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Geometric properties of metric measure spaces and Sobolev-type inequalities

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Riikka Korte: *Geometric properties of metric measure spaces and Sobolev-type inequalities*; Helsinki University of Technology, Institute of Mathematics, Research Reports A540 (2008).

Abstract: *This dissertation studies analysis in metric spaces that are equipped with a doubling measure and satisfy a Poincaré inequality. The treatise consists of four articles in which we discuss geometric implications of the Poincaré inequality as well as equivalent characterizations of capacities, Sobolev inequalities and conditions for the thickness of a boundary of sets. We also study the size of exceptional sets for Newtonian functions.*

AMS subject classifications: 46E35, 31C15.

Keywords: boxing inequality, capacity, doubling measure, functions of bounded variation, Hausdorff content, Lebesgue points, metric spaces, modulus, Newtonian spaces, Poincaré inequality, quasiconvexity, Sobolev–Poincaré inequality, Sobolev spaces

Riikka Korte: *Metristen mitta–avaruuksien geometriset ominaisuudet ja Sobolev–tyyppiset epäyhtälöt*; Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja A540 (2008).

Tiivistelmä: *Väitöskirjassa tutkitaan metrisiä avaruuksia, joissa on tuplaava mitta ja jotka kantavat Poincarén epäyhtälön. Työ koostuu neljästä artikkelista, jotka käsittelevät muun muassa Poincarén epäyhtälön geometrisia ominaisuuksia sekä yhtäpitäviä karakterisaatioita kapasiteeteille, Sobolev–tyyppisille epäyhtälöille ja joukon reunan paksuutta mittaaville ehdoille. Työssä tutkitaan myös Newtonin funktioiden epäsäännöllisten pisteiden joukon suuruutta.*

Avainsanat: boxing epäyhtälö, BV–funktiot, Hausdorff–kontentti, kapasiteetti, kvasikonveksisuus, Lebesguen pisteet, metriset avaruudet, Newtonin avaruudet, Poincarén epäyhtälö, Sobolevin avaruudet, Sobolev–Poincaré epäyhtälö, tuplaava mitta

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Working with my co–authors NAGESWARI SHANMUGALINGAM and HELI TUOMINEN has been very rewarding and enjoyable. It has also been a privilege to be a member of the *Nonlinear PDE* group. I have really appreciated the good spirit and teamwork in our research group.

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Espoo, January 2008

Riikka Korte

List of included articles

This dissertation consists of this overview and the following publications:

[I] R. Korte, *Geometric implications of the Poincaré inequality*, Results Math., 50 (2007), pp. 93–107.

[II] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen, *Lebesgue points and capacities via boxing inequality in metric spaces*, (to appear in) Indiana U. Math. J., 57 (2008).

[III] R. Korte and N. Shanmugalingam, *Equivalence and self–improvement of p–fatness and Hardy's inequality, and association with uniform perfectness.* arXiv:0709.1097v1 [math.FA].

[IV] J. Kinnunen and R. Korte, *Characterizations of Sobolev inequalities on metric spaces.* arXiv:0709.2013v1 [math.AP].

Author's contribution

The work presented in this dissertation has been mainly carried out at the Institute of Mathematics, Helsinki University of Technology, during the period 2005–2007. The writing and analysis of [I] was in part completed while the author was visiting University of Cincinnati in fall 2005.

The author has had a central role in all aspects of the work reported in this dissertation. In [I], the author's independent research is reported, while, in [II]–[IV], the author is responsible for a substantial part of writing and analysis.

Geometric properties of metric measure spaces and Sobolev-type inequalities

Riikka Korte

1 Introduction

This dissertation is about the analysis on metric measure spaces with the standard setting, where the measure is doubling and some sort of a Sobolev–Poincaré inequality is valid. More precisely, we discuss the geometry of spaces supporting a Poincaré inequality and we present equivalent characterisations of Sobolev–Poincaré inequalities. We also discuss regularity of Sobolev functions and the relationship between several conditions describing thickness of the boundary of a set in a metric space.

In this section, we give a short overview of the theory of metric spaces with a doubling measure and a Poincaré inequality, and we discuss how the theory of Sobolev functions can be extended to these kind of spaces. Sections 2–5 include an overview of papers [I]–[IV].

1.1 Doubling measure

Requiring that the measure is doubling is a common way to ensure that the space has a finite dimensional nature. The doubling condition is not strong enough to guarantee an unambiguous dimension for the space. Nevertheless, it is enough to provide many classical results of zeroth order calculus. These results include, for example, the Vitali covering theorem, Lebesgue differentiation theorem and Hardy–Littlewood maximal theorem.

