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A NONSTANDARD MIXED FINITE ELEMENT FAMILY

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Abstract: *We show that standard mixed finite element methods for second order elliptic equations can be modified by imposing additional continuity conditions for the flux, which reduces the dimension of the space. This reduced space still gives a stable method with an optimal order of convergence. We recall our postprocessing method and the a posteriori error estimator based on this.*

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1 Introduction

Mixed finite elements are well established methods for second order elliptic equations and numerous families are known for both two and three dimensional problems [7, 4].

The analysis of mixed methods is traditionally based on the use of two interpolation operators (for the vector and scalar unknowns, respectively) which together with the divergence operator satisfy a commuting diagram property. Recently, the analysis has also been given a differential geometric framework [2].

The purpose of this paper is to show that for quadratic and higher order spaces one can modify the methods by imposing the condition that the vector unknown is continuous at the vertices of the mesh. With this modification the (local) commuting diagram property is lost, but we show that the method is still stable and optimally convergent. We perform the analysis starting from the two dimensional Brezzi-Douglas-Marini spaces (BDM) [8] and the three dimensional Brezzi-Douglas-Duran-Fortin spaces (BDDF) [6].

The plan of the paper is the following. In the next section we first recall the mixed formulation and the BDM and BDDF spaces. Then we introduce our reduced spaces and show the stability and optimal order convergence. In the final section we recall our postprocessing method and the a posteriori error analysis based on this.

2 The finite element spaces

We consider the the Poisson problem as an elliptic system

$$\begin{aligned} \boldsymbol{\sigma} - \nabla u &= \mathbf{0}, \\ \operatorname{div} \boldsymbol{\sigma} + f &= 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

and the mixed finite element formulation: *find* $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{S}_h \times V_h \subset \boldsymbol{H}(\mathrm{div}:\Omega) \times L^2(\Omega)$ *such that*

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u_h) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{S}_h, \tag{2.2}$$

$$(\operatorname{div} \boldsymbol{\sigma}_h, v) + (f, v) = 0 \quad \forall v \in V_h. \tag{2.3}$$

Given an an integer $k \geq 1$, the BDM and BDDF spaces are

$$\boldsymbol{S}_h = \{\, \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}:\Omega) \mid \boldsymbol{\tau}|_K \in [P_k(K)]^n \ \forall K \in \mathcal{C}_h \,\}, \tag{2.4}$$

$$V_h = \{\, v \in L^2(\Omega) \mid v|_K \in P_{k-1}(K) \ \forall K \in \mathcal{C}_h \,\}. \tag{2.5}$$

Here $\mathcal{C}_h$ is a partitioning of Ω into triangles/tetrahedrons for which we use the generic notation K. The inter element edges (for $n = 2$ or faces (for $n = 3$) we use the generic notation S and with Γ_h we denote the collection of them. The notation E is reserved for edges of the tetrahedrons in the three dimensional problem.

For the analysis the following degrees of freedom are used [7]:

$$\int_S \boldsymbol{\tau}\cdot\boldsymbol{n}\, z \quad \forall z\in P_k(S),\quad \forall S\subset\partial K, \tag{2.6}$$

$$\int_K \boldsymbol{\tau}\cdot\nabla z \quad \forall z\in P_{k-1}(K), \tag{2.7}$$

$$\int_K \boldsymbol{\tau}\cdot\boldsymbol{z} \qquad \forall \boldsymbol{z}\in \boldsymbol{\Psi}_k(K), \tag{2.8}$$

with

$$\boldsymbol{\Psi}_k(K)=\begin{cases}\{\,\boldsymbol{z}\,|\,\boldsymbol{z}=\mathbf{curl}(b_K w),\ w\in P_{k-2}(K)\}\ \text{for } n=2,\\ \{\,\boldsymbol{z}\in[P_k(K)]^3\,|\,\operatorname{div}\boldsymbol{z}=0,\ \boldsymbol{z}\cdot\boldsymbol{n}|_{\partial K}=0\}\ \text{for } n=3,\end{cases} \tag{2.9}$$

where b_K is the cubic bubble on the triangle K.

