Helsinki University of Technology Institute of Mathematics Research Reports

Espoo 2008 A547

A GENERALIZATION OF AN INEQUALITY BY N. V. KRYLOV

Wolfgang Desch **Stig-Olof Londen**

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2008 A547

A GENERALIZATION OF AN INEQUALITY BY N. V. KRYLOV

Wolfgang Desch Stig-Olof Londen

Helsinki University of Technology
Faculty of Information and Natural Sciences
Department of Mathematics and Systems Analysis

Wolfgang Desch, Stig-Olof Londen: *A generalization of an inequality by N. V. Krylov*; Helsinki University of Technology Institute of Mathematics Research Reports A547 (2008).

Abstract: *In a paper by Krylov [6], a parabolic Littlewood-Paley inequality and its application to an* $\mathbf{L}^p$*-estimate of the gradient of the heat kernel are proved. These estimates are crucial tools in the development of a theory of parabolic stochastic partial differential equations constructed by Krylov [7]. We generalize these inequalities so that they can be applied to stochastic integrodifferential equations.*

AMS subject classifications: 60H15, 45K05

Keywords: Littlewood-Paley inequality, heat kernel, resolvent operators, stochastic integral equations

Correspondence

Institut für Mathematik und Wissenschaftliches Rechnen
Karl-Franzens-Universität Graz
Heinrichstrasse 36
8010 Graz
Austria

Stig-Olof Londen
Institute of Mathematics
Helsinki University of Technology
P.O. Box 1100
FI-02015 TKK
Finland

georg.desch@uni-graz.at, slonden@math.hut.fi

ISBN 978-951-22-9441-1 (print)
ISBN 978-951-22-9442-8 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)

Helsinki University of Technology
Faculty of Information and Natural Sciences
Department of Mathematics and Systems Analysis
P.O. Box 1100, FI-02015 TKK, Finland
email: math@tkk.fi http://math.tkk.fi/

1 Introduction

In [7], an $\mathbf{L}^p$-theory for parabolic stochastic partial differential equations is developed. Although the theory is applicable to a far more general class of equations, the starting point is a thorough discussion of the stochastic heat equation

$$dv(t,x,\omega) = \Delta v(t,x,\omega)\,dt + \sum_{i=1}^{\infty} g_i(t,x,\omega)\,dw_i(t,\omega). \tag{1.1}$$

The solution v is scalar valued and defined for $t \in [0,\infty)$, $x \in \mathbb{R}^d$ for some integer $d \geq 1$, and for ω in a probability space Ω. The forcing terms w_i are independent scalar valued Brownian motions, and for fixed t, x, ω, the sequence $g(t,x,\omega) = (g_i(t,x,\omega))_{i=1,\cdots,\infty}$ is in $\ell^2(\mathbb{R})$. As usual, Δ denotes the Laplace operator on $\mathbb{R}^d$. Once existence and uniqueness of solutions to (1.1) is established, these results are extended to more general equations. In particular, in [7] very sharp estimates on the regularity of solutions are obtained. Krylov's approach relies heavily on an estimate which he has proved in a separate paper [6] and which we will state below as Theorem 1.1. Our paper aims at a generalization of this crucial estimate.

Our intention is to apply Krylov's method to a stochastic partial differential-integral equation:

$$\begin{aligned} y(t,x,\omega) \;-\; \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\Delta y(s,x,\omega)\,ds \\ = \int_0^t \sum_{i=1}^{\infty} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\,g_i(s,x,\omega)\,dw_i(s,\omega). \end{aligned} \tag{1.2}$$

We always assume at least $0 < \alpha < 2$, $\frac{1}{2} < \beta < 2$. Again, $x \in \mathbb{R}^d$, $t \geq 0$, ω is in some probability space Ω. For fixed t, x, ω, the sequence $g(t,x,\omega) = (g_i(t,x,\omega))_{i=1\cdots\infty}$ is in $\ell^2(\mathbb{R})$.

To get a better understanding of the role of the parameters α and β, we proceed heuristically and assume for a moment that $g(t,x,\omega)$ is independent of t. In this case, formally (1.2) can be rewritten in the language of fractional derivatives

$$\frac{d^\alpha}{dt^\alpha} y(t,x,\omega) = \Delta y(t,x,\omega) + \sum_{i=1}^{\infty} g_i(x,\omega)\,dw_{i,\beta-\alpha}(s,\omega), \tag{1.3}$$

$$\text{with}\quad w_{i,\mu}(t,\omega) = \int_0^t \frac{(t-s)^\mu}{\Gamma(\mu+1)}\,dw_i(s,\omega).$$

This is a fractional differential equation forced by some noise. If $\beta = \alpha$ the forcing term $dw_{i,0} = dw_i$ is just white noise. If $\beta > \alpha$, the forcing term is a fractional integral of white noise (thus smoother), otherwise it is a fractional derivative of white noise (thus rougher). In fact, $w_{i,\beta-\alpha}$ is a Riemann-Liouville process with Hurst index $H := \beta - \alpha + \frac{1}{2} > 0$ (see [5]). Notice that in the case $\alpha = 1$, $\beta = 1$ we obtain the stochastic heat equation (1.1).

We have made a first attempt to use Krylov's approach to handle (1.2) in [3]. In [3], we have replaced the analogue of Theorem 1.1 by some easier estimates, with not quite as sharp results on regularity. To pave the way to regularity results as sharp as [7], in the present paper we generalize Theorem 1.1 so that it is suited to treat the integrodifferential equation. The actual application to (1.2) will be given in a forthcoming paper.

Before we can state our results in more detail, we need to provide some notation and background about the solution operators for the deterministic heat equation

$$\frac{\partial}{\partial t} v(t,x) = \Delta v(t,x) + g(t,x), \quad v(0,x) = h(x), \tag{1.4}$$

and its generalization to an integral equation

$$y(t,x) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \Delta y(s,x)\, ds = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,x)\, ds. \tag{1.5}$$

Let the forcing term be a function $g \in \mathbf{L}^p([0,\infty) \times \mathbb{R}^d, H)$, and the initial function $h \in \mathbf{L}^p(\mathbb{R}^d, H)$. For fixed $t \geq 0$, the solution $y(t)$ will be a function in $\mathbf{L}^p(\mathbb{R}^d, H)$. Here H is a separable Hilbert space. (We have in mind $H = \ell^2(\mathbb{R})$.)

If the initial function h and the forcing term g are sufficiently smooth and satisfy appropriate size conditions, then it is well known that the solution v of (1.4) can be described by the heat kernel u_t and the heat semigroup $T(t)$, namely

$$v(t,x) = [T(t)h](x) + \int_0^t [T(t-s)g(s,\cdot)](x)\, ds,$$

where

$$[T(t)h](x) = \int_{\mathbb{R}^d} u_t(x-y)h(y)\, dy.$$

Now let $p \in [1,\infty)$. It is well known that $T(t)$ can be extended to a bounded linear operator $T(t) : \mathbf{L}^p(\mathbb{R}^d, H) \to \mathbf{L}^p(\mathbb{R}^d, H)$.

The analogue of the heat semigroup for the integrodifferential equation (1.5) is its resolvent operator $S_{\alpha,\beta}(t) : \mathbf{L}^p(\mathbb{R}^d, H) \to \mathbf{L}^p(\mathbb{R}^d, H)$, which satisifies

$$S_{\alpha,\beta}(t)h - \Delta \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_{\alpha,\beta}(s)h\, ds = \frac{t^{\beta-1}}{\Gamma(\beta)} h. \tag{1.6}$$

Using the resolvent operator, the solution to (1.5) is given by a variation of parameters formula

$$y(t,\cdot) = \int_0^t S_{\alpha,\beta}(t-s)g(s,\cdot)\, ds. \tag{1.7}$$

The theory of resolvent operators for integral equations is well understood. In fact, for $\beta = 1$, Equation (1.5) is a parabolic integral equation as treated in [8, Chapter 3]. By Laplace transform methods it is shown that such equations admit a resolvent operator on $\mathbf{L}^p(\mathbb{R}^d, H)$. For fixed $x \in H$, the function $S_{\alpha,1}(t)x$ is continuous with respect to t in $[0, \infty)$ and infinitely continuously differentiable with respect to t for $t > 0$ (see [8, Theorem 3.1]). Given $S_{\alpha,1}$ and $\beta > 0$, at least formally the operator $S_{\alpha,\beta}$ could be obtained as a fractional integral or derivative of $S_{\alpha,1}$, depending on whether β is less or larger than 1. However, it is easy to obtain $S_{\alpha,\beta}$ directly by adapting the Laplace transform approach to the case $\beta \neq 1$: Formally, the Laplace transform of $S_{\alpha,\beta}h$ is

$$\hat{S}_{\alpha,\beta}(s)h = s^{-\beta}(1 - s^{-\alpha}\Delta)^{-1}h = s^{\alpha-\beta}(s^\alpha - \Delta)^{-1}. \tag{1.8}$$

Thus $S_{\alpha,\beta}(t)$ can be defined by the contour integral

$$S_{\alpha,\beta}(t)h = \frac{1}{2\pi i}\int_{\tilde{C}} e^{st} s^{\alpha-\beta}(s^\alpha - \Delta)^{-1}h\, ds, \tag{1.9}$$

where the contour $\tilde{C}$ consists of the three curves

$$\begin{cases} \sigma \mapsto -r\sigma e^{-i\rho} & \text{for } \sigma \in (-\infty, -1], \\ \sigma \mapsto re^{i\sigma\rho} & \text{for } \sigma \in [-1, 1], \\ \sigma \mapsto r\sigma e^{i\rho} & \text{for } \sigma \in [1, \infty). \end{cases}$$

Here $r > 0$ is an arbitrary constant, and ρ is such that $\frac{\pi}{2} < \rho$ and $\alpha\rho < \pi$. The following estimates, for $r = 1/t$, show that the integral (1.9) exists for $t > 0$ and that with suitable constants M, M_1 and $\gamma = -\cos(\rho) > 0$

$$\begin{aligned} &\|S_{\alpha,\beta}(t)h\|_{\mathbf{L}^2(\mathbb{R}^d,H)} \\ \leq\; & \frac{1}{2\pi}\int_1^\infty e^{-\gamma\sigma rt}(r\sigma)^{\alpha-\beta}\frac{M}{(r\sigma)^\alpha}\|h\|_{\mathbf{L}^2(\mathbb{R}^d,H)}\, r d\sigma \\ & + \frac{1}{2\pi}\int_{-1}^1 e^{rt} r^{\alpha-\beta}\frac{M}{r^\alpha}\|h\|_{\mathbf{L}^2(\mathbb{R}^d,H)} r\rho\, d\sigma \\ & + \frac{1}{2\pi}\int_1^\infty e^{-\gamma\sigma rt}(r\sigma)^{\alpha-\beta}\frac{M}{(r\sigma)^\alpha}\|h\|_{\mathbf{L}^2(\mathbb{R}^d,H)}\, r d\sigma \\ =\; & \frac{1}{\pi}\int_1^\infty e^{-\gamma\sigma} t^{\beta-\alpha}\sigma^{\alpha-\beta}\frac{Mt^\alpha}{\sigma^\alpha}\|h\|_{\mathbf{L}^2(\mathbb{R}^d,H)}\, t^{-1} d\sigma \\ & + \frac{1}{2\pi}\int_{-1}^1 e t^{\beta-\alpha} M t^\alpha \|h\|_{\mathbf{L}^2(\mathbb{R}^d,H)} t^{-1}\rho\, d\sigma \\ =\; & M_1 t^{\beta-1}\|h\|_{\mathbf{L}^2(\mathbb{R}^d,H)}. \end{aligned}$$

In particular, $S_{\alpha,\beta}(t)h$ admits a Laplace transform. Proceeding along these lines, one sees that

$\hat{S}_{\alpha,\beta}(s) = s^{-\beta}(1 - s^{-\alpha}\Delta)^{-1}$,

$S_{\alpha,\beta}(t)$ is analytic for t in a suitable sector,

$$S_{\alpha,\beta}(t) - \Delta\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} S_{\alpha,\beta}(s)\, ds = \frac{t^{\beta-1}}{\Gamma(\beta)}.$$

With this background we return to the topic of our paper to create tools for the existence theory of the stochastic partial differential and integral equations (1.1) and (1.2): In [7] the variation of parameters formula with the heat semigroup $T(t)$ is utilized. At least formally

$$v(t) = T(t)h + \int_0^t T(t-s) \sum_{i=1}^{\infty} g_i(s)\, dw_i(s). \tag{1.10}$$

Suitable estimates are needed to control the effects of the stochastic forcing. A crucial step is the following estimate, which is proved in a separate paper. (Krylov's original version is more general, for the purpose of an introduction we give a somewhat abbreviated version):

Theorem 1.1 (Krylov, [6], Theorem 2.1)**.** *Let $T(t)$ denote the heat semigroup on $\mathbf{L}^p(\mathbb{R}^d, H)$, where H is a separable Hilbert space. Let $-\infty \le a < b \le \infty$, let $p \in [2,\infty)$. Then there exists a constant M such that for any $g \in \mathbf{L}^p((a,b)\times\mathbb{R}^d, H)$ we have*

$$\int_{\mathbb{R}^d}\int_a^b \left[\int_a^t \left\|[\nabla T(t-s)g(s,\cdot)](x)\right\|_H^2 ds\right]^{\frac{p}{2}} dt\, dx \le M \int_{\mathbb{R}^d}\int_a^b \left\|g(s,x)\right\|_H^p ds\, dx. \tag{1.11}$$

Notice that this is a deterministic result, although it is the crucial lemma to estimate the effects of the stochastic forcing in [7]. We adapt this theorem to fit the needs of integral equation (1.2). Here the variation of parameters formula reads

$$y(t) = \int_0^t S_{\alpha,\beta}(t-s) \sum_{i=1}^{\infty} g_i(s)\, dw_i(s). \tag{1.12}$$

To handle the stochastic integral, we will prove the following estimate:

Theorem 1.2. *Let $\alpha \in (0,2)$, $\beta > \frac{1}{2}$, $\gamma \in (0,1)$ be such that $\beta - \alpha\gamma = \frac{1}{2}$. Let H be a separable Hilbert space, $2 \le p < \infty$, $b \in \mathbb{R}$ and $g \in \mathbf{L}^p((-\infty,b]\times\mathbb{R}^d, H)$. Let $S_{\alpha,\beta}(t)$ be the resolvent operator given by (1.6). Then there exists some constant M such that*

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{-\infty}^b \left[\int_{-\infty}^t \left\|[(-\Delta)^\gamma S_{\alpha,\beta}(t-s)g(s,\cdot)](x)\right\|_H^2 ds\right]^{\frac{p}{2}} dt\, dx \quad (1.13)\\ \le\ & M \int_{\mathbb{R}^d}\int_{-\infty}^b \left\|g(s,y)\right\|_H^p ds\, dy. \end{aligned}$$

In the theorem above, we deal with resolvent operators instead of the heat semigroup, and the regularity has changed. Instead of taking the gradient, we take a fractional derivative $(-\Delta)^\gamma$ where

$$\gamma = \frac{\beta}{\alpha} - \frac{1}{2\alpha}. \tag{1.14}$$

To understand the meaning of this relation heuristically, we refer to the fractional differential version (1.3): Here the parameter $\beta-\alpha$ determines the smoothness of the driving noise $w_{i,\beta-\alpha}$. Inequality (1.13) gives an estimate for $(-\Delta)^\gamma S_{\alpha,\beta}(t)$. Therefore, in applications to (1.3), Theorem 1.2 yields estimates for the solution in the Bessel potential space $D(-\Delta)^\gamma$. The relation between time smoothness of the forcing noise and space smoothness of the solution is expressed by (1.14): Increasing the smoothness of the forcing noise by one unit of time regularity corresponds to an increase of the smoothness of the solution by $\frac{2}{\alpha}$ units of space regularity.

