb{R}$ such that $\lim_{t\to 0^+} a(t) > 0$ it holds that

$$\begin{aligned} &\lim_{t\to 0^+} \tfrac{||a(t)x+ty||-||a(t)x||}{t} = \lim_{t\to 0^+} \tfrac{||a(t)x+\frac{a(t)}{a(t)}ty||-||a(t)x||}{t} \\ = \ &\lim_{t\to 0^+} \tfrac{||x+\frac{t}{a(t)}y||-||x||}{\frac{t}{a(t)}} = \lim_{t\to 0^+} \tfrac{||x+ty||-||x||}{t}. \end{aligned} \tag{21}$$

We will also apply the fact that

$$\lim_{t\to 0^+} \frac{t(\lim_{t\to 0^+} \frac{||x-ty||-||x+ty||}{2t||x||}) - t\frac{||x-ty||-||x+ty||}{2t||x||}}{t} = 0. \tag{22}$$

The claimed one-sided limits are calculated as follows:

$$\begin{aligned} &\lim_{t\to 0^+} \frac{||x+t(y+\theta x)||-||x||}{t} \\ = \ &\lim_{t\to 0^+} \frac{||(1+\frac{||x-ty||-||x+ty||}{2||x||})x+ty||-||x||}{t} \\ = \ &\lim_{t\to 0^+} \frac{||(1+\frac{||x-ty||-||x+ty||}{2||x||})x+ty||-(1+\frac{||x-ty||-||x+ty||}{2||x||})||x||}{t} \\ + \ &\lim_{t\to 0^+} \frac{(1+\frac{||x-ty||-||x+ty||}{2||x||})||x||-||x||}{t} \\ = \ &\lim_{t\to 0^+} \frac{||x+ty||-||x||}{t} + \lim_{t\to 0^+} \frac{||x-ty||-||x+ty||}{2t} \\ = \ &\lim_{t\to 0^+} \frac{||x+ty||+||x-ty||-2||x||}{2t}. \end{aligned}$$

In the first equality above we applied the fact (22), and in the third equality the fact (21). The calculations for the equation

$$\lim_{t\to 0^+} \frac{||x-t(y+\theta x)||-||x||}{t} = \lim_{t\to 0^+} \frac{||x+ty||-||x-ty||-2||x||}{2t}$$

are similar. □

Proof of Theorem 4.1. The equivalence of conditions (1) and (2) is well-known (see e.g. [9] or Remark 6.1 in [4]). The direction (1) $\Longrightarrow$ (3) is established by using the Hahn-Banach Theorem to obtain suitable rank-1 projections P. Towards the implication (3) $\Longrightarrow$ (2), suppose that $\rho(\mathrm{X}) = 2$. Given $\delta > 0$ there exists a projection $P \colon \mathrm{X} \to \mathrm{Y}$, which satisfies $||P|| = 1$ and $||\mathbf{I} - P|| > 2 - \frac{\delta}{2}$. Choose $x \in \mathbf{S}_{\mathrm{X}}$ such that $||x - P(x)|| > 2 - \frac{\delta}{2}$. This gives that $||P(x)|| \geq 1 - \frac{\delta}{2}$. Put $y = \frac{P(x)}{||P(x)||}$ and note that $y \in \mathbf{S}_{\mathrm{X}}$ and $||y - P(x)|| < \frac{\delta}{2}$. Moreover,

$$||x - y|| \geq ||x - P(x)|| - ||y - P(x)|| > 2 - \delta > 2(1 - \delta)$$

and

$$||x + y|| \geq ||x + P(x)|| - ||y - P(x)|| > ||x + P(x)|| - \frac{\delta}{2}$$

$$= \quad ||2x + P(x) - x||\ ||P|| - \frac{\delta}{2}$$

$$\geq \quad ||P(2x + P(x) - x)|| - \frac{\delta}{2} = ||P(2x)|| - \frac{\delta}{2} > 2 - \delta - \frac{\delta}{2} > 2(1 - \delta).$$

Thus X is not uniformly non-square.

To verify the last sentence in the theorem, an application of Lemma 4.2 yields that if $\sup_{x \in \mathbf{S}_{\mathrm{X}}} \eta(x, \mathrm{X}) = 2$, then X is not uniformly non-square. Alternatively, this can be seen by modifying the argument in Remark 1 of [3]. We obtain that $\rho(\mathrm{X}) = 2$. $\square$

The extreme roughness of X is a tremendously stronger condition than $\rho(\mathrm{X}) = 2$. For example, if (F_n) is a sequence of finite-dimensional smooth spaces such that $\rho(F_n) \to 2$ as $n \to \infty$, then the space

$$\mathrm{X} = \bigoplus_{n \in \mathbb{N}} F_n \qquad \text{(summation in } \ell^2\text{−sense)}$$

is Fréchet-smooth but $\rho(\mathrm{X}) = 2$.

However, for convex-transitive spaces X the condition of being extremely rough is equivalent to the condition $\rho(\mathrm{X}) = 2$. Indeed, if a convex-transitive space is not extremely rough then, by [5, Thm. 6.8], it must be uniformly convex and thus $\rho(\mathrm{X}) < 2$. It is unknown to us whether a convex-transitive Banach space is reflexive if it does not contain an isomorphic copy of ℓ^1.

In the same spirit as in this section, the projection constants of L^p spaces were discussed in [21].

5 Final Remarks: On the universality of transitivity properties

The well-known Banach-Mazur problem mentioned in the introduction asks whether every transitive, separable Banach space must be linearly isometric to a Hilbert space. It is well-known that all such (transitive+separable) spaces must be smooth; otherwise, not much is known. Even adding some properties like being a dual space or even reflexivity has not sufficed, to date, for proving that the norm is Hilbertian.

Let us make a few remarks on the universality of some spaces of continuous functions. It is well-known that $C(\Delta)$ contains $C([0,1])$ isometrically; hence, the former space is universal for the property of being uniformly convex-transitive and separable. However, it is not almost transitive.

To get a space which is universal for the property of being almost transitive and separable, just consider the almost transitive space $X = C_0^{\mathbb{C}}(L)$ where L is the pseudo-arc with one point removed ([16] or [19]). Since $[0,1]$ is a continuous image of L, every separable space is isometrically contained in X (complex case) or $X_{\mathbb{R}}$ (real case). Finally, note that the almost transitivity of a Banach space implies that of the real underlying space.

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