Definition 1.1. We say that μ is a *doubling measure* if there exists a constant $c_D \geq 1$, called the *doubling constant of* μ, such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r)).$$

Here $B(x, r) = \{y \in X : d(x, y) < r\}$.

The quasimetric spaces with a doubling measure are also sometimes called *spaces of homogeneous type*. Roughly speaking, the doubling condition gives an upper bound for the dimension of the metric space. Indeed, if $0 < r < R < \infty$ and $x \in B(y, R)$, then

$$\frac{\mu(B(x, r))}{\mu(B(y, R))} \geq c \left(\frac{r}{R}\right)^s,$$

where $s = \log_2 c_D$ and c is a constant that depends only on c_D. By rather straightforward calculations, we can show that the Hausdorff dimension of the space cannot exceed s.

A metric space is *doubling* if there exists a constant $N < \infty$ such that every ball of radius r may be covered by N balls of radii $r/2$. Note that, if a metric space X has a doubling measure, then X is doubling, but the converse does not hold in general. However, it is possible to construct a doubling measure to every complete doubling space, see Luukkainen–Saksman [39] and Vol′berg–Konyagin [55].

The doubling condition is general enough to allow for a wide number of spaces. However, every doubling metric space can be quasisymmetrically embedded into some finite dimensional Euclidean space. The argument is attributable to Assouad [5].

Sometimes we need stronger growth restrictions than the doubling condition. For example, we may assume that there exist $0 < Q_1 \le Q_2 < \infty$ and $C > 1$ such that

$$\frac{1}{C}\left(\frac{r}{R}\right)^{Q_2} \le \frac{\mu(B(x,r))}{\mu(B(y,R))} \le C\left(\frac{r}{R}\right)^{Q_1} \tag{1.1}$$

for every $y \in X$, $0 < r < R < \infty$ and $x \in B(y,R)$. Here the lower bound is equivalent to the doubling condition. The upper bound in (1.1) gives a lower bound for the dimension of the space. By rather simple calculations, we can show that, if the space is connected, or even uniformly perfect, and the measure is doubling, then the measure satisfies (1.1) with some positive constants. However, some phenomena occur only when $Q_1 > 1$. If $Q_1 = Q_2 = Q$, we say that the space is *Ahlfors Q–regular*. Only the Ahlfors regularity condition is strong enough to allow global estimates for the size of balls. If $Q_1 < Q_2$, we only gain information about the relative size of balls that are near each other. Consider, for example, $\mathbb{R}^n$ equipped with Euclidean metric and the measure

$$\mu(E) = \int_E w(x)\,\mathrm{d}x = \int_E |x|^{\delta-n}\,\mathrm{d}x$$

with $0 < \delta < 1$. Since $w(x)$ is a Muckenhoupt A_1–weight, the measure μ is doubling and the space supports a $(1,1)$–Poincaré inequality. However, it satisfies the condition (1.1) only with $Q_1 \le \delta$ and $Q_2 \ge n$, because the measure of balls depends not only on their radius but also on their distance to the origin.

1.2 Poincaré inequality

The Poincaré inequality creates a link between the metric, the measure and the gradient, and it provides a way to pass from the infinitesimal information of the gradient to larger scales. If the integral of the gradient is small, then the function does not oscillate too much. The Poincaré inequality also guarantees the existence of a number of rectifiable paths connecting any two points in the space. The doubling condition is enough to allow zeroth order calculus, but the Poincaré inequality lets us proceed to first order calculus.

It is not obvious to generalise the concept of the gradient to general metric measure spaces, because it involves directions. Therefore, the gradient is often replaced by an upper gradient. A nonnegative Borel–measurable function g is an *upper gradient* of a function $u : X \to [-\infty, \infty]$ if for all rectifiable paths $\gamma : [a, b] \to X$,

$$|u(\gamma(b)) - u(\gamma(a))| \leq \int_\gamma g \, \mathrm{d}s.$$

In Euclidean spaces, the minimal upper gradient corresponds to the norm of the gradient. However, the upper gradient is not unique. Namely, if g is an upper gradient and h is any nonnegative Borel–measureable function, then also $g+h$ is an upper gradient. Upper gradient is a local concept in the sense that the minimal upper gradient is zero almost everywhere in the set where the function is constant. The upper gradient also has some linear nature, but is not linear itself. This can be seen by considering upper gradients of two functions. The sum of the upper gradients is an upper gradient of the sum of the functions, but the analogous result does not hold for their difference.