Since larger β means smoother input, while smaller γ means less requirement on the space regularity, one expects a similar result for the case

$$\gamma < \frac{\beta}{\alpha} - \frac{1}{2\alpha}.$$

In fact, such a result can be given, with the slight modification that subexponential growth at $t=\infty$ is possible. Therefore we need to introduce an exponential weight $e^{-\epsilon t}$:

Corollary 1.3. *Let $\alpha \in (0,2)$, $\gamma \in (0,1)$, and $\theta > \beta := \frac{1}{2} + \alpha\gamma$. Let $\epsilon > 0$. Let H be a separable Hilbert space, $2 \le p < \infty$, $b \in (-\infty,\infty]$ and g such that $e^{-\epsilon t} g \in \mathbf{L}^p((-\infty,b]\times\mathbb{R}^d, H)$. Let $S_{\alpha,\theta}(t)$ be the resolvent operator given by* (1.6) *(with θ instead of β). Then there exists some constant M such that*

$$\int_{\mathbb{R}^d}\int_{-\infty}^b \left[\int_{-\infty}^t \left\| e^{-\epsilon t}[(-\Delta)^\gamma S_{\alpha,\theta}(t-s) g(s,\cdot)](x)\right\|_H^2 ds\right]^{\frac{p}{2}} dt\, dx$$
$$\le M \int_{\mathbb{R}^d}\int_{-\infty}^b \left\| e^{-\epsilon s} g(s,y)\right\|_H^p ds\, dy. \quad (1.15)$$

We turn now to the question how to prove these estimates. In [6], Theorem 1.1 is obtained as a straightforward corollary from a more general inequality, applied to the gradient of the heat kernel $\psi = \nabla u_1$:

Theorem 1.4 (Krylov, [6], Theorem 1.1)**.** *Let K be a constant, let d be a positive integer, and $\psi : \mathbb{R}^d \to \mathbb{R}$ be infinitely differentiable and such that*

$$\int_{\mathbb{R}^d} \psi(x)\, dx = 0,$$
$$\|\psi\|_{\mathbf{L}^1(\mathbb{R}^d)} + \|\,|x|\,\psi\,\|_{\mathbf{L}^1(\mathbb{R}^d)} + \|\nabla\psi\|_{\mathbf{L}^1(\mathbb{R}^d)} + \|x\cdot\nabla\psi\|_{\mathbf{L}^1(\mathbb{R}^d)} \le K. \quad (1.16)$$

Let H be a separable Hilbert space and $p \in [2,\infty)$. For $h \in \mathbf{L}^2(\mathbb{R}^d, H)$ let

$$\Psi_t h = t^{-\frac{d}{2}} \psi(t^{-\frac{1}{2}} x) * h \quad (1.17)$$

where $*$ denotes convolution in $\mathbb{R}^d$.

Then there exists a constant M depending only on d, p, K such that for all $-\infty < a < b \le \infty$, $g \in \mathbf{L}^p((a,b)\times\mathbb{R}^d, H)$

$$\int_{\mathbb{R}^d}\int_a^b \left[\int_a^t \|\Psi_{t-s} g(s,x)\|_H^2 \frac{ds}{t-s}\right]^{\frac{p}{2}} dt\, dx \le M \int_{\mathbb{R}^d}\int_a^b \|g(s,x)\|_H^p\, ds\, dx.$$

The keys to the application of Theorem 1.4 to the heat semigroup are the self-similarity properties of the heat kernel and its rapid decay at infinity. The kernel functions of resolvent operators exhibit also self-similarity, but the exponents in (1.17) need to be adjusted. Moreover, when we treat the integral equation, we have to deal with fractional derivatives instead of the plain gradient. While the heat kernel and all its derivatives of integer order are rapidly decreasing in space, its fractional derivatives are not. (This can be seen easily from (3.2) below, since the convolution kernel involved decays only like $|x|^{2-2\gamma-d}$.) Instead of estimates on the gradient like (1.16) the best we can achieve is Hölder continuity, and we only have that $|x|^{\epsilon}\psi \in \mathbf{L}^1(\mathbb{R}^d)$ with some $\epsilon < 1$ instead of $\epsilon = 1$. Fortunately, these conditions are sufficient and we can generalize Theorem 1.4 in the following form, which will be sufficient to derive Theorem 1.2:

Theorem 1.5. *Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a measurable function with the following properties:*

$$\psi \in \mathbf{L}^1(\mathbb{R}^d) \text{ with } \int_{\mathbb{R}^d} |\psi(x)|\, dx \le M_1. \tag{1.18}$$

There exist $\epsilon_2 \in (0,1]$, $M_2 > 0$ such that

$$\int_{\mathbb{R}^d} |x^{\epsilon_2}\psi(x)|\, dx \le M_2. \tag{1.19}$$

There exist $\epsilon_3 \in (0,1]$, $M_3 > 0$, $\delta_3 > 0$ such that for $y \in \mathbb{R}^d$ with $|y| < \delta_3$,

$$\int_{\mathbb{R}^d} |\psi(x+y) - \psi(x)|\, dx \le M_3|y|^{\epsilon_3}. \tag{1.20}$$

There exist $\epsilon_4 \in (0,1]$, $M_4 > 0$, $\delta_4 \in (0,1)$ such that for $\lambda \in (\delta_4, \frac{1}{\delta_4})$,

$$\int_{\mathbb{R}^d} |\psi(\lambda x) - \psi(x)|\, dx \le M_4|1-\lambda|^{\epsilon_4}. \tag{1.21}$$

$$\int_{\mathbb{R}^d} \psi(x)\, dx = 0. \tag{1.22}$$

Let $(H, \|\cdot\|_H)$ be a separable Hilbert space. Let $2 \le p < \infty$, $\alpha \in (0,2)$, and $-\infty < b \le \infty$. For $g \in \mathbf{L}^p((-\infty,b] \times \mathbb{R}^d, H)$ we define $Pg : (-\infty,b] \times \mathbb{R}^d \to [0,\infty]$ by

$$(Pg)(t,x) := \left[\int_{-\infty}^{t} \left\| \int_{\mathbb{R}^d} (t-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi\left((t-s)^{-\frac{\alpha}{2}}(x-y)\right) g(s,y)\, dy \right\|_H^2 ds \right]^{\frac{1}{2}}. \tag{1.23}$$

Then there exists a constant M depending on ψ, d, α, and p, such that for all $g \in \mathbf{L}^p(\mathbb{R} \times \mathbb{R}^d, H)$

$$\int_{\mathbb{R}^d} \int_{-\infty}^{b} [(Pg)(t,x)]^p\, dt\, dx \le M \int_{\mathbb{R}^d} \int_{-\infty}^{b} \|g(s,y)\|_H^p\, ds\, dy. \tag{1.24}$$

Remark 1.6. *If* (1.20) *holds with some* $\delta_3 > 0$*, then by standard arguments, for any* $\delta > 0$ *there exists some constant* M *such that* (1.20) *holds with* δ *and* M *instead of* δ_3 *and* M_3*.*
Similarly, if (1.21) *holds with some* $\delta_4 < 1$*, then for any* $\delta \in (0,1)$ *there exists* M *such that* (1.21) *holds with* δ *and* M *instead of* δ_4 *and* M_4*.*

Remark 1.7. *It is easily seen that any function* ψ *satisfying the conditions of Theorem 1.4 also satisfies* (1.20) *and* (1.21)*.*

Thus, the purpose of this paper is to prove Theorem 1.5 and subsequently Theorem 1.2. The proof of Theorem 1.5 is given in Section 2. It follows very closely the lines of [6], with obvious modifications of the exponents, but some nontrivial refinements of the estimates for ψ at infinity are required. The main ingredients of the proof are a straightforward L^2-estimate and a sophisticated BMO-estimate based on the rescaling properties of Ψ_t. In the end, L^p-estimates are obtained by interpolation.

Once Theorem 1.5 is proved, we need to show that the convolution kernel of $(-\Delta)^\gamma S_{\alpha,\beta}(t)$ satisfies its assumptions, in particular (1.19), (1.20), and (1.21). It turns out that this is rather involved, even if the resolvent operator is replaced by the heat semigroup. We give the proof for the heat kernel in Section 3. In Section 4 we proceed to the resolvent kernel in two steps. First, $(-\Delta)^\gamma(s-\Delta)^{-1}$ is handled by integrating the heat semigroup, and finally $(-\Delta)^\gamma S_{\alpha,\beta}(t)$ is handled by the contour integral (1.9). Finally, in Section 5 we prove Corollary 1.3.

2 Proof of Theorem 1.5

The proof follows the ideas of [6] with some nontrivial modifications. The inequality is obtained for general $p \in [2,\infty)$ by interpolation between the case $p = 2$ and a BMO-estimate. We start out with the case $p = 2$, which will be finished in Lemma 2.2:

Lemma 2.1. *With the assumptions of Theorem 1.5 let* $\tilde{\psi}$ *be the Fourier transform of* ψ*. Then*

1) $\tilde{\psi}$ *is bounded and continuous on* $\mathbb{R}^d$*.*

2) There exists a constant $M = M(d,\psi)$ *such that for all* $\xi \in \mathbb{R}^d$*,*

$$|\tilde{\psi}(\xi)| \leq M|\xi|^{-\epsilon_3}. \tag{2.1}$$

3) There exists a constant $M = M(d,\psi)$ *such that for all* $\xi \in \mathbb{R}^d$*,*

$$|\tilde{\psi}(\xi)| \leq M|\xi|^{\epsilon_2}. \tag{2.2}$$

4) There exists a constant $M = M(d, \psi, \alpha)$ such that for all $\xi \in \mathbb{R}^d$,

$$\int_0^\infty |\tilde{\psi}(t^{\frac{\alpha}{2}}\xi)| \frac{dt}{t} \le M. \tag{2.3}$$

Proof. (1) is an immediate consequence of the assumption that $\psi \in \mathbf{L}^1(\mathbb{R}^d)$. Since $\tilde{\psi}$ is bounded, it is sufficient to prove (2) for large ξ. Notice that the Fourier transform of the shifted function satisfies

$$\widetilde{\psi(\cdot + y)}(\xi) = e^{i\langle \xi, y\rangle}\tilde{\psi}(\xi).$$

Using (1.20) we obtain for all y with $|y| \le \delta_3$ and all $\xi \in \mathbb{R}^d$:

$$\begin{aligned} & \left|(e^{i\langle \xi, y\rangle} - 1)\tilde{\psi}(\xi)\right| = \left|\widetilde{\psi(\cdot + y)}(\xi) - \tilde{\psi}(\xi)\right| \\ = \ & (2\pi)^{-\frac{d}{2}} \left|\int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle}(\psi(x+y) - \psi(x))\, dx\right| \le (2\pi)^{-\frac{d}{2}} M_3 |y|^{\epsilon_3}. \end{aligned}$$

Now let $|\xi| \ge \frac{\pi}{\delta_3}$, so that $y := \frac{\pi}{|\xi|^2}\xi$ satisfies $|y| < \delta_3$. Then

$$|-2\tilde{\psi}(\xi)| \le (2\pi)^{-\frac{d}{2}} M_3 \left(\frac{\pi}{|\xi|^2}|\xi|\right)^{\epsilon_3} \le M|\xi|^{-\epsilon_3}$$

with a suitable constant M.
For the proof of (3) we utilize the inequality

$$\left|e^{i\langle \xi, x\rangle} - 1\right| \le M\left(|\xi||x|\right)^{\epsilon_2}$$

which follows easily from the fact that e^{it} is both bounded and globally Lipschitz in t. Moreover, by (1.22), we have $\tilde{\psi}(0) = 0$. Thus for $\xi \in \mathbb{R}^d$ we have by (1.19)

$$\begin{aligned} & \left|\tilde{\psi}(\xi)\right| = \left|\tilde{\psi}(\xi) - \tilde{\psi}(0)\right| \le (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \left|e^{-i\langle \xi, x\rangle} - 1\right| \, |\psi(x)|\, dx \\ \le \ & (2\pi)^{-\frac{d}{2}} M |\xi|^{\epsilon_2} \int_{\mathbb{R}^d} |x|^{\epsilon_2} \, |\psi(x)|\, dx \le (2\pi)^{-\frac{d}{2}} M M_2 |\xi|^{\epsilon_2}. \end{aligned}$$

To prove (4), use (2) and (3) and make a transform $s = t|\xi|^{\frac{2}{\alpha}}$:

$$\begin{aligned} & \int_0^\infty |\tilde{\psi}(t^{\alpha/2}\xi)| \frac{dt}{t} \le M \int_0^\infty \min\left(\left(t^{\alpha/2}|\xi|\right)^{-\epsilon_3}, \left(t^{\alpha/2}|\xi|\right)^{\epsilon_2}\right) \frac{dt}{t} \\ = \ & M \int_0^\infty \min\left(s^{-\epsilon_3\alpha/2}, s^{\epsilon_2\alpha/2}\right) \frac{ds}{s} < \infty. \end{aligned}$$