In the family we now present we replace the degrees of freedom (2.6) with the following:

In two space dimensions, for $n=2$,

$$\boldsymbol{\tau}(\boldsymbol{x}) \quad \forall \textit{ vertices } \boldsymbol{x} \textit{ of } K, \tag{2.10}$$

$$\int_S \boldsymbol{\tau}\cdot\boldsymbol{n}\, z \quad \forall z\in P_{k-2}(S),\quad \forall S\subset\partial K, \tag{2.11}$$

In three space dimensions, for $n=3$, we choose the degrees of freedom in the following way.

$$\boldsymbol{\tau}(\boldsymbol{x}) \quad \forall \textit{ vertices } \boldsymbol{x} \textit{ of } K. \tag{2.12}$$

For each face $S\subset\partial K$:

$$\begin{aligned}&\textit{the value of } \boldsymbol{\tau}\cdot\boldsymbol{n} \ \textit{ at } k-2\\ &\textit{distinct interior points of every edge } E\subset S,\end{aligned} \tag{2.13}$$

and

$$\int_S \boldsymbol{\tau}\cdot\boldsymbol{n}\, z \quad \forall z\in P_{k-3}(S). \tag{2.14}$$

For the collection of vertices in the mesh $\mathcal{C}_h$ we introduce the notations $\mathcal{V}_h$. The space for vector variable we now define as follows for $k\ge 2$

$$\begin{aligned}\boldsymbol{S}_h=\{\,\boldsymbol{\tau}\in \quad & \boldsymbol{H}(\operatorname{div}:\Omega)\,|\,\boldsymbol{\tau}|_K\in[P_k(K)]^n\ \forall K\in\mathcal{C}_h\\ & \boldsymbol{\tau} \text{ is continuous at each vertex } \boldsymbol{x}\in\mathcal{V}_h\,\}.\end{aligned} \tag{2.15}$$

Remark 2.1. *The above degrees of freedom are chosen for the analysis. In an implementation the most straightforward choice would be to start from the Lagrange nodes and then make coordinate transforms so that the normal component of the vector is a degree on each node, except the vertices, on each edge/face is a degree of freedom.*

The space for the scalar variable is kept as (2.5). With this the equilibrium property

$$\operatorname{div} \boldsymbol{S}_h \subset V_h. \tag{2.16}$$

is valid. When denoting by $P_h : L^2(\Omega) \to V_h$ the L^2-projection, this implies that

$$(\operatorname{div} \boldsymbol{\tau}, u - P_h u) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{S}_h. \tag{2.17}$$

This property is crucial for the error analysis. The other is the stability which we prove using the mesh dependent norms

$$\|v\|_{1,h}^2 = \sum_{K \in \mathcal{C}_h} \|\nabla v\|_{0,K}^2 + \sum_{S \in \Gamma_h} h_S^{-1} \|[\![v]\!]\|_{0,S}^2 \tag{2.18}$$

and

$$\|\boldsymbol{\tau}\|_{0,h}^2 = \|\boldsymbol{\tau}\|_0^2 + \sum_{S \in \Gamma_h} h_S \|\boldsymbol{\tau} \cdot \boldsymbol{n}\|_{0,S}^2, \tag{2.19}$$

where $[\![v]\!]$ is the jump in v along interior edges/faces and v on edges/faces on $\partial\Omega$.

By partial integration we have

$$|(\operatorname{div} \boldsymbol{\tau}, v)| \leq \|\boldsymbol{\tau}\|_{0,h} \|v\|_{1,h} \qquad \forall (\boldsymbol{\tau}, v) \in \boldsymbol{S}_h \times V_h. \tag{2.20}$$

We recall that in the subspace the norm for the flux is equivalent to the L^2 norm:

$$C\|\boldsymbol{\tau}\|_{0,h} \leq \|\boldsymbol{\tau}\|_0 \leq \|\boldsymbol{\tau}\|_{0,h} \quad \forall \boldsymbol{\tau} \in \boldsymbol{S}_h. \tag{2.21}$$

Now we prove the Babuška-Brezzi condition for our choice of spaces. Here and in the sequel we denote with C a generic constant independent of the mesh parameter h, which may take different values in different occurrences.

Theorem 2.1. *There is a positive constant C such that*

$$\sup_{\boldsymbol{\tau} \in \boldsymbol{S}_h} \frac{(\operatorname{div} \boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{0,h}} \geq C\|v\|_{1,h} \quad \forall v \in V_h \tag{2.22}$$

is valid for the finite element spaces (2.15) *and* (2.5).