Definition 1.2. We say that X supports *a weak* $(1, p)$*–Poincaré inequality* if there exist constants $c_P > 0$ and $\lambda \geq 1$ such that, for all $x \in X$ and $r > 0$, for all measurable functions u on X and for all upper gradients g of u,

$$⨍_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq c_P r \left(⨍_{B(x,\lambda r)} g^p \, \mathrm{d}\mu \right)^{1/p}. \tag{1.2}$$

Here we used the notation

$$f_B = ⨍_B f \, \mathrm{d}\mu = \frac{1}{\mu(B)} \int_B f \, \mathrm{d}\mu.$$

The word *weak* in the definition above refers to the possibility that λ may be strictly greater than 1. In $\mathbb{R}^n$ we obtain $\lambda = 1$.

There are several possible definitions for the Poincaré inequality. We can vary the class of functions for which the Poincaré inequality is required to hold, and replace the upper gradient by other substitutes for the gradient. Most of the reasonable definitions coincide when the measure is doubling and the space is complete, see Keith [30, 31] and Keith–Rajala [32]. For example, instead of all measurable functions, it is enough to require inequality (1.2) for compactly supported Lipschitz functions with Lipschitz upper gradients. Or we may replace the upper gradient by the local Lipschitz constant.

It is common to all of these definitions that they are invariant under changes of coordinates in bi–Lipschitz mappings. This is a characteristic of all analysis in metric spaces, since, in general, we do not have any linear structure and thus the concepts should not depend on such a structure even if it exists in some special cases.

By Hölder inequality, it is clear that, if X supports a weak $(1, p)$–Poincaré inequality, then it supports also a weak $(1, q)$–Poincaré inequality for all $q > p$. Both sides of the Poincaré inequality also enjoy a self–improving property: if the space supports a $(1, p)$–Poincaré inequality, then it supports also an (s, p)–Poincaré inequality and, if $p > 1$, a $(1, p - \varepsilon)$–Poincaré inequality for some $s > p$ and $\varepsilon > 0$. The first one is attributable to Hajłasz–Koskela [22] and Bakry–Coulhon–Ledoux–Saloff-Coste [6], see also Heinonen [24] and Franchi–Pérez–Wheeden [15]. The latter is a recent result by Keith–Zhong [33].

There is a wide variety of spaces equipped with a doubling measure and satisfying a weak Poincaré inequality. These include Euclidean spaces with Lebesgue measure and weighted Euclidean spaces with Muckenhoupt weights, as well as graphs, complete Riemannian manifolds with nonnegative Ricci curvature, Heisenberg groups and more general Carnot–Carathéodory spaces, see [25], [52], [47] and [24].

Not only the topological structure, but also the choice of the metric, is essential for the validity of the Poincaré inequality. For example, consider the *snowflaking* identity map $(X, d) \to (X, d^\alpha)$ with $0 < \alpha < 1$. It is a quasisymmetric map, but snowflaked metric spaces (X, d^α) have no rectifiable paths and consequently cannot support a Poincaré inequality.

Although a Poincaré inequality is a widely used assumption, in many cases it is hard to check whether a certain space supports a Poincaré inequality. Some of the few results concerning sufficient conditions are by Semmes. He studied this problem first in Ahlfors regular spaces in [48] and then in more general metric spaces in [49]. The conditions restrict the behaviour of the Hausdorff measure of the space and the behaviour of its topology. Separately these assumptions permit fractal phenomena, which are incompatible with analytical results like Sobolev embedding theorems. On Riemannian manifolds, Grigor'yan and Saloff-Coste observed that the doubling condition and the Poincaré inequality are not only sufficient, but also necessary, conditions for a scale-invariant parabolic Harnack principle for the heat equation, see [46], [47] and [17].

1.3 Sobolev functions

Sobolev spaces are classically defined in $\mathbb{R}^n$ and its subsets as pth power integrable functions whose distributional derivatives are integrable to power p, $1 \le p < \infty$. For more details about classical Sobolev spaces, see, for example, Adams [1], Evans–Gariepy [11], Maz′ya [43] and Ziemer [56]. Since distributional derivatives are defined via integration by parts, another means is needed to define Sobolev functions for general metric spaces.