□

The following lemma is the special case of Theorem 1.5 for $p = 2$:

Lemma 2.2. *Suppose all assumptions of Theorem 1.5 hold. Then there exists a constant M depending only on α, d, and ψ such that for all $g \in \mathbf{L}^2(\mathbb{R}\times\mathbb{R}^d, H)$*

$$\int_{\mathbb{R}^d}\int_{-\infty}^{b} [(Pg)(t,x)]^2\,dt\,dx \le M \int_{\mathbb{R}^d}\int_{-\infty}^{b} \|g(s,y)\|_H^2\,ds\,dy.$$

Proof. By Plancherel's Theorem (which holds also for H-valued functions) we may switch to Fourier transforms. Recall the convolution theorem for Fourier transforms $\widetilde{f*g} = (2\pi)^{\frac{d}{2}}\tilde{f}\tilde{g}$, and the rescaling formula $\widetilde{\psi(\lambda\cdot)}(\xi) = \frac{1}{\lambda^d}\tilde{\psi}(\frac{1}{\lambda}\xi)$. Using these transformations and, in the end, (2.3) and again Plancherel, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{-\infty}^{b}\int_{-\infty}^{t}\left\|\int_{\mathbb{R}^d}(t-s)^{-\frac{1}{2}-\frac{\alpha d}{2}}\psi\left((t-s)^{-\frac{\alpha}{2}}(x-y)\right)g(s,y)\,dy\right\|_H^2 ds\,dt\,dx\\
&=\int_{-\infty}^{b}\int_{-\infty}^{t}(t-s)^{-1-\alpha d}\int_{\mathbb{R}^d}\left\|\left[\psi\left((t-s)^{-\frac{\alpha}{2}}\cdot\right)*g(s,\cdot)\right](x)\right\|_H^2 dx\,ds\,dt\\
&=(2\pi)^{\frac{d}{2}}\int_{-\infty}^{b}\int_{-\infty}^{t}(t-s)^{-1-\alpha d}\int_{\mathbb{R}^d}\left\|\widetilde{\psi\left((t-s)^{-\frac{\alpha}{2}}\cdot\right)}(\xi)\tilde{g}(s,\cdot)(\xi)\right\|_H^2 d\xi\,ds\,dt\\
&=(2\pi)^{\frac{d}{2}}\int_{-\infty}^{b}\int_{-\infty}^{t}(t-s)^{-1-\alpha d}\int_{\mathbb{R}^d}\left\|(t-s)^{\frac{d\alpha}{2}}\tilde{\psi}\left((t-s)^{\frac{\alpha}{2}}\xi\right)\tilde{g}(s,\cdot)(\xi)\right\|_H^2 d\xi\,ds\,dt\\
&=(2\pi)^{\frac{d}{2}}\int_{\mathbb{R}^d}\int_{-\infty}^{b}\|\tilde{g}(s,\xi)\|_H^2\int_{s}^{b}(t-s)^{-1}\left|\tilde{\psi}\left((t-s)^{\frac{\alpha}{2}}\xi\right)\right|^2 dt\,ds\,d\xi\\
&=(2\pi)^{\frac{d}{2}}\int_{\mathbb{R}^d}\int_{-\infty}^{b}\|\tilde{g}(s,\xi)\|_H^2\int_{0}^{b-s}t^{-1}\left|\tilde{\psi}(t^{\frac{\alpha}{2}}\xi)\right|^2 dt\,ds\,d\xi\\
&\le(2\pi)^{\frac{d}{2}}M\int_{\mathbb{R}^d}\int_{-\infty}^{b}\|\tilde{g}(s,\xi)\|_H^2\,ds\,d\xi\\
&=(2\pi)^{\frac{d}{2}}M\int_{-\infty}^{b}\int_{\mathbb{R}^d}\|g(s,x)\|_H^2\,dx\,ds
\end{aligned}$$

□

This finishes the case $p = 2$ and we set out for the BMO-estimate. The following definition is a slight modification of the definition of $Q(r)$ given in [6] in order that the rescaling argument in Lemma 2.8 below can be reproduced in our setting:

Definition 2.3. *For $r > 0$ we set $Q(r) = (-r^{2/\alpha}, 0) \times B(0, r) \subset \mathbb{R}\times\mathbb{R}^d$. Here $B(0, r)$ is the open ball in $\mathbb{R}^d$ with center 0 and radius r.*

We begin investigating the case $b = 0$. The case of general b will be settled later by a rescaling method.

Definition 2.4. *As in [6] we split the operator P into two parts: For $(t,x) \in Q(1)$ and $g \in \mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d,H)$ we define*

$$(P_1 g)(t,x) = \left[\int_{-\infty}^{-2} \left\| \int_{\mathbb{R}^d} (t-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi\left((t-s)^{-\frac{\alpha}{2}}(x-y)\right) g(s,y)\, dy \right\|_H^2 ds \right]^{\frac{1}{2}},$$
$$(P_2 g)(t,x) = \left[\int_{-2}^{t} \left\| \int_{\mathbb{R}^d} (t-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi\left((t-s)^{-\frac{\alpha}{2}}(x-y)\right) g(s,y)\, dy \right\|_H^2 ds \right]^{\frac{1}{2}}.$$

Obviously, with this notation, for $f_0 \geq 0$ we have

$$\left|(Pg)(t,x) - f_0\right| \leq \left|(P_1 g)(t,x) - f_0\right| + \left|(P_2 g)(t,x)\right|.$$

Lemma 2.5. *Let the assumptions of Theorem 1.5 hold. Then there exists a constant M such that for all $g \in \mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d,H)$,*

$$\int_{Q(1)} \left|(P_1 g)(t,x) - (P_1 g)(0,0)\right| dx\, dt \leq M \|g\|_{\mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d,H)}.$$

Proof. This is the part of the proof which deviates most from Krylov's paper [6], since our assumptions on the behavior of ψ at infinity are much weaker. Since $Q(1)$ has finite measure, it is sufficient to show that

$$\int_{Q(1)} \left|(P_1 g)(t,x) - (P_1 g)(0,0)\right|^2 dx\, dt \leq M \|g\|^2_{\mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d,H)}.$$

We use the triangle inequality in $\mathbf{L}^2((-\infty,-2),\mathbb{R})$. Subsequently we apply the following transforms of variables: $\sigma = -s$, $\tau = -t/\sigma$, $\xi = \sigma^{-\alpha/2}x$, and $\eta = -\sigma^{-\alpha/2}y$. Notice that in the integrals below $t \in [-1,0]$ and $s < -2$, so that $\tau \in [0, \frac{1}{2}]$ and $1-\tau \in [\frac{1}{2}, 1]$.

$$
\begin{aligned}
& \int_{-1}^{0} \int_{B(0,1)} \big|(P_1 g)(t,x) - (P_1 g)(0,0)\big|^2 \, dx\, dt \\
= & \int_{-1}^{0} \int_{B(0,1)} \Big| \Big[\int_{-\infty}^{-2} \Big\| \int_{\mathbb{R}^d} (t-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi[(t-s)^{-\frac{\alpha}{2}}(x-y)]\, g(s,y)\, dy \Big\|_H^2 ds \Big]^{\frac{1}{2}} \\
& - \Big[\int_{-\infty}^{-2} \Big\| \int_{\mathbb{R}^d} (-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi[(-s)^{-\frac{\alpha}{2}}(-y)]\, g(s,y)\, dy \Big\|_H^2 ds \Big]^{\frac{1}{2}} \Big|^2 dx\, dt \\
\le & \int_{-1}^{0} \int_{B(0,1)} \int_{-\infty}^{-2} \Big[\Big\| \int_{\mathbb{R}^d} (t-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi[(t-s)^{-\frac{\alpha}{2}}(x-y)]\, g(s,y)\, dy \Big\|_H \\
& - \Big\| \int_{\mathbb{R}^d} (-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi[(-s)^{-\frac{\alpha}{2}}(-y)]\, g(s,y)\, dy \Big\|_H \Big]^2 ds\, dx\, dt \\
\le & \|g\|_{\mathbf{L}^\infty}^2 \int_{-\infty}^{-2} \int_{-1}^{0} \int_{B(0,1)} \Big[\int_{\mathbb{R}^d} \big| (t-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi[(t-s)^{-\frac{\alpha}{2}}(x-y)] \\
& - (-s)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi[(-s)^{-\frac{\alpha}{2}}(-y)] \big|\, dy \Big]^2 dx\, dt\, ds \\
= & \|g\|_{\mathbf{L}^\infty}^2 \int_{2}^{\infty} \int_{0}^{1/\sigma} \int_{B(0,\sigma^{-\alpha/2})} \sigma^{\frac{\alpha d}{2}} \\
& \Big[\int_{\mathbb{R}^d} \big| (1-\tau)^{-\frac{\alpha d}{2}-\frac{1}{2}} \psi[(1-\tau)^{-\frac{\alpha}{2}}(\eta+\xi)] - \psi[\eta] \big|\, d\eta \Big]^2 d\xi\, d\tau\, d\sigma \\
\le & 3 \|g\|_{\mathbf{L}^\infty}^2 \big(I_1 + I_2 + I_3 \big).
\end{aligned}
$$

with

$$
\begin{aligned}
I_1 &= \int_{2}^{\infty} \sigma^{\frac{\alpha d}{2}} \int_{0}^{1/\sigma} \int_{B(0,\sigma^{-\alpha/2})} \Big[\int_{\mathbb{R}^d} (1-\tau)^{-\frac{\alpha d}{2}-\frac{1}{2}} \\
& \qquad \big| \psi[(1-\tau)^{-\frac{\alpha}{2}}(\eta+\xi)] - \psi[(1-\tau)^{-\frac{\alpha}{2}}\eta] \big|\, d\eta \Big]^2 d\xi\, d\tau\, d\sigma, \\
I_2 &= M_U \int_{2}^{\infty} \int_{0}^{1/\sigma} \Big[\int_{\mathbb{R}^d} (1-\tau)^{-\frac{\alpha d}{2}-\frac{1}{2}} \big| \psi[(1-\tau)^{-\frac{\alpha}{2}}(\eta)] - \psi[\eta] \big|\, d\eta \Big]^2 d\tau\, d\sigma, \\
I_3 &= M_U \int_{2}^{\infty} \int_{0}^{1/\sigma} \Big[\int_{\mathbb{R}^d} \big| (1-\tau)^{-\frac{\alpha d}{2}-\frac{1}{2}} - 1 \big|\, |\psi(\eta)|\, d\eta \Big]^2 d\tau\, d\sigma.
\end{aligned}
$$

In the equations above, M_U is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Now we estimate the three integrals separately. In the following estimates, M will denote a generic constant which may vary from line to line.

For I_1 we make a transform of variables $\xi_1 = (1-\tau)^{-\alpha/2}\xi$, $\eta_1 = (1-\tau)^{-\alpha/2}\eta$, and utilize Hypothesis (1.20). By Remark 1.6 we may assume that

$\delta_3 > 1$.

$$
\begin{aligned}
I_1 &= \int_2^\infty \sigma^{\frac{\alpha d}{2}} \int_0^{1/\sigma} \int_{B(0,[(1-\tau)\sigma]^{-\alpha/2})} \Big[\int_{\mathbb{R}^d} (1-\tau)^{-\frac{1}{2}} \\
&\qquad \big|\psi(\eta_1+\xi_1) - \psi(\eta_1)\big| \, d\eta_1 \Big]^2 (1-\tau)^{\alpha d/2} \, d\xi_1 \, d\tau \, d\sigma, \\
&\leq M \int_2^\infty \sigma^{\frac{\alpha d}{2}} \int_0^{1/\sigma} \int_{B(0,\sigma^{-\alpha/2})} |\xi_1|^{2\epsilon_3} \, d\xi_1 \, d\tau \, d\sigma \\
&= M \int_2^\infty \sigma^{\frac{\alpha d}{2}} \int_0^{1/\sigma} \int_0^{\sigma^{-\alpha/2}} r^{2\epsilon_3+d-1} \, dr \, d\tau \, d\sigma \\
&= M \int_2^\infty \sigma^{\frac{\alpha d}{2}} \sigma^{-1} \sigma^{-\frac{\alpha}{2}(2\epsilon_3+d)} \, d\sigma \\
&= M \int_2^\infty \sigma^{-1-\epsilon_3 \alpha} \, d\sigma < \infty.
\end{aligned}
$$

To estimate I_2 we use Hypothesis (1.21). By Remark 1.6 we may assume without loss of generality that $\delta_4 < \frac{1}{2}$. Notice also that $|(1-\tau)^{-\frac{\alpha}{2}} - 1| \leq M\tau$ with a suitable constant M, since $0 \leq \tau \leq \frac{1}{2}$.

$$
\begin{aligned}
I_2 &\leq M \int_2^\infty \int_0^{1/\sigma} \Big[\int_{\mathbb{R}^d} \big|\psi[(1-\tau)^{-\frac{\alpha}{2}}(\eta)] - \psi[\eta]\big| \, d\eta \Big]^2 d\tau \, d\sigma \\
&\leq M \int_2^\infty \int_0^{1/\sigma} \tau^{2\epsilon_4} \, d\tau \, d\sigma = M \int_2^\infty \sigma^{-2\epsilon_4 - 1} \, d\sigma < \infty.
\end{aligned}
$$

Finally, to estimate I_3 we use Hypothesis (1.18) and the fact that $|(1-\tau)^{-\frac{\alpha d}{2}-\frac{1}{2}} - 1| \leq M\tau$. Thus

$$
\begin{aligned}
I_3 &\leq M \int_2^\infty \int_0^{1/\sigma} \Big[\int_{\mathbb{R}^d} \tau |\psi(\eta)| \, d\eta \Big]^2 d\tau \, d\sigma \\
&\leq M \int_2^\infty \int_0^{1/\sigma} \tau^2 \, d\tau \, d\sigma = M \int_2^\infty \sigma^{-3} \, d\sigma < \infty.
\end{aligned}
$$

This finishes the proof of Lemma 2.5. □

Lemma 2.6. *Let the assumptions of Theorem 1.5 hold. Then there exists a constant M such that for all $g \in \mathbf{L}^\infty((-\infty,0] \times \mathbb{R}^d, H)$,*

$$
\int_{Q(1)} (P_2 g)(t,x) \, dx \, dt \leq M \|g\|_{\mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d,H)}.
$$

Proof. The proof is the same as in [6] with some very small modifications. We split the function g in two parts

$$
\begin{aligned}
g_1(t,x) &= \begin{cases} g(t,x) & \text{if } \ (t,x) \in [-2,0] \times B(0,2), \\ 0 & \text{else,} \end{cases} \\
g_2(t,x) &= g(t,x) - g_1(t,x).
\end{aligned}
$$

Of course, it is sufficient to prove the lemma separately for the two special cases $g = g_1$ and $g = g_2$.