Proof. We use the macroelement technique and use macroelements consisting of two elements. To this end, let $M = K_1 \cup K_2$ be an arbitrary macroelement consisting of the elements K_1 and K_2 with one common edge/face. Define

$$\begin{aligned} \boldsymbol{S}_{0,M} = \{ \boldsymbol{\tau} \in \quad & \boldsymbol{H}(\operatorname{div}\!: M) \mid \boldsymbol{\tau}|_{K_i} \in [P_k(K_i)]^n,\ i = 1,2,\ \boldsymbol{\tau} \cdot \boldsymbol{n}|_{\partial M} = 0, \\ & \boldsymbol{\tau} \text{ vanish at each vertex in } M \} \end{aligned} \tag{2.23}$$

$$P_M = \{ v \in L^2(M) \mid v|_{K_i} \in P_{k-1}(K_i),\ i = 1,2 \} \tag{2.24}$$

and

$$N_M = \{ v \in P_M \mid (\operatorname{div} \boldsymbol{\sigma}, v)_M = 0\ \forall \boldsymbol{\sigma} \in \boldsymbol{S}_{0,M} \}. \tag{2.25}$$

The stability now follows if N_M is one-dimensional consisting of the functions constant on M, cf. [10, 12, 13, 14].

Hence, let $v \in P_M$. For $i = 1, 2$, choose $\boldsymbol{\sigma}_i \in \boldsymbol{S}_{0,M}$ such that its support is in K_i and all other degrees of freedom except the ones given by (2.7). The condition $v \in N_M$ leads to

$$0 = (\operatorname{div} \boldsymbol{\sigma}, v)_M = (\operatorname{div} \boldsymbol{\sigma}, v)_{K_i} = -(\boldsymbol{\sigma}, \nabla v)_{K_i}. \tag{2.26}$$

From (2.7) it now follows that v is constant on K_i,

$$v|_{K_i} = c_i, \quad i = 1, 2. \tag{2.27}$$

For $v \in N_M$ it thus holds

$$0 = (\operatorname{div} \boldsymbol{\sigma}, v)_M = (c_1 - c_2) \int_{K_1 \cap K_2} \boldsymbol{\sigma} \cdot \boldsymbol{n}_1. \tag{2.28}$$

For the two dimensional case the degrees of freedom (2.11) immediately implies that $c_1 = c_2$. For $n = 3$, we use (2.13) to choose $\boldsymbol{\sigma}$ such that its restriction of its normal component (i.e. $\boldsymbol{n}_1$) to $K_1 \cap K_2$ is the quadratic polynomial that is equal to one at the midpoints of each edge common to K_1 andK_2 and vanish at the vertices. Then it holds

$$\int_{K_1 \cap K_2} \boldsymbol{\sigma} \cdot \boldsymbol{n}_1 = \operatorname{area}(K_1 \cap K_2) \tag{2.29}$$

which implies $c_1 = c_2$. The macroelement condition is hence fulfilled. □

The stability and equilibrium property give the following quasi-optimal error estimate.

Theorem 2.2. *Suppose that the solution to* (2.1) *satisfies* $u \in H^{k+2}(\Omega)$. *Then we have the error estimate*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|P_h u - u_h\|_{1,h} \leq C h^{k+1} |u|_{k+2}. \tag{2.30}$$

Proof. Let $\boldsymbol{I}_h \boldsymbol{\sigma} \in \boldsymbol{S}_h \cap [C(\Omega)]^n$ be the Lagrange interpolant to $\boldsymbol{\sigma}$. From the inf-sup condition (2.22) and the consistency it follows that there exists $(\boldsymbol{\tau}, v) \in \boldsymbol{S}_h \times V_h$, with $\|\boldsymbol{\tau}\|_{0,h} + \|v\|_{1,h} \leq C$, such that

$$\begin{aligned} \|\boldsymbol{\sigma}_h \quad - \quad & \boldsymbol{I}_h \boldsymbol{\sigma}\|_{0,h} + \|u_h - P_h u\|_{1,h} \\ & \leq (\boldsymbol{\sigma} - \boldsymbol{I}_h \boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u - P_h u) + (\operatorname{div} (\boldsymbol{\sigma} - \boldsymbol{I}_h \boldsymbol{\sigma}), v). \end{aligned} \tag{2.31}$$

Using (2.17), (2.20), (2.21) and the triangle inequality we get

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u_h - P_h u\|_{1,h} \leq C \|\boldsymbol{\sigma} - \boldsymbol{I}_h \boldsymbol{\sigma}\|_{0,h}. \tag{2.32}$$

Standard interpolation estimates then give

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u_h - P_h u\|_{1,h} \leq C h^{k+1} \|\boldsymbol{\sigma}\|_{k+1} \leq C h^{k+1} \|u\|_{k+2}. \tag{2.33}$$

□

We end this section with some remarks on the method.