There are several alternative definitions available for the generalisation of

Sobolev spaces to the metric measure space setting. We use a geometric definition based on upper gradients (see [26]), which gives the standard Sobolev space in the Euclidean case with Lebesgue measure also for $p = 1$. The theory of these function spaces has been developed in Shanmugalingam [51].

Definition 1.3. Let $1 \le p < \infty$. If u is a function that is integrable to power p in X, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, \mathrm{d}\mu + \inf_g \int_X g^p \, \mathrm{d}\mu \right)^{\frac{1}{p}},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(x) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

If X does not contain any rectifiable paths, then $N^{1,p}(X)$ is just the set of L^p–integrable functions. For example, the Poincaré inequality is enough to guarantee the existence of a plenitude of rectifiable paths that is enough to guarantee the development of an interesting theory of Sobolev spaces.

Note that it follows from the definition that Newtonian functions are defined a priori quasieverywhere while Sobolev functions are defined only almost everywhere. Thus the spaces $N^{1,p}(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$ are the same in the sense that every function in $N^{1,p}(\mathbb{R}^n)$ is a quasicontinuous representative of a Sobolev function and vice versa, see [7].

Another candidate for Sobolev spaces in metric measure spaces is the *Hajłasz space* $M^{1,p}(X)$, which is defined via a pointwise definition and whose upper gradient corresponds to the maximal function of the gradient. When $p > 1$, both $M^{1,p}(\mathbb{R}^n)$ and $N^{1,p}(\mathbb{R}^n)$ coincide with the standard Sobolev space $W^{1,p}(\mathbb{R}^n)$. In general, the space $M^{1,p}(X)$ may be strictly smaller than $N^{1,p}(X)$. This is the case when $p = 1$, or when the space X does not contain any rectifiable paths. For more information about various Sobolev spaces on metric spaces, see also [9], [14], [21], [27], [37] and the references therein.

2 Geometry of metric spaces supporting a Poincaré inequality

If a complete doubling metric measure space supports a weak $(1, p)$–Poincaré inequality, then the space is *quasiconvex*; that is, there exists a constant such that every pair of points can be connected with a path whose length is at most the constant times the distance between the points, see Semmes [50], Cheeger [9], Keith [30] and [I]. Since the Poincaré inequality is invariant under bi–Lipschitz mappings, it follows that every complete doubling metric measure space supporting a $(1,p)$–Poincaré inequality can be turned into a geodesic space by a bi–Lipschitz change of the metric, i.e.

$$d_{NEW}(x,y) = \inf \operatorname{length}(\gamma_{xy}),$$

where the infimum is taken over all rectifiable paths γ_{xy} joining x and y.

In [I], we improve the quasiconvexity result. We show that, if the space supports a weak $(1,p)$–Poincaré inequality with p smaller than the lower bound of the dimension, then annuli are almost quasiconvex; that is, there exists a constant $C \geq 1$ such that, if $x, y \in B(x_0, r_0) \setminus B(x_0, r_0/2)$, then there exists a path γ whose length is at most the constant C times the distance between the points and $\gamma \subset B(x_0, Cr_0) \setminus B(x_0, r_0/C)$. This means that the paths connecting any pair of points include a set of paths that are fairly short and broadly distributed. For example, the space cannot become disconnected by removing one point.

This condition is related to a weaker condition, which is called *local linear connectivity (LLC)*, see, for example, Heinonen–Koskela [26]. A space satisfies the LLC–condition, if any two points in an annulus can be joined with a path that does not intersect a small neighbourhood of the centre and with a path that does not go too far from the annulus. For example, Björn–MacManus–Shanmugalingam [8] and Holopainen–Shanmugalingam [29] assume that the space is locally linearly connected and that it supports a $(1,p)$–Poincaré inequality. It follows from our results that the assumption of local linear connectivity can be removed in many cases.

Our result is of a quantitative nature. It is based on estimating the modulus of path families joining small neighbourhoods of a pair of points. The *p–modulus* of a family of paths Γ is defined as

$$\operatorname{mod}_p \Gamma = \inf \int_X \rho^p \, \mathrm{d}\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho : X \to [0,\infty]$ satisfying

$$\int_\gamma \rho \, \mathrm{d}s \geq 1$$

for all rectifiable paths $\gamma \in \Gamma$. The Poincaré inequality gives a lower bound for the modulus of a path family of all paths joining neighbourhoods of a pair of points. Using appropriate test functions, we are able to get upper bounds for the modulus of paths that are very long compared with r_0 or that intersect $B(x_0, r_0/C)$. A combination of these estimates gives a quantitative estimate for the modulus of relatively short paths that do not leave the annulus. The classical proof of quasiconvexity gives only one path joining any pair of points.