For g_1 we utilize the $\mathbf{L}^2$-estimate Lemma 2.2. Notice that the support of g_1 is contained in $[-2, 0] \times B(0, 2)$ and the domain of integration is $Q(1) = (-1, 0) \times B(0, 1)$. Both have finite measure such that the embedding $\mathbf{L}^\infty \subset \mathbf{L}^2 \subset \mathbf{L}^1$ holds on both domains. The constant M in the following estimates may change from line to line.

$$\begin{aligned}
&\int_{Q(1)} |(P_2 g_1)(t,x)| \, dx \, dt \leq \sqrt{|Q(1)|} \left[\int_{Q(1)} (P_2 g_1)(t,x)^2 \, dx \, dt \right]^{\frac{1}{2}} \\
\leq\ & M \left[\int_{-\infty}^0 \int_{\mathbb{R}^d} (P g_1)(t,x)^2 \, dx \, dt \right]^{\frac{1}{2}} \leq M \left[\int_{-\infty}^0 \int_{\mathbb{R}^d} \|g_1(t,x)\|_H^2 \, dx \, dt \right]^{\frac{1}{2}} \\
\leq\ & M \|g_1\|_{\mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d, H)}.
\end{aligned}$$

Now we consider the case $g = g_2$. Let $(t, x) \in Q(1)$. If $s \geq -2$, then either $g_2(s, y) = 0$ or $|y| \geq 2$. The latter implies $|x - y| \geq 1$. Using this observation together with Hypothesis (1.19) and some easy transforms of variables, we obtain

$$\begin{aligned}
& [(P_2 g_2)(t,x)]^2 \\
=\ & \int_{-2}^t \left\| \int_{|y|\geq 2} (t-s)^{-\frac{\alpha d}{2} - \frac{1}{2}} \psi[(t-s)^{-\frac{\alpha}{2}}(x-y)] g_2(s,y) \, dy \right\|_H^2 ds \\
\leq\ & \|g_2\|_{\mathbf{L}^\infty}^2 \int_{-2}^t \left[\int_{|x-y|\geq 1} (t-s)^{-\frac{\alpha d}{2} - \frac{1}{2}} \left| \psi[(t-s)^{-\frac{\alpha}{2}}(x-y)]) \right| dy \right]^2 ds \\
=\ & \|g_2\|_{\mathbf{L}^\infty}^2 \int_0^{t+2} \left[\int_{|x-y|\geq 1} \tau^{-\frac{\alpha d}{2} - \frac{1}{2}} \left| \psi[\tau^{-\frac{\alpha}{2}}(x-y)] \right| dy \right]^2 d\tau \\
=\ & \|g_2\|_{\mathbf{L}^\infty}^2 \int_0^{t+2} \left[\int_{|z|\geq \tau^{-\alpha/2}} \tau^{-\frac{1}{2}} |\psi(z)| \, dz \right]^2 d\tau \\
\leq\ & \|g_2\|_{\mathbf{L}^\infty}^2 \int_0^{t+2} \left[\int_{|z|\geq \tau^{-\alpha/2}} |z|^\epsilon |\psi(z)| \, dz \right]^2 \tau^{-1+\alpha\epsilon} \, d\tau \\
\leq\ & \|g_2\|_{\mathbf{L}^\infty}^2 \, \left\| |z|^\epsilon \psi \right\|_{\mathbf{L}^1(\mathbb{R}^d,\mathbb{R})}^2 \int_0^2 \tau^{-1+\alpha\epsilon} \, d\tau \\
\leq\ & M \|g_2\|_{\mathbf{L}^\infty}^2.
\end{aligned}$$

□

The remainder of the proof of Theorem 1.5 follows exactly the lines of [6]. By self-similarity, the estimates above can be rescaled for $Q(r)$ with arbitrary $r > 0$. In the end, an interpolation argument completes the proof:

Definition 2.7. *For a measurable function* $f : \mathbb{R}\times\mathbb{R}^d \to [0, \infty]$ *and* $(t_0, x_0) \in$

$\mathbb{R} \times \mathbb{R}^d$, *we define*

$$\mathbf{M}f(t_0, x_0) = \sup \frac{1}{|Q(r)|} \int_{(t_1,x_1)+Q(r)} f(t,x)\,dt\,dx. \tag{2.4}$$

$$f^\sharp(t_0, x_0) = \sup \inf_{f_0 \in \mathbb{R}} \frac{1}{|Q(r)|} \int_{(t_1,x_1)+Q(r)} |f(t,x) - f_0|\,dt\,dx. \tag{2.5}$$

where in both cases the supremum is taken over all $(t_1, x_1) \in \mathbb{R} \times \mathbb{R}^d$ *and all* $r > 0$ *such that* $(t_0, x_0) \in (t_1, x_1) + Q(r)$. *Here* $|Q(r)|$ *is the Lebesgue measure of* $Q(r)$.

Lemma 2.8. *Let the assumptions of Theorem 1.5 hold. Then there exists a constant* M *such that for all* $g \in \mathbf{L}^\infty((-\infty, b] \times \mathbb{R}^d, H)$ *and all* $t_0 \in (-\infty, b]$, $x_0 \in \mathbb{R}^d$,

$$|(Pg)^\sharp(t_0, x_0)| \leq M \|g\|_{\mathbf{L}^\infty((-\infty,b]\times\mathbb{R}^d,H)}.$$

Proof. Combining Lemmas 2.5 and 2.6 we obtain for all $g_1 \in \mathbf{L}^\infty((-\infty, 0] \times \mathbb{R}^d, H)$

$$\frac{1}{|Q(1)|} \int_{Q(1)} \big|(Pg_1)(t,x) - f_0\big|\,dx\,dt \leq M \|g_1\|_{\mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d,H)},$$

with $f_0 = (P_1 g_1)(0,0)$. We generalize this estimate by a simple rescaling procedure: Let $g \in \mathbf{L}^\infty((-\infty, b] \times \mathbb{R}^d, H)$ with general $b \in \mathbb{R}$. Fix $r > 0$ and $(t_1, x_1) \in (-\infty, b] \times \mathbb{R}^d$, such that $(t_0, x_0) \in (t_1, x_1) + Q(r)$. We want to show that

$$\frac{1}{|Q(r)|} \int_{(t_1,x_1)+Q(r)} |(Pg)(t,x) - f_0|\,dx\,dt \leq M \|g\|_{\mathbf{L}^\infty((-\infty,b]\times\mathbb{R}^d,H)}$$

with a suitable f_0. In the following estimates we use the transforms $\tau = r^{-\frac{2}{\alpha}}(t - t_1)$, $\xi = r^{-1}(x - x_1)$, $\sigma = r^{-\frac{2}{\alpha}}(s - t_1)$, $\eta = r^{-1}(y - x_1)$, and $g_1(\sigma, \eta) = g(t_1 + r^{\frac{2}{\alpha}}\sigma, x_1 + r\eta)$. This substitution is constructed in a way such that in the computation all powers of r cancel. We put $f_0 = (P_1 g_1)(0,0)$.

$$\begin{aligned}
&\frac{1}{|Q(r)|} \int_{(t_1,x_1)+Q(r)} \big|f_0 - (Pg)(t,x)\big|\,dx\,dt \\
&= \frac{1}{r^{d+\frac{2}{\alpha}}|Q(1)|} \int_{t_1 - r^{2/\alpha}}^{t_1} \int_{B(x_1,r)} \Big| f_0 - \\
&\qquad \Big[\int_{-\infty}^{t} \Big\| \int_{\mathbb{R}^d} (t-s)^{-\frac{\alpha d}{2} - \frac{1}{2}} \psi[(t-s)^{-\frac{\alpha}{2}}(x-y)]\, g(s,y)\,dy \Big\|_H^2 ds\Big]^{\frac{1}{2}} \Big|\,dx\,dt \\
&= \frac{1}{|Q(1)|} \int_{-1}^{0} \int_{B(0,1)} \Big| (P_1 g_1)(0,0) - \\
&\qquad \Big[\int_{-\infty}^{0} \Big\| \int_{\mathbb{R}^d} (\tau-\sigma)^{-\frac{\alpha d}{2} - \frac{1}{2}} \psi[(\tau-\sigma)^{-\frac{\alpha}{2}}(\xi-\eta)]\, g_1(\sigma,\eta)\,d\eta \Big\|_H^2 d\sigma\Big]^{\frac{1}{2}} \Big|\,d\xi\,d\tau \\
&\leq M \|g_1\|_{\mathbf{L}^\infty((-\infty,0]\times\mathbb{R}^d,H)} \\
&= M \|g\|_{\mathbf{L}^\infty((-\infty,t_1]\times\mathbb{R}^d,H)}.
\end{aligned}$$

Thus the lemma is proved. □

Proof of Theorem 1.5, completed: For $g \in \mathbf{L}^2(\mathbb{R} \times \mathbb{R}^d, H) + \mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}^d, H)$ we define $P^\sharp g = (Pg)^\sharp$. We have noticed in Lemma 2.2 that P maps $\mathbf{L}^2(\mathbb{R} \times \mathbb{R}^d, H)$ continuously into $\mathbf{L}^2(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Moreover, the operator $\mathbf{M}$ maps $\mathbf{L}^\infty$ into $\mathbf{L}^\infty$ and is weak (1,1) by [2, Théorème 2.1]. Therefore, similarly as in [4, Theorem 2.5], we infer that $\mathbf{M}$ maps $\mathbf{L}^p$ into $\mathbf{L}^p$ for $1 < p \leq \infty$. This holds in particular for $p = 2$, so that $P^\sharp g \leq \mathbf{M}Pg \in \mathbf{L}^2$. By Lemma 2.8, the operator $P^\sharp$ maps $\mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}^d, H)$ continuously into $\mathbf{L}^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Now Marcinkiewicz's interpolation theorem [4, Theorem 2.4] implies that $P^\sharp$ maps $\mathbf{L}^p(\mathbb{R} \times \mathbb{R}^d, H)$ into $\mathbf{L}^p(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ for all $p \in [2, \infty)$. (We remark that in [4] Marcinkiewicz's interpolation theorem is stated and proved for real valued functions, but the proof carries over to Banach space valued functions literally.) For any nonnegative function f we have $f(t,x) \leq \mathbf{M}f(t,x)$ almost everywhere. Moreover, from [1, Théorème 2] we infer that $\|\mathbf{M}Pg\|_{\mathbf{L}^p} \leq c\|P^\sharp g\|_{\mathbf{L}^p}$ with a constant c dependent on d and α, but not on g. Therefore we have

$$\begin{aligned} &\|Pg\|_{\mathbf{L}^p((-\infty,b]\times\mathbb{R}^d,\mathbb{R})} \leq \|\mathbf{M}Pg\|_{\mathbf{L}^p((-\infty,b]\times\mathbb{R}^d,\mathbb{R})} \\ \leq\ & c\|P^\sharp g\|_{\mathbf{L}^p((-\infty,b]\times\mathbb{R}^d,\mathbb{R})} \leq M\|g\|_{\mathbf{L}^p((-\infty,b]\times\mathbb{R}^d,H)} \end{aligned}$$

with a suitable constant M. This proves Theorem 1.5. □

3 Estimates for the heat kernel

In this section let u_t denote the heat kernel on $\mathbb{R}^d$, i.e., for $x \in \mathbb{R}^d$,

$$u_t(x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}. \tag{3.1}$$

Here t may be any complex number with positive real part. Δ is the Laplacian in $\mathbb{R}^d$, and its fractional powers are denoted by $(-\Delta)^\gamma$ for $\gamma \in (0,1)$. By $T(t) = e^{t\Delta}$ we denote the heat semigroup on $\mathbf{L}^p(\mathbb{R}^d, H)$. Then for $|\arg(t)| \leq \phi < \frac{\pi}{2}$ and for $h \in \mathbf{L}^p(\mathbb{R}^d, H)$,

$$[T(t)h](x) = \int_{\mathbb{R}^d} u_t(x-y)h(y)\,dy.$$

It is known ([9, p.117]) that for rapidly decreasing f (such as the heat kernel)

$$[(-\Delta)^\gamma f](x) = c\int_{\mathbb{R}^d} |x-y|^{2-2\gamma-d}[-\Delta f](y)\,dy, \tag{3.2}$$

with a constant c depending on γ and d. Notice also that

$$[\Delta u_t](x) = (4\pi t)^{-d/2}\left(\frac{|x|^2}{4t^2} - \frac{d}{2t}\right)e^{-|x|^2/4t}. \tag{3.3}$$

We will prove the following result:

Proposition 3.1. *Let $d \in \mathbb{N}$, $\phi \in (0, \frac{\pi}{2})$ and $\gamma \in (0,1)$. As in (3.1), u_t is the heat kernel in $\mathbb{R}^d$. For shorthand we denote $v_t = (-\Delta)^\gamma u_t$. Let $\epsilon \in [0, 2\gamma)$ and $\eta \in (0, 2-2\gamma) \cap (0,1)$. Then there exists a constant M depending only on d, γ, ϵ, η, ϕ, such that for all $t \in \mathbb{C}$ with $|\arg(t)| \le \phi$ the following estimates hold:*

$$\int_{\mathbb{R}^d} |x|^\epsilon \, |v_t(x)| \, dx \le M |t|^{\frac{\epsilon}{2} - \gamma}, \tag{3.4}$$

$$\text{for all } \; z \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} |v_t(x+z) - v_t(x)| \, dx \le M |t|^{-\frac{\eta}{2} - \gamma} \, |z|^\eta, \tag{3.5}$$

$$\text{for all } \; \lambda \in (\frac{7}{8}, \frac{9}{8}), \quad \int_{\mathbb{R}^d} |v_t(\lambda x) - v_t(x)| \, dx \le M |t|^{-\gamma} \, |1 - \lambda|^\eta. \tag{3.6}$$

We give the proof in several steps. We notice that v_t as well as $[v_t(\cdot + z) - v_t(\cdot)]$ and $[v_t(\lambda \cdot) - v_t(\cdot)]$ are obtained by convolution of Δu_t with suitable functions f, f_z, and f_λ (see, e.g., (3.2)):

$$\int_{\mathbb{R}^d} f(x,y)(-\Delta u_t)(y) \, dy.$$

We prove a general result (Lemma 3.2) for such integrals. Since $f(x,y)$, $f_\lambda(x,y)$, $f_z(x,y)$ may blow up at $x = y$, we need two sets of assumptions on f etc., namely $\mathbf{L}^1$-assumptions near $x = y$ and $\mathbf{L}^\infty$-estimates where x is bounded away from y. Subsequently we will prove that f, f_z, and f_λ satisfy the assumptions of Lemma 3.2. This requires elementary but tedious calculations, summed up in Lemma 3.7 which is nothing less than Proposition 3.1 for the case of $|t| = 1$. In the end, a self-similarity argument yields the result for general t.