Remark 2.2. *Note that all degrees of freedom of the type* (2.7) *and* (2.8) *can be locally condensed.*

Remark 2.3. *For the lowest order method with* $k = 2$ *the decrease of degrees of freedom, after condensing, is near one half. This is roughly estimated as follows.*

For $n = 2$ *let* T *be the number of triangles. Since each edge is shared by two triangles, the degrees of freedom* (2.6) *for BDM are of the order* $\mathcal{O}(9T/2)$. *A vertex, however, is shared by an average of six triangles. This leads the order* $\mathcal{O}(5T/2)$ *for the present method.*

In three dimensions the number of degrees of freedom for the BDDF family is $\mathcal{O}(12T)$. *For our method the estimate is obtained by using Eulers formula:* $V + F = E + T + 1$, *for the number of vertices, faces, edges and triangles, respectively. Again, each face is shared by two tetrahedrons, i.e.* $F = \mathcal{O}(2T)$. *This gives* $\mathcal{O}(6T)$ *degrees of freedom of the type* (2.13). *Now,* $E = \mathcal{O}(7V)$. *Substituting into Eulers formula and solving gives* $V = \mathcal{O}(T/6)$. *The number of vertex degrees of freedom is hence* $\mathcal{O}(T/2)$, *which gives a total of* $\mathcal{O}(13T/2)$.

Remark 2.4. *Due to the imposing of the continuity at the vertices the Fraijs de Veubeke hybridization cannot be used for solving the discretized equations. There exist, however, other techniques for solving saddle point problems, such as the preconditioning by the interior penalty method [11] (see also the survey [3]), and then the lower number of degrees of freedom can be an advantage.*

3 Postprocessing and a posteriori estimates

The estimate (2.30) for the deflection is a supercovergence result. This, together with the fact that $\boldsymbol{\sigma}_h$ is a good approximation of ∇u, implies that an improved approximation for the displacement can be constructed by local postprocessing [1, 5, 15, 14].

In the method of [15, 14] the postprocessed displacement is sought in the FE space

$$V_h^* = \{\, v \in L^2(\Omega) \mid v|_K \in P_{k+1}(K)\ \forall K \in \mathcal{C}_h \,\}. \tag{3.1}$$

Postprocessing method. *Find* $u_h^* \in V_h^*$ *such that*

$$P_h u_h^* = u_h \tag{3.2}$$

and

$$(\nabla u_h^*, \nabla v)_K = (\boldsymbol{\sigma}_h, \nabla v)_K \quad \forall v \in (I - P_h)V_h^*|_K. \tag{3.3}$$

The error analysis of [15, 14] directly applies to the present method and it gives the following error estimate.

Theorem 3.1. *Suppose that the solution to* (2.1) *satisfies* $u \in H^{k+2}(\Omega)$. *Then it holds*

$$\|u - u_h^*\|_{1,h} \le C h^{k+1} |u|_{k+2}. \tag{3.4}$$

In [9] it was shown that the postprocessed solution can be used for constructing a posteriori estmators. Also, this analysis covers our method and we will summarize the results.

We define the following local error indicators on the elements

$$\eta_{1,K} = \|\nabla u_h^* - \boldsymbol{\sigma}_h\|_{0,K}, \quad \eta_{2,K} = h_K \|f - P_h f\|_{0,K}, \tag{3.5}$$

and on the edges

$$\eta_S = h_S^{-1/2} \|[\![u_h^*]\!]\|_{0,S}. \tag{3.6}$$

Using these quantities, the global estimator is

$$\eta = \Big(\sum_{K \in \mathcal{C}_h} \big(\eta_{1,K}^2 + \eta_{2,K}^2 \big) + \sum_{S \in \Gamma_h} \eta_S^2 \Big)^{1/2}. \tag{3.7}$$

The efficiency of the estimator is given by the following lower bounds, which directly follow from (2.1) using the triangle inequality, and from (3.6) noting that $[\![u]\!] = 0$ on each edge S.

Theorem 3.2. *It holds*

$$\begin{aligned} \eta_{1,K} &\le \|\nabla (u - u_h^*)\|_{0,K} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K}, \\ \eta_S &= h_S^{-1/2} \|[\![u - u_h^*]\!]\|_{0,S}. \end{aligned} \tag{3.8}$$

The upper bound is given by the following theorem.

Theorem 3.3. *There exists a positive constant* C *such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u - u_h^*\|_{1,h} \le C\eta. \tag{3.9}$$

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