Using similar methods, we also obtain lower bounds for the Hausdorff content and the diameter of spheres. If p is greater than the dimension, then even a path family going through one point may have positive p–modulus. This shows that our results are optimal in the sense that any weaker Poincaré inequality is not enough to guarantee quasiconvexity of annuli or positive Hausdorff s–content for any $s > 0$.

3 Boxing inequality and Lebesgue points

Paper [II] has been written jointly with Juha Kinnunen, Nageswari Shanmugalingam and Heli Tuominen. The purpose of the paper is to study regularity of Newtonian functions $N^{1,1}(X)$ and different characterisations of 1–capacity.

Our proofs are based on the theory of functions of bounded variation. The functions of bounded variation are studied in Miranda [45] and Ambrosio–Miranda–Pallara [3] in the setting of doubling metric measure spaces supporting a weak $(1,1)$–Poincaré inequality. See also Giusti [16], Ambrosio–Franchi–Pallara [2] and Evans–Gariepy [11] for the classical theory of functions of bounded variation. Following [45], for $u \in L^1_{loc}(X)$, we define the total variation of u in X

$$\|Du\|(X) = \inf \left\{ \liminf_{i\to\infty} \int_X \operatorname{Lip} u_i \, d\mu \right\},$$

where the infimum is taken over all sequences $\{u_i\}$ of locally Lipschitz functions such that $u_i \to u$ in $L^1_{loc}(X)$ and

$$\operatorname{Lip} u(x) = \limsup_{y\to x} \frac{|u(x) - u(y)|}{d(x,y)}$$

is the local Lipschitz constant of u. We say that a function $u \in L^1(X)$ is of *bounded variation*, $u \in BV(X)$, if $\|Du\|(X) < \infty$. From Theorem 3.4 in [45], we have that $\|Du\|$ is a Borel regular measure restricted to the open sets of X with finite mass whenever $u \in BV(X)$. We also have the *coarea formula*, i.e.

$$\|Du\|(A) = \int_{-\infty}^{\infty} \|D\chi_{\{u>\lambda\}}\|(A) \, d\lambda$$

for every $u \in BV(X)$ and open subset A of X.

One other important tool for us is a metric space version of a so-called boxing inequality. The boxing inequality originates with Gustin [18]. In the Euclidean case, it states that every compact set $K \subset \mathbb{R}^n$ can be covered by balls (or boxes) $B(x_i, r_i)$, $i = 1, 2, \ldots$, in such a way that

$$\sum_{i=1}^{\infty} r_i^{n-1} \le c H^{n-1}(\partial K)$$

where the constant c depends only on the dimension n. Here H^{n-1} refers to the $(n-1)$–dimensional Hausdorff measure. Because doubling metric spaces do not usually have an unambiguous dimension, we need to have a version of the boxing inequality that does not depend on the dimension of the space. It is obtained by replacing r_i^{n-1} by $\mu(B(x_i, r_i))/r_i$ and the Hausdorff measure of ∂K by the perimeter measure of K. We prove this metric space version of boxing inequality by using a Calderón–Zygmund-type covering and the relative isoperimetric inequality.

Then we use the metric space version of the boxing inequality and the theory of BV–functions to prove that the Hausdorff content of codimension one and 1–capacity are quantitatively equivalent. The coarea formula and the fact that the BV norm defines a measure are essential in our arguments.

Exceptional sets for Sobolev functions are measured in terms of the p–capacity, since Sobolev functions have a pointwise representative that has Lebesgue points outside a set of p–capacity zero. In [II], we extend this result to cover the case $p = 1$ in metric spaces. The classical proof is based on the Besicovitch covering theorem, extension results or representation formulas, see Evans–Gariepy [11] and Federer–Ziemer [13]. However, we have none of these tools available. For $p > 1$, the result was extended to cover metric measure spaces by Kinnunen–Latvala [35], but the same method does not work when $p = 1$, because it requires that the Hardy–Littlewood maximal operator is bounded in $L^p(X)$, which is not true for $p = 1$. Our proof is based on a capacitary weak-type estimate

$$\operatorname{cap}_1(\{x \in X : Mu(x) > \lambda\}) \leq \frac{c}{\lambda}\|Du\|(X).$$

The estimate is proved by first using a standard 5–covering theorem for the balls B with