Lemma 3.2. *Let $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfy the following assumptions: There exist $0 < \alpha_1 < \alpha_2 < 1$, $0 < \beta_1 < \beta_2$, $K > 0$, $\delta > 0$, $\kappa \ge 0$, such that*

1) If $\alpha_1 |x| + \beta_1 < |x - y|$, then $f(x,y)$ is twice continuously differentiable with respect to y, and

$$|f''(x,y)| \le K(1 + |x|)^{-d-\delta}.$$

2) If $\alpha_1 |x| + \beta_1 < |x - y| \le \alpha_2 |x| + \beta_2$, then

$$|f(x,y)| \le K(1 + |x|)^{-d-\delta+2},$$
$$|f'(x,y)| \le K(1 + |x|)^{-d-\delta+1}.$$

3) For all $x \in \mathbb{R}^d$,

$$\int_{B(x, \alpha_2 |x| + \beta_2)} |f(x,y)| \, dy \le K(1 + |x|)^\kappa.$$

(Here f' is the gradient and $f''(x,y)$ is the Hessian of f with respect to y, and $|f''(x,y)|$ is its matrix norm.)
Let $\epsilon \in [0,\delta)$, $\phi \in (0,\frac{\pi}{2})$. Let u_t denote the heat kernel as in (3.1), and put

$$w_t(x) = \int_{\mathbb{R}^d} f(x,y)[\Delta u_t](y)\,dy.$$

Then there exists a constant M depending only on d, β_1, β_2, α_1, α_2, δ, ϵ, κ, ϕ, such that for t with $|\arg(t)| \le \phi$, $|t| = 1$,

$$\int_{\mathbb{R}^d} |x|^\epsilon\,|w_t(x)|\,dx \le MK. \tag{3.7}$$

Proof. By a standard partition-of-unity procedure we can decompose $f = f_1 + f_2$ such that f_1, f_2 satisfy the Assumptions 1 and 3 of the lemma (possibly with modified constants), and in addition

$$\begin{aligned} f_1(x,y) = 0 \quad &\text{if} \quad |x-y| \ge \alpha_2|x| + \beta_2, \\ f_2(x,y) = 0 \quad &\text{if} \quad |x-y| \le \alpha_1|x| + \beta_1. \end{aligned}$$

(In fact, Assumption 2 is only needed to prove Assumption 1 for f_1 and f_2.) It is sufficient to prove Lemma 3.2 for the two special cases $f = f_1$ and $f = f_2$. In the following computations, M will denote a generic constant which may vary from line to line, and which depends only on d, β_1, β_2, α_1, α_2, δ, ϵ, κ, ϕ. Let $t \in \mathbb{C}$ with $|t| = 1$, $|\arg(t)| \le \phi$.

To treat the case $f = f_1$, notice that $f(x,y) \ne 0$ implies $|y| \ge (1 - \alpha_2)|x| - \beta_2$, so that

$$e^{-|y|^2\cos(\phi)/4} \le g(|x|)e^{-|y|^2\cos(\phi)/8}$$

with

$$g(|x|) = \begin{cases} e^{-[(1-\alpha_2)|x|-\beta_2]^2\cos(\phi)/8} & \text{if } \ (1-\alpha_2)|x| \ge \beta_2, \\ 1 & \text{else.} \end{cases}$$

Remember that $|t| = 1$. We estimate

$$\begin{aligned} |w_t(x)| &= M\left|\int_{\mathbb{R}^d} f(x,y)(\frac{d}{2t} - \frac{|y|^2}{4t^2})\,e^{-|y|^2/4t}\,dy\right| \\ &\le M\int_{|x-y|\le\alpha_2|x|+\beta_2} |f(x,y)|\,(1+|y|^2)\,e^{-|y|^2\cos(\phi)/4}\,dy \\ &\le Mg(|x|)\int_{|x-y|\le\alpha_2|x|+\beta_2} |f(x,y)|\,(1+|y|^2)\,e^{-|y|^2\cos(\phi)/8}\,dy \\ &\le Mg(|x|)\int_{|x-y|\le\alpha_2|x|+\beta_2} |f(x,y)|\,dy. \end{aligned}$$

Now we use Assumption 3 of the Lemma to obtain

$$\int_{\mathbb{R}^d} |x|^\epsilon\,|w_t(x)|\,dx \le M\int_{\mathbb{R}^d} |x|^\epsilon g(|x|)K(1+|x|)^\kappa\,dx \le MK.$$

To treat the case $f = f_2$ we use that $[\Delta u_t](y) = [\Delta u_t](-y)$ and $\int_{\mathbb{R}^d} [\Delta u_t](y)\, dy = 0$. Notice that in this case also $|f''(x,y)| \le MK(1+|x|)^{-d-\delta}$ for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, so that Assumption 1 implies for all (x,y)

$$|\frac{1}{2}f(x,y) + \frac{1}{2}f(x,-y) - f(x,0)| \le MK(1+|x|)^{-d-\delta}|y|^2.$$

Therefore we have

$$\begin{aligned} |w_t(x)| &= \left| \int_{\mathbb{R}^d} [\frac{1}{2}f(x,y) + \frac{1}{2}f(x,-y) - f(x,0)]\, [\Delta u_t](y)\, dy \right| \\ &\le M \int_{\mathbb{R}^d} K(1+|x|)^{-d-\delta} |y|^2\, (\frac{d}{2} + \frac{|y|^2}{4}) e^{-|y|^2 \cos(\phi)/4}\, dy \\ &\le KM(1+|x|)^{-d-\delta}, \end{aligned}$$

and since $\epsilon < \delta$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} |x|^\epsilon\, |w_t(x)|\, dx \le KM \int_{\mathbb{R}^d} |x|^\epsilon\, (1+|x|)^{-d-\delta}\, dx \\ = & \; KM \int_0^\infty r^\epsilon (1+r)^{-1-\delta}\, dr \le KM. \end{aligned}$$

□

Lemma 3.3. *Let $f : \mathbb{R}^d \times \mathbb{R}^d$ be defined by $f(x,y) = |x-y|^{2-2\gamma-d}$ with $\gamma \in (0,1)$. Then, for $y \neq x$, f is twice continuously differentiable with respect to y, with the following gradient and Hessian (with respect to y):*

$$\begin{aligned} f'(x,y) &= (2-2\gamma-d)\, |x-y|^{-2\gamma-d}\, (y-x)^T, \\ f''(x,y) &= (2-2\gamma-d)\, |x-y|^{-2\gamma-d}\, \mathbf{1} \\ &\quad -(2-2\gamma-d)(2\gamma+d)\, |x-y|^{-2-2\gamma-d}\, (y-x)(y-x)^T. \end{aligned}$$

(Here $\mathbf{1}$ is the $d \times d$ unit matrix.)

The proof is straightforward computation.

Lemma 3.4. *Let $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be defined by $f(x,y) = |x-y|^{2-2\gamma-d}$ with $\gamma \in (0,1)$. Then f satisfies the assumptions of Lemma 3.2 with arbitrary $0 < \alpha_1 < \alpha_2 < 1$, $0 < \beta_1 < \beta_2$, $\delta = 2\gamma$, $\kappa = 2 - 2\gamma$ and a constant K depending on d, α_1, α_2, β_1, β_2.*

Proof. Assumptions 1 and 2 of Lemma 3.2 are obtained from Lemma 3.3 by straightforward estimates. To obtain Assumption 3 we estimate

$$\begin{aligned} & \int_{B(x,\alpha_2|x|+\beta_2)} |f(x,y)|\, dy = \int_{B(0,\alpha_2|x|+\beta_2)} |z|^{2-2\gamma-d}\, dz \\ = & \; M \int_0^{\alpha_2|x|+\beta_2} r^{2-2\gamma-d}\, r^{d-1}\, dr = M(\beta_2 + \alpha_2|x|)^{2-2\gamma}. \end{aligned}$$

□

Lemma 3.5. *With z in $\mathbb{R}^d$, $\gamma \in (0,1)$, we consider the function*

$$f_z : \begin{cases} \mathbb{R}^d \times \mathbb{R}^d & \to \mathbb{R}, \\ (x,y) & \mapsto \left[|x+z-y|^{2-d-2\gamma} - |x-y|^{2-d-2\gamma}\right]. \end{cases}$$

Then f_z is twice continuously differentiable with respect to y whenever $y \notin \{x, x+z\}$. Moreover, there exists a constant M depending only on d and γ such that for all $x, y, z \in \mathbb{R}^d$ with

$$|y-x| \geq \frac{1}{4}|x| + 2 \text{ and } |z| \leq \frac{1}{8}|x| + 1 \tag{3.8}$$

the following estimates hold:

$$\begin{aligned} |f_z(x,y)| &\leq M|z|\,(1+|x|)^{-d-2\gamma+1}, \\ |f_z'(x,y)| &\leq M|z|\,(1+|x|)^{-d-2\gamma}, \\ |f_z''(x,y)| &\leq M|z|\,(1+|x|)^{-d-2\gamma-1}. \end{aligned}$$

(Here f' denotes the gradient and f'' denotes the Hessian with respect to y.)

Proof. From Lemma 3.3 we know that f_z is twice differentiable with respect to y if $y \neq x$ and $y \neq x+z$ with

$$\begin{aligned} f_z'(x,y) &= (2-d-2\gamma)\left[|x+z-y|^{-d-2\gamma}(y-x-z)^T - |x-y|^{-d-2\gamma}(y-x)^T\right], \\ f_z''(x,y) &= (2-d-2\gamma)\left[|x+z-y|^{-d-2\gamma} - |x-y|^{-d-2\gamma}\right]\mathbf{1} \\ &\quad - (2-d-2\gamma)(d+2\gamma)\big[|x+z-y|^{-2-d-2\gamma}(x+z-y)(x+z-y)^T \\ &\quad - |x-y|^{-2-d-2\gamma}(x-y)(x-y)^T\big]. \end{aligned}$$

Notice that (3.8) implies that both, $|x-y|$ and $|x+z-y|$ are bounded away from 0, in particular,

$$\frac{1}{8}|x| + 1 \leq \frac{1}{2}|x-y| \leq |x-y| - |z| \leq |x+z-y| \leq |x-y| + |z| \leq \frac{3}{2}|x-y|.$$

In this case, for any power $\theta \leq 1$, the mean value theorem implies the following estimate:

$$\begin{aligned} \left||x+z-y|^\theta - |x-y|^\theta\right| &\leq |\theta| \max\left(|x+z-y|^{\theta-1}, |x-y|^{\theta-1}\right) |z| \\ &\leq 2^{1-\theta}\,|\theta|\,|x-y|^{\theta-1}\,|z|. \end{aligned} \tag{3.9}$$

We use (3.9) repeatedly and obtain

$$\begin{aligned}
|f_z(x,y)| &= \Big||x+z-y|^{2-d-2\gamma} - |x-y|^{2-d-2\gamma}\Big| \\
&\le M|x-y|^{1-d-2\gamma}\,|z| \\
&\le M(1+|x|)^{1-d-2\gamma}|z|, \\
|f_z'(x,y)| &\le M\Big||x+z-y|^{-d-2\gamma}(y-x-z)^T - |x-y|^{-d-2\gamma}(y-x)^T\Big| \\
&\le M\Big[\big||x+z-y|^{-d-2\gamma} - |x-y|^{-d-2\gamma}\big|\,|(y-x-z)| \\
&\qquad + |x-y|^{-d-2\gamma}\,\big|(x+z-y)-(x-y)\big|\Big] \\
&\le M\Big[|x-y|^{-d-2\gamma-1}\,|z|\,|y-x-z| + |x-y|^{-d-2\gamma}\,|z|\Big] \\
&\le M\,|x-y|^{-d-2\gamma}\,|z| \\
&\le M\,(1+|x|)^{-d-2\gamma}\,|z|, \\
|f_z''(x,y)| &\le M\Big[\Big||x+z-y|^{-d-2\gamma} - |x-y|^{-d-2\gamma}\Big| \\
&\qquad + \Big||x+z-y|^{-d-2\gamma-2} - |x-y|^{-2-d-2\gamma}\Big|\;|x+z-y|^2 \\
&\qquad + |x-y|^{-d-2\gamma-2}\;\Big|(x+z-y)(x+z-y)^T - (x-y)(x-y)^T\Big|\Big] \\
&\le M\Big[|x-y|^{-d-2\gamma-1}\,|z| \\
&\qquad + |x-y|^{-d-2\gamma-3}\,|z|\,|x-y|^2 \\
&\qquad + |x-y|^{-d-2\gamma-2}\,\big(2|x-y|\,|z| + |z|^2\big)\Big] \\
&\le M\,|x-y|^{-d-2\gamma-1}\,|z| \\
&\le M\,(1+|x|)^{-d-2\gamma-1}\,|z|.
\end{aligned}$$

□

Lemma 3.6. *As in Lemma 3.5 we consider the function*

$$f_z : \begin{cases} \mathbb{R}^d \times \mathbb{R}^d & \to \mathbb{R}, \\ (x,y) & \mapsto \big[|x+z-y|^{2-d-2\gamma} - |x-y|^{2-d-2\gamma}\big]. \end{cases}$$

with z in $\mathbb{R}^d$, $\gamma \in (0,1)$. Let $x \in \mathbb{R}^d$ be such that $|z| \le \frac{1}{8}|x| + 1$. Let $\eta \in (0,1)$ such that $\eta < 2-2\gamma$. Then there exists a constant M, depending only on d, γ, η such that

$$\int_{B(x,\frac{1}{4}|x|+2)} |f_z(x,y)|\,dy \le M\,|z|^\eta\,(1+|x|)^{2-2\gamma-\eta}.$$

Proof. For shorthand we write $R = \frac{1}{4}|x| + 2$, thus $|z| \le \frac{R}{2} \le R \le M(1+|x|)$.

First we consider the case $d \ge 2$. Without loss of generality we assume $z = (0, \cdots, 0, \zeta)^T$ with $\zeta > 0$, in particular, $\zeta = |z|$. We decompose $x - y =$

$\begin{pmatrix} \overline{y} \\ \xi \end{pmatrix}$ where $\overline{y}$ consists of the first $d-1$ coefficients of $x-y$. We denote $\mu = |\overline{y}|$. Then

$$\begin{aligned}
&\int_{B(x,\frac{1}{4}|x|+2)} |f_z(x,y)|\, dy \\
\leq\ & \int_{-R}^{R}\int_{|\overline{y}|\leq R} \big|(|\overline{y}|^2+|\xi+\zeta|^2)^{\frac{2-d-2\gamma}{2}} - (|\overline{y}|^2+|\xi|^2)^{\frac{2-d-2\gamma}{2}}\big|\, d\overline{y}\, d\xi \\
=\ & M\int_{-R}^{R}\int_0^R \mu^{d-2}\big|(\mu^2+(\xi+\zeta)^2)^{\frac{2-d-2\gamma}{2}} - (\mu^2+\xi^2)^{\frac{2-d-2\gamma}{2}}\big|\, d\mu\, d\xi \\
=\ & M\int_0^R \mu^{d-2}\Big[\int_{-R}^{-\zeta/2}\big[(\mu^2+(\xi+\zeta)^2)^{\frac{2-d-2\gamma}{2}} - (\mu^2+\xi^2)^{\frac{2-d-2\gamma}{2}}\big]\, d\xi \\
&\quad + \int_{-\zeta/2}^{R}\big[(\mu^2+\xi^2)^{\frac{2-d-2\gamma}{2}} - (\mu^2+(\xi+\zeta)^2)^{\frac{2-d-2\gamma}{2}}\big]\, d\xi\Big]\, d\mu \\
=\ & M\int_0^R \mu^{d-2}\Big[-\int_{-R}^{-\zeta/2} + \int_{-R+\zeta}^{\zeta/2} - \int_{\zeta/2}^{R+\zeta} + \int_{-\zeta/2}^{R}\Big](\mu^2+\xi^2)^{\frac{2-d-2\gamma}{2}}\, d\xi\, d\mu \\
=\ & M\int_0^R \mu^{d-2}\Big[-\int_{-R}^{-R+\zeta} + 2\int_{-\zeta/2}^{\zeta/2} - \int_{R}^{R+\zeta}\Big](\mu^2+\xi^2)^{\frac{2-d-2\gamma}{2}}\, d\xi\, d\mu \\
\leq\ & 4M\int_0^R \mu^{d-2}\int_0^{\zeta/2}(\mu^2+\xi^2)^{\frac{2-d-2\gamma}{2}}\, d\xi\, d\mu \\
=\ & 4M\int_0^R \mu^{1-2\gamma}\int_0^{\zeta/2\mu}(1+\tau^2)^{\frac{2-d-2\gamma}{2}}\, d\tau\, d\mu \\
=\ & 4M\Big[\int_0^{\zeta/2R}(1+\tau^2)^{\frac{2-d-2\gamma}{2}}\int_0^R \mu^{1-2\gamma}\, d\mu\, d\tau \\
&\quad + \int_{\zeta/2R}^{\infty}(1+\tau^2)^{\frac{2-d-2\gamma}{2}}\int_0^{\zeta/2\tau}\mu^{1-2\gamma}\, d\mu\, d\tau\Big] \\
=\ & 4M[I_1+I_2].
\end{aligned}$$

The first integral is

$$\begin{aligned}
I_1 &= MR^{2-2\gamma}\int_0^{\zeta/2R}(1+\tau^2)^{\frac{2-d-2\gamma}{2}}\, d\tau \leq MR^{1-2\gamma}\zeta \\
&= MR^{1-2\gamma}\zeta^{1-\eta}\zeta^{\eta} \leq M(1+|x|)^{2-2\gamma-\eta}\,|z|^{\eta}.
\end{aligned}$$

The second integral is

$$\begin{aligned}
I_2 &= M\int_{\zeta/2R}^{\infty}(1+\tau^2)^{\frac{2-d-2\gamma}{2}}\zeta^{2-2\gamma}\tau^{2\gamma-2}\, d\tau \\
&\leq M\zeta^{\eta}\int_{\zeta/2R}^{1}\zeta^{2-2\gamma-\eta}\tau^{2\gamma-2}\, d\tau + M\zeta^{2-2\gamma}\int_1^{\infty}\tau^{-d}\, d\tau \\
&\leq M\zeta^{\eta}\int_0^1 (2R)^{2-2\gamma-\eta}\tau^{-\eta}\, d\tau + M\zeta^{\eta}R^{2-2\gamma-\eta} \\
&\leq MR^{2-2\gamma-\eta}\zeta^{\eta} \leq M\,(1+|x|)^{2-2\gamma-\eta}\,|z|^{\eta}.
\end{aligned}$$

The case $d = 1$ needs to be treated separately, since some exponents have opposite sign, so that the inequalities are reverted. In this case we assume without loss of generality that $z > 0$ and obtain

$$\begin{aligned}
&\int_{x-R}^{x+R} |f_z(x,y)|\,dy \\
= &\int_{-R}^{R} \Big| |y+z|^{1-2\gamma} - |y|^{1-2\gamma} \Big|\,dy \\
= &\pm \int_{-R}^{-z/2} \left(|y+z|^{1-2\gamma} - |y|^{1-2\gamma}\right) dy \mp \int_{-z/2}^{R} \left(|y+z|^{1-2\gamma} - |y|^{1-2\gamma}\right) dy
\end{aligned}$$

Here the sign depends on whether $1-2\gamma$ is positive or negative. We continue the estimate:

$$\begin{aligned}
&\int_{x-R}^{x+R} |f_z(x,y)|\,dy \\
= &\pm \left(\int_{-R+z}^{z/2} - \int_{-R}^{-z/2} - \int_{z/2}^{R+z} + \int_{-z/2}^{R} \right) |y|^{1-2\gamma}\,dy \\
= &\pm \left(- \int_{-R}^{-R+z} - \int_{R}^{R+z} + 2\int_{-z/2}^{z/2} \right) |y|^{1-2\gamma}\,dy \\
\leq &\int_{R-z}^{R+z} y^{1-2\gamma}\,dy + 4\int_{0}^{z/2} y^{1-2\gamma}\,dy \\
\leq &\; MzR^{1-2\gamma} + Mz^{2-2\gamma} \\
\leq &\; Mz^{\eta}R^{2-2\gamma-\eta} + Mz^{\eta}R^{2-2\gamma-\eta} \\
\leq &\; Mz^{\eta}(1+|x|)^{2-2\gamma-\eta}.
\end{aligned}$$

□

Lemma 3.7. *Suppose that the assumptions of Proposition 3.1 hold. Then there exists a constant M depending only on d, γ, ϵ, η, ϕ, such that for all $t \in \mathbb{C}$ with $|t| = 1$ and $|\arg(t)| \leq \phi$ the following estimates hold:*

$$\int_{\mathbb{R}^d} |x|^{\epsilon}\,|v_t(x)|\,dx \leq M, \tag{3.10}$$

$$\text{for all } z \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} |v_t(x+z) - v_t(x)|\,dx \leq M|z|^{\eta}, \tag{3.11}$$

$$\text{for all } \lambda \in \left(\frac{7}{8}, \frac{9}{8}\right), \quad \int_{\mathbb{R}^d} |v_t(\lambda x) - v_t(x)|\,dx \leq M|1-\lambda|^{\eta}. \tag{3.12}$$

Proof. To prove (3.10), put $f(x,y) = |x-y|^{2-2\gamma-d}$ (as in Lemma 3.4) and notice that $v_t(x) = M\int_{\mathbb{R}^d} f(x,y)\Delta u_t(y)\,dy$ with a suitable constant M. Take $0 < \alpha_1 < \alpha_2$ and $0 < \beta_1 < \beta_2$ arbitrary, $\delta = 2\gamma$, $\kappa = 2-2\gamma$. Lemma 3.4 states that f satisfies the conditions of Lemma 3.2 so that (3.10) holds with $\epsilon \in [0, 2\gamma)$.

To prove (3.11) we distinguish the cases $|z| < 1$ and $|z| \geq 1$. The case $|z| \geq 1$ is an easy consequence of (3.10) with $\epsilon = 0$:

$$\int_{\mathbb{R}^d} |v_t(x+z) - v_t(x)|\, dx \leq 2 \int_{\mathbb{R}^d} |v_t(x)|\, dx \leq 2M \leq 2M|z|^\eta.$$

For $|z| < 1$, we put $f_z(x,y) = \left[|x+z-y|^{2-2\gamma-d} - |x-y|^{2-2\gamma-d}\right]$ as in Lemma 3.5. Then $v_t(x+z) - v_t(x) = M \int_{\mathbb{R}^d} f_z(x,y) \Delta u_t(y)\, dy$. Take $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = 2$, $\beta_2 = 4$, $\delta = 1 + 2\gamma$, $\kappa = 2 - 2\gamma - \eta$, where $0 < \eta < 2 - 2\gamma$. Lemmas 3.5 and 3.6 imply that f_z satisfies the assumptions of Lemma 3.2 with $K = M_0|z|^\eta$, where M_0 is a suitable constant independent of z. Notice that the condition $|z| \leq \frac{1}{8}|x|+1$ in (3.8) and in Lemma 3.6 is trivially satisfied for any x, since $|z| \leq 1$. We apply Lemma 3.2 with $\epsilon = 0$ and obtain (3.11).

Finally let $\lambda \in (\frac{7}{8}, \frac{9}{8})$ and put $f_\lambda(x,y) = \left[|\lambda x - y|^{2-2\gamma-d} - |x-y|^{2-2\gamma-d}\right]$ so that $v_t(\lambda x) - v_t(x) = M \int_{\mathbb{R}^d} f_\lambda(x,y) \Delta u_t(y)\, dy$. Notice that for each fixed x we have $f_\lambda(x,y) = f_z(x,y)$ with $z = (\lambda - 1)x$ and f_z defined as in Lemma 3.5. The restrictions on λ imply the estimate $|z| \leq |\lambda - 1|(1 + |x|) \leq \frac{1}{8}|x| + 1$. Again, put $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = 2$, $\beta_2 = 4$, $\delta = 2\gamma$, $\kappa = 2 - 2\gamma - \eta$, and choose $0 < \eta < 2 - 2\gamma$. Lemmas 3.5 and 3.6 imply that f_λ satisfies the assumptions of Lemma 3.2 with $K = M_0|\lambda - 1|^\eta$. Again we use Lemma 3.2 with $\epsilon = 0$. Thus (3.12) holds. □

The last ingredient for the proof of Proposition 3.1 is the self-similarity of v_t:

Proposition 3.8. *Let $\mu > 0$ be some constant and $s \in \mathbb{C}$ with $\Re s > 0$. Let $v_s = (-\Delta)^\gamma u_s$ be given as in Proposition 3.1. Then for all $x \in \mathbb{R}^d$*

$$v_{\mu s}(x) = \mu^{\frac{-2\gamma-d}{2}} v_s(\mu^{-\frac{1}{2}} x) \tag{3.13}$$

Proof. We start with the self-similarity of the heat kernel:

$$\begin{aligned} u_{\mu s}(y) &= (4\pi\mu s)^{-\frac{d}{2}} e^{-|y|^2/4\mu s} \\ &= \mu^{-\frac{d}{2}} (4\pi s)^{-\frac{d}{2}} e^{-|\mu^{-1/2} y|^2/4s} \\ &= \mu^{-\frac{d}{2}} u_s(\mu^{-\frac{1}{2}} y). \end{aligned}$$

Therefore

$$[\Delta u_{\mu s}](y) = \mu^{\frac{-d-2}{2}} [\Delta u_s](\mu^{-\frac{1}{2}} y),$$

and consequently (with some given constant c and with $\xi = \mu^{-1/2} x$)

$$\begin{aligned} v_{\mu s}(x) &= c \int_{\mathbb{R}^d} |x-y|^{2-2\gamma-d} [\Delta u_{\mu s}](y)\, dy \\ &= c\mu^{\frac{-d-2}{2}} \int_{\mathbb{R}^d} |x-y|^{2-2\gamma-d} [\Delta u_s](\mu^{-\frac{1}{2}} y)\, dy \\ &= c\mu^{\frac{-d-2}{2}} \int_{\mathbb{R}^d} \mu^{\frac{2-2\gamma-d}{2}} |\xi - \eta|^{2-2\gamma-d} [\Delta u_s](\eta)\, \mu^{\frac{d}{2}}\, d\eta \\ &= \mu^{\frac{-2\gamma-d}{2}} v_s(\mu^{-\frac{1}{2}} x). \end{aligned}$$

□

Proof of Proposition 3.1, completed: Now let $t \in \mathbb{C}$ with $|\arg(t)| \le \phi$, and $t = \mu s$ with $\mu = |t|$. The estimates (3.10), (3.11), and (3.12) can be applied with s instead of t. From (3.10) we infer

$$\begin{aligned} & \int_{\mathbb{R}^d} |x|^\epsilon \, |v_t(x)| \, dx \\ = \; & \mu^{\frac{-2\gamma-d}{2}} \int_{\mathbb{R}^d} |x|^\epsilon \, |v_s(\mu^{-\frac{1}{2}} x)| \, dx \\ = \; & \mu^{\frac{\epsilon-2\gamma}{2}} \int_{\mathbb{R}^d} |\xi|^\epsilon \, |v_s(\xi)| \, d\xi \le \mu^{\frac{\epsilon-2\gamma}{2}} M, \end{aligned}$$

which proves (3.4). From (3.11) we infer

$$\begin{aligned} & \int_{\mathbb{R}^d} |v_t(x+z) - v_t(x)| \, dx \\ = \; & \mu^{\frac{-2\gamma-d}{2}} \int_{\mathbb{R}^d} |v_s(\mu^{-\frac{1}{2}} x + \mu^{-\frac{1}{2}} z) - v_s(\mu^{-\frac{1}{2}} x)| \, dx \\ = \; & \mu^{\frac{-2\gamma-d}{2}} \int_{\mathbb{R}^d} |v_s(\xi + \mu^{-\frac{1}{2}} z) - v_s(\xi)| \, \mu^{\frac{d}{2}} \, d\xi \\ \le \; & M \mu^{-\gamma} |\mu^{-\frac{1}{2}} z|^\eta \\ = \; & M \mu^{\frac{-2\gamma-\eta}{2}} |z|^\eta, \end{aligned}$$

which proves (3.5). Finally (3.12) implies

$$\begin{aligned} & \int_{\mathbb{R}^d} |v_t(\lambda x) - v_t(x)| \, dx \\ = \; & \mu^{\frac{-2\gamma-d}{2}} \int_{\mathbb{R}^d} |v_s(\lambda \mu^{-\frac{1}{2}} x) - v_s(\mu^{-\frac{1}{2}} x)| \, dx \\ = \; & \mu^{\frac{-2\gamma-d}{2}} \int_{\mathbb{R}^d} |v_s(\lambda \xi) - v_s(\xi)| \, \mu^{\frac{d}{2}} \, d\xi \\ \le \; & \mu^{-\gamma} M \, |\lambda - 1|^\eta, \end{aligned}$$

which proves (3.6). Now the proof of Proposition 3.1 is finished. □

4 Proof of Theorem 1.2

We proceed by two steps. In the following Lemmas 4.1 and 4.2 we investigate the kernel of $(-\Delta)^\gamma (s - \Delta)^{-1}$ for s with $|\arg(s)| \le \phi + \frac{\pi}{2}$, where $\phi \in (0, \frac{\pi}{2})$. For this purpose we use that $(s - \Delta)^{-1}$ can be obtained by integrating the heat semigroup. In the subsequent Propositions 4.3 and 4.4, the properties of the convolution kernel of $(-\Delta)^\gamma S_{\alpha,\beta}(t)$ are discussed. These will be derived from the properties of $(-\Delta)^\gamma (s - \Delta)^{-1}$ utilizing the contour integral (1.9). Finally we will use these results to apply Theorem 1.5, where ψ is the convolution kernel of the operator $(-\Delta)^\gamma S_{\alpha,\beta}(1)$.

Lemma 4.1. *For $t \in \mathbb{C}$ with $\Re(t) > 0$ we denote by u_t the heat kernel in $\mathbb{R}^d$ as in (3.1). Let $\gamma \in (0,1)$, $\phi \in (0, \frac{\pi}{2})$, $\epsilon \in [0, 2\gamma)$, $\eta \in (0, 2-2\gamma) \cap (0,1)$. For $s \in \mathbb{C}$ with $\arg s = \omega$, $|\omega| \le 2\phi$, we put*

$$w_s = e^{-i\omega/2} \int_0^\infty e^{-|s| e^{i(\omega/2)} t} \, [(-\Delta)^\gamma u_{te^{-i\omega/2}}] \, dt. \tag{4.1}$$

Then the integral in (4.1) exists as a Bochner integral in $\mathbf{L}^1(\mathbb{R}^d)$. Moreover, there exists a constant M depending only on d, γ, ϕ, ϵ, η, such that the following estimates hold:

$$\int_{\mathbb{R}^d} |x|^\epsilon \, |w_s(x)| \, dx \le M \, |s|^{-1+\gamma-\frac{\epsilon}{2}}, \tag{4.2}$$

$$\text{for all } z \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} |w_s(x+z) - w_s(x)| \, dx \le M \, |s|^{-1+\frac{\eta}{2}+\gamma} \, |z|^\eta, \tag{4.3}$$

$$\text{for all } \lambda \in \left(\frac{7}{8}, \frac{9}{8}\right), \quad \int_{\mathbb{R}^d} |w_s(\lambda x) - w_s(x)| \, dx \le M \, |s|^{\gamma-1} \, |\lambda - 1|^\eta. \tag{4.4}$$

Moreover, for all $p \in [1, \infty)$ and all $h \in \mathbf{L}^p(\mathbb{R}^d, H)$ we have

$$[(-\Delta)^\gamma (s - \Delta)^{-1} h](x) = \int_{\mathbb{R}^d} w_s(x-y) h(y) \, dy. \tag{4.5}$$

Proof. For shorthand we write $\nu = e^{-i\omega/2}$, so that $s = |s|\nu^{-2}$ and

$$w_s = \nu \int_0^\infty e^{-|s|t/\nu} v_{\nu t} \, dt$$

where we use the notation of Proposition 3.1:

$$v_{\nu t} = (-\Delta)^\gamma u_{\nu t}.$$

Of course, $|\nu| = 1$ and $|\arg(\nu)| \le \phi$. We start by proving (4.2). The same estimate (for the special case $\epsilon = 0$) will also prove that the integral in (4.1) converges as an integral in $\mathbf{L}^1(\mathbb{R}^d)$. Using (3.4) we obtain with a suitable constant M

$$\int_{\mathbb{R}^d} |x|^\epsilon \, |w_s(x)| \, dx \le \int_0^\infty \int_{\mathbb{R}^d} |e^{-|s|t/\nu}| \, |x|^\epsilon \, |v_{\nu t}(x)| \, dx \, dt$$
$$\le \; M \int_0^\infty e^{-|s|\cos(\phi) t} \, |t|^{\frac{\epsilon}{2}-\gamma} \, dt = M \, |s|^{\gamma - \frac{\epsilon}{2} - 1} \int_0^\infty e^{-\tau \cos(\phi)} \, \tau^{\frac{\epsilon}{2}-\gamma} \, d\tau.$$

This proves (4.2).

Equations (4.3) and (4.4) are derived similarly from (3.5) and (3.6), respectively.

Now notice that $t \mapsto T(\nu t)$ is a bounded analytic semigroup generated by $\nu\Delta$. Therefore,

$$(s - \Delta)^{-1} h = \nu(|s|\nu^{-1} - \nu\Delta)^{-1} h = \nu \int_0^\infty e^{-|s|t/\nu} T(\nu t) h \, dt.$$

Since $\gamma < 1$ and $T(\nu t)$ is a bounded analytic semigroup, we have that

$$\int_0^\infty |e^{-|s|t/\nu}| \, \|(-\Delta)^\gamma T(\nu t)\| \, dt < \infty.$$

Using the closedness of $(-\Delta)^\gamma$, we see therefore that

$$(-\Delta)^\gamma (s-\Delta)^{-1} h = \nu \int_0^\infty e^{-|s|t/\nu} (-\Delta)^\gamma T(\nu t) h \, dt$$

as a Bochner integral in $\mathbf{L}^p(\mathbb{R}^n, H)$. On the other hand the heat semigroup is given by convolution

$$\nu e^{-|s|t/\nu} (-\Delta)^\gamma T(\nu t) h = \nu e^{-|s|t/\nu} v_{\nu t} * h.$$

We have shown that

$$w_s = \nu \int_0^\infty e^{-|s|t/\nu} v_{\nu t} \, dt$$

as a Bochner integral in $\mathbf{L}^1(\mathbb{R}^d)$, thus

$$w_s * h = \nu \int_0^\infty e^{-|s|t/\nu} v_{\nu t} * h \, dt = (-\Delta)^\gamma (s-\Delta)^{-1} h,$$

where the integral is a Bochner integral in $\mathbf{L}^p(\mathbb{R}^d, H)$, and (4.5) holds. □

Lemma 4.2. *For $s \in \mathbb{C}$ with $s \notin (-\infty, 0]$ and $\gamma \in (0,1)$, let w_s be the kernel of $(-\Delta)^\gamma (s-\Delta)^{-1}$ as in Lemma 4.1. Let $\mu > 0$ be some constant. Then for all $x \in \mathbb{R}^d$,*

$$w_{\mu s}(x) = \mu^{\frac{2\gamma+d-2}{2}} w_s(\mu^{\frac{1}{2}} x). \tag{4.6}$$

Proof. As in the proof of Lemma 4.1, let $\arg(s) = \omega$ and $\nu = e^{-i\omega/2}$. Using Proposition 3.8 and the definition of w_s, we have

$$\begin{aligned} w_{\mu s}(x) &= \nu \int_0^\infty e^{-\mu|s|t/\nu} v_{\nu t}(x) \, dt \\ &= \mu^{-1} \nu \int_0^\infty e^{-|s|\tau/\nu} v_{\nu \mu^{-1} \tau}(x) \, d\tau \\ &= \mu^{-1+\frac{2\gamma+d}{2}} \nu \int_0^\infty e^{-|s|\tau/\nu} v_{\nu\tau}(\mu^{\frac{1}{2}} x) \, d\tau \\ &= \mu^{\frac{2\gamma+d-2}{2}} w_s(\mu^{\frac{1}{2}} x). \end{aligned}$$

□

Proposition 4.3. *Let $\alpha \in (0,2)$, $\beta \geq 0$, $\gamma \in (0,1)$ be such that $1+\alpha\gamma > \beta$. Let $\epsilon \in [0, 2\frac{\alpha\gamma-\beta+1}{\alpha})$, $\eta \in (0, 2-2\gamma) \cap (0,1)$. As in Lemma 4.1, u_t denotes the heat kernel in $\mathbb{R}^d$, and w_s is given by (4.1). Let $\rho \in (\frac{\pi}{2}, \pi)$ be such that $\alpha\rho < \pi$. We define the contour $C = \{(\tau, s(\tau)) \mid \tau \in (-\infty, \infty)\}$ by*

$$s(\tau) = \begin{cases} -e^{-i\rho}\tau & \text{if } \tau \leq 0, \\ e^{i\rho}\tau & \text{if } \tau > 0. \end{cases}$$

Then for each $t > 0$, the following integral exists as a Bochner integral in $\mathbf{L}^1(\mathbb{R}^d)$:

$$\psi_t = \frac{1}{2\pi i} \int_C e^{st} s^{\alpha-\beta} w_{s^\alpha}\, ds. \tag{4.7}$$

There exists a constant M depending only on d, α, β, γ, ϵ, η, such that

$$\int_{\mathbb{R}^d} |x|^\epsilon\, |\psi_t(x)|\, dx \leq M\, |t|^{\beta-\alpha\gamma+\frac{\alpha\epsilon}{2}-1}, \tag{4.8}$$

$$\text{for all } z \in \mathbb{R}^d \quad \int_{\mathbb{R}^d} |\psi_t(x+z) - \psi_t(x)|\, dx \leq M\, |t|^{\beta-\alpha\gamma-\frac{\alpha\eta}{2}-1} |z|^\eta, \tag{4.9}$$

$$\text{for all } \lambda \in (\frac{7}{8}, \frac{9}{8}), \quad \int_{\mathbb{R}^d} |\psi_t(\lambda x) - \psi_t(x)|\, dx \leq M\, |t|^{\beta-\alpha\gamma-1}\, |\lambda - 1|^\eta. \tag{4.10}$$

Moreover, let H be a separable Hilbert space, $p \in [1,\infty)$. Let $S_{\alpha,\beta}$ be the resolvent operator to (1.6) and $h \in \mathbf{L}^p(\mathbb{R}^d, H)$. Then

$$[(-\Delta)^\gamma S_{\alpha,\beta}(t)h](x) = \int_{\mathbb{R}^d} \psi_t(x-y) h(y)\, dy. \tag{4.11}$$

Proof. We start with proving (4.8). With $\epsilon = 0$, our computations will also imply that the integral in (4.7) converges in $\mathbf{L}^1(\mathbb{R}^d)$. Let M be the constant from (4.2) in Lemma 4.1.

$$\begin{aligned} & \int_C |e^{st} s^{\alpha-\beta}| \int_{\mathbb{R}^d} |x|^\epsilon\, |w_{s^\alpha}(x)|\, dx\, |ds| \\ \leq\ & 2 \int_0^\infty e^{\cos(\rho)\sigma t} \sigma^{\alpha-\beta} M\, \sigma^{\alpha(-1+\gamma-\frac{\epsilon}{2})}\, d\sigma \\ =\ & 2M \int_0^\infty e^{\cos(\rho)\sigma t} \sigma^{\alpha\gamma-\beta-\frac{\alpha\epsilon}{2}}\, d\sigma \\ =\ & 2M t^{-\alpha\gamma+\beta+\frac{\alpha\epsilon}{2}-1} \int_0^\infty e^{\cos(\rho)\tau} \tau^{\alpha\gamma-\beta-\frac{\alpha\epsilon}{2}}\, d\tau. \end{aligned}$$

The last integral is a finite constant, since $\alpha\gamma - \beta - \frac{\alpha\epsilon}{2} > -1$ and $\cos(\rho) < 0$. Using the same technique and Equations (4.3) and (4.4), respectively, we obtain (4.9) and (4.10).

Now we use the contour integration formula (1.9). Since Δ generates an analytic semigroup, we know that

$$\|s^{-\beta+\alpha}(-\Delta)^\gamma (s^\alpha - \Delta)^{-1}\| \leq M|s|^{-\beta+\alpha}|s^\alpha|^{-1+\gamma} = |s|^{-\beta+\gamma\alpha}. \tag{4.12}$$

Since $-\beta+\gamma\alpha>-1$, we see easily that the integral

$$\frac{1}{2\pi i}\int_{\tilde{C}} e^{st}s^{-\beta+\alpha}(-\Delta)^{\gamma}(s^{\alpha}-\Delta)^{-1}\,ds$$

converges even in operator norm, and using the closedness of $(-\Delta)^{\gamma}$, we see that

$$\begin{aligned}(-\Delta)^{\gamma}S_{\alpha,\beta}(t)h &= \frac{1}{2\pi i}(-\Delta)^{\gamma}\int_{\tilde{C}} e^{st}s^{-\beta+\alpha}(s^{\alpha}-\Delta)^{-1}h\,ds\\ &= \frac{1}{2\pi i}\int_{\tilde{C}} e^{st}s^{-\beta+\alpha}(-\Delta)^{\gamma}(s^{\alpha}-\Delta)^{-1}h\,ds\\ &= \frac{1}{2\pi i}\int_{C} e^{st}s^{-\beta+\alpha}(-\Delta)^{\gamma}(s^{\alpha}-\Delta)^{-1}h\,ds.\end{aligned}$$

In fact, the contour $\tilde{C}$ can be replaced by the contour C (with radius $r=0$) because of (4.12) and Cauchy's Theorem. On the other hand, by (4.5) we know that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C} e^{st}s^{-\beta+\alpha}[(-\Delta)^{\gamma}(s^{\alpha}-\Delta)^{-1}h](x)\,ds\\ =\;& \frac{1}{2\pi i}\int_{C} e^{st}s^{-\beta+\alpha}\int_{\mathbb{R}^d} w_{s^{\alpha}}(x-y)h(y)\,dy\\ =\;& \int_{\mathbb{R}^d}\psi_t(x-y)h(y)\,dy.\end{aligned}$$

Here we have used that the integral in (4.7) converges in $\mathbf{L}^1(\mathbb{R}^d)$. This proves (4.11). □

Proposition 4.4. *Let α, β, γ be as in Proposition 4.3, and let ψ_t be the kernel of $(-\Delta)^{\gamma}S_{\alpha,\beta}(t)$ as defined by (4.7). Let $\mu>0$. Then for all $x\in\mathbb{R}^d$, $t>0$, we have*

$$\psi_{\mu t}(x)=\mu^{\beta-\gamma\alpha-\frac{\alpha d}{2}-1}\psi_t(\mu^{-\frac{\alpha}{2}}x). \tag{4.13}$$

Proof. We use Lemma 4.2 and some elementary computations:

$$\begin{aligned}\psi_{\mu t}(x) &= \frac{1}{2\pi i}\int_C e^{s\mu t}s^{\alpha-\beta}w_{s^{\alpha}}(x)\,ds\\ &= \frac{1}{2\pi i}\int_C e^{\sigma t}\left(\frac{\sigma}{\mu}\right)^{\alpha-\beta} w_{(\sigma/\mu)^{\alpha}}(x)\,\frac{1}{\mu}\,d\sigma\\ &= \frac{1}{2\pi i}\mu^{\beta-\alpha-1}\int_C e^{\sigma t}\sigma^{\alpha-\beta}\mu^{-\alpha\frac{2\gamma+d-2}{2}}w_{\sigma^{\alpha}}(\mu^{-\frac{\alpha}{2}}x)\,d\sigma\\ &= \frac{1}{2\pi i}\mu^{\beta-\gamma\alpha-\frac{d\alpha}{2}-1}\int_C e^{\sigma t}\sigma^{\alpha-\beta}w_{\sigma^{\alpha}}(\mu^{-\frac{\alpha}{2}}x)\,d\sigma\\ &= \mu^{\beta-\gamma\alpha-\frac{d\alpha}{2}-1}\psi_t(\mu^{-\frac{\alpha}{2}}x).\end{aligned}$$

□

Proof of Theorem 1.2, completed: Now let $\beta - \alpha\gamma = \frac{1}{2}$. We put $\psi = \psi_1$ as defined in Proposition 4.3. By (4.11) and Proposition 4.4 we have

$$\begin{aligned}
& [(-\Delta)^\gamma S_{\alpha,\beta}(t-s)g(s,\cdot)](x) \\
= & \int_{\mathbb{R}^d} \psi_{t-s}(x-y)g(s,y)\,dy \\
= & \int_{\mathbb{R}^d} (t-s)^{\beta-\gamma\alpha-\frac{\alpha d}{2}-1}\psi_1((t-s)^{-\frac{\alpha}{2}}(x-y))g(s,y)\,dy \\
= & \int_{\mathbb{R}^d} (t-s)^{-\frac{1}{2}-\frac{\alpha d}{2}}\psi((t-s)^{-\frac{\alpha}{2}}(x-y))g(s,y)\,dy.
\end{aligned}$$

By Proposition 4.3, the function ψ satisfies all requirements of Theorem 1.5. Putting

$$[Pg](t,x) = \left[\int_{-\infty}^{t} \left\|[(-\Delta)^\gamma S_{\alpha,\beta}(t-s)g(s,\cdot)](x)\right\|_H^2 ds\right]^{\frac{1}{2}}$$

we obtain Theorem 1.2 as a corollary of Theorem 1.5. □

5 Proof of Corollary 1.3

Now let g be such that the function $e^{-\epsilon t}g(t,x)$ is in $\mathbf{L}^p((-\infty,b]\times\mathbb{R}^n,H)$ with some $\epsilon > 0$. Let $\alpha \in (0,2)$, $\theta > \frac{1}{2}$, $\gamma \in (0,1)$ be such that $\theta - \alpha\gamma > \frac{1}{2}$. Put $\beta = \frac{1}{2} + \alpha\gamma$ and $\mu = \theta - \beta > 0$. We want to find an estimate

$$\begin{aligned}
& \int_{\mathbb{R}^d}\int_{-\infty}^{b} \left[\int_{-\infty}^{t} \left\|e^{-\epsilon t}[(-\Delta)^\gamma S_{\alpha,\beta+\mu}(t-s)g(s,\cdot)](x)\right\|_H^2 ds\right]^{\frac{p}{2}} dt\,dx \\
\leq\ & M \int_{\mathbb{R}^d}\int_{-\infty}^{b} \left\|e^{-\epsilon s}g(s,y)\right\|_H^p ds\,dy.
\end{aligned}$$

Of course it is sufficient to find a proof for $b < \infty$, provided the constant M is independent of b. A trivial limiting argument settles then the case $b = \infty$.

For shorthand we write

$$\begin{aligned}
g_1(t,x) &= e^{-\epsilon t}g(t,x), \\
W(t,s,x) &= \left\|[(-\Delta)^\gamma S_{\alpha,\beta}(t)g_1(s,.)]\,(x)\right\|_H, \\
V(t,x)^2 &= \int_{-\infty}^{t} \left\|[(-\Delta)^\gamma S_{\alpha,\beta}(t-s)g_1(s,.)]\,(x)\right\|_H^2 ds, \\
U(t,x)^2 &= \int_{-\infty}^{t} \left\|[(-\Delta)^\gamma S_{\alpha,\beta+\mu}(t-s)e^{\epsilon s}g_1(s,.)]\,(x)\right\|_H^2 ds.
\end{aligned}$$

With this notation, Theorem 1.2 (applied to g_1 instead of g) says that for some constant M we have

$$\int_{\mathbb{R}^d}\int_{-\infty}^{b} V^p(t,x)\,dt\,dx \leq M \int_{\mathbb{R}^d}\int_{-\infty}^{b} \|g_1(s,y)\|_H^p\,ds\,dy. \tag{5.1}$$

Corollary 1.3 is proved if we can show that

$$\int_{\mathbb{R}^d}\int_{-\infty}^{b}[e^{-\epsilon t}U(t,x)]^p\,dt\,dx \le M\epsilon^{-\mu p}\int_{\mathbb{R}^d}\int_{-\infty}^{b}\|g_1(s,y)\|_H^p\,ds\,dy. \tag{5.2}$$

We prove first

$$U(t,x) \le e^{\epsilon t}\int_0^\infty \frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{-\epsilon\sigma}V(t-\sigma,x)\,d\sigma. \tag{5.3}$$

In fact, from (1.8) we obtain

$$S_{\alpha,\beta+\mu}(t)h = \int_0^t \frac{\sigma^{\mu-1}}{\Gamma(\mu)}S_{\alpha,\beta}(t-\sigma)h\,d\sigma.$$

Now,

$$U(t,x) = \sup\int_{-\infty}^{t} f(s)\left\|\left[(-\Delta)^\gamma S_{\alpha,\beta+\mu}(t-s)e^{\epsilon s}g_1(s,.)\right](x)\right\|_H\,ds$$

where the supremum is taken over all $f\in\mathbf{L}^2((-\infty,t],\mathbb{R})$ such that

$$\int_{-\infty}^{t} f(s)^2\,ds = 1.$$

For such f, we estimate

$$\begin{aligned}
&\int_{-\infty}^{t} f(s)\left\|\left[(-\Delta)^\gamma S_{\alpha,\beta+\mu}(t-s)e^{\epsilon s}g_1(s,.)\right](x)\right\|_H\,ds\\
\le\ &\int_{-\infty}^{t} f(s)\int_0^{t-s}\frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{\epsilon s}W(t-s-\sigma,s,x)\,d\sigma\,ds\\
\le\ &\int_0^\infty\frac{\sigma^{\mu-1}}{\Gamma(\mu)}\int_{-\infty}^{t-\sigma} f(s)e^{\epsilon s}W(t-s-\sigma,s,x)\,ds\,d\sigma\\
\le\ &\int_0^\infty\frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{\epsilon(t-\sigma)}\left[\int_{-\infty}^{t-\sigma} f(s)^2\,ds\right]^{1/2}\left[\int_{-\infty}^{t-\sigma} W(t-s-\sigma,s,x)^2\,ds\right]^{1/2}d\sigma\\
\le\ & e^{\epsilon t}\int_0^\infty\frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{-\epsilon\sigma}V(t-\sigma,x)\,d\sigma.
\end{aligned}$$

This proves (5.3).

Next we prove

$$\int_{-\infty}^{b}\left[e^{-\epsilon t}U(t,x)\right]^p\,dt \le \epsilon^{-\mu p}\int_{\infty}^{b} V(t,x)^p\,dt. \tag{5.4}$$

We utilize the same trick as above: Let $\frac{1}{p}+\frac{1}{q}=1$ and $f\in\mathbf{L}^q((-\infty,b],\mathbb{R})$ such that $f(t)\ge 0$ and

$$\int_{-\infty}^{b} f(t)^q\,dt = 1.$$

Then, using (5.3) and Hölder's inequality we have

$$\begin{aligned}
&\int_{-\infty}^{b} f(t)e^{-\epsilon t}U(t,x)\,dt \\
\leq\ &\int_{-\infty}^{b} f(t)\int_0^\infty \frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{-\epsilon\sigma}V(t-\sigma,x)\,d\sigma\,dt \\
=\ &\int_0^\infty \frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{-\epsilon\sigma}\int_{-\infty}^{b} f(t)V(t-\sigma,x)\,dt\,d\sigma \\
\leq\ &\int_0^\infty \frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{-\epsilon\sigma}\left[\int_{-\infty}^{b} f(t)^q\,dt\right]^{1/q}\left[\int_{-\infty}^{b-\sigma} V(t,x)^p\,dt\right]^{1/p} d\sigma \\
\leq\ &\int_0^\infty \frac{\sigma^{\mu-1}}{\Gamma(\mu)}e^{-\epsilon\sigma}\,d\sigma\ \left[\int_{-\infty}^{b} V(t,x)^p\,dt\right]^{1/p} \\
=\ &\epsilon^{-\mu}\left[\int_{-\infty}^{b} V(t,x)^p\,dt\right]^{1/p}.
\end{aligned}$$

Finally, integrating (5.4) over $\mathbb{R}^d$ and using (5.1) we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{-\infty}^{b}\left[e^{-\epsilon t}U(t,x)\right]^p\,dt\,dx \\
\leq\ &\epsilon^{-\mu p}\int_{\mathbb{R}^d}\int_{-\infty}^{b}[V(t,x)]^p\,dt\,dx \\
\leq\ &\epsilon^{-\mu p}M\int_{\mathbb{R}^d}\int_{-\infty}^{b}\|g_1(s,y)\|_H^p\,ds\,dy.
\end{aligned}$$

This is (5.2), and the proof of Corollary 1.3 is finished. □

References

[1] N. Burger, Espaces des fonctions à variation bornée sur un espace de nature homogène, C. R. Acad. Sc. Paris, Sèrie A, 286 (1978), 139–142.

[2] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, Lecture Notes in Mathematics 242, Springer, Berlin, Heidelberg, New York 1971.

[3] W. Desch and S. O. Londen, On a stochastic parabolic integral equation, in *Functional Analysis and Evolution Equations. The Günter Lumer Volume*, H. Amann, W. Arendt, M. Hieber, F. Neubrander, S. Nicaise, J. von Below, eds., Birkhäuser, Basel 2007, 157–169.

[4] J. Duoandikoetxea, *Fourier Analysis*, Graduate Studies in Mathematics, American Mathematical Society, Providence R. I. 2001.

[5] C. Jost, Transformation formulas for fractional Brownian motion, Stochatic Processes and their Applications 116 (2006), 1341–1357.

[6] N. V. Krylov, A parabolic Littlewood-Paley inequality with applications to parabolic equations, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center 4 (1994), 355–364.

[7] N. V. Krylov, An analytic approach to SPDEs, in *Stochastic Partial Differential Equations: Six Perspectives*, R. A. Carmona and B. Rozovskii, eds., A.M.S. Mathematical Surveys and Monographs vol. 64 (1999), 185–242.

[8] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel 1993.

[9] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton N. J., 1970.

(continued from the back cover)

A541 Teemu Lukkari
Nonlinear potential theory of elliptic equations with nonstandard growth
February 2008

A540 Riikka Korte
Geometric properties of metric measure spaces and Sobolev-type inequalities
January 2008

A539 Aly A. El-Sabbagh, F.A. Abd El Salam, K. El Nagaar
On the Spectrum of the Symmetric Relations for The Canonical Systems of Differential Equations in Hilbert Space
December 2007

A538 Aly A. El-Sabbagh, F.A. Abd El Salam, K. El Nagaar
On the Existence of the selfadjoint Extension of the Symmetric Relation in Hilbert Space
December 2007

A537 Teijo Arponen, Samuli Piipponen, Jukka Tuomela
Kinematic analysis of Bricard's mechanism
November 2007

A536 Toni Lassila
Optimal damping set of a membrane and topology discovering shape optimization
November 2007

A535 Esko Valkeila
On the approximation of geometric fractional Brownian motion
October 2007

A534 Jarkko Niiranen
A priori and a posteriori error analysis of finite element methods for plate models
October 2007

A533 Heikki J. Tikanmäki
Edgeworth expansion for the one dimensional distribution of a Lévy process
September 2007

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The reports are available at *http://math.tkk.fi/reports/* .
The list of reports is continued inside the back cover.

A546 Olavi Nevanlinna
Resolvent and polynomial numerical hull
May 2008

A545 Ruth Kaila
The integrated volatility implied by option prices, a Bayesian approach
April 2008

A544 Stig-Olof Londen, Hana Petzeltová
Convergence of solutions of a non-local phase-field system
March 2008

A543 Outi Elina Maasalo
Self-improving phenomena in the calculus of variations on metric spaces
February 2008

A542 Vladimir M. Miklyukov, Antti Rasila, Matti Vuorinen
Stagnation zones for A-harmonic functions on canonical domains
February 2008

ISBN 978-951-22-9441-1 (print)
ISBN 978-951-22-9442-8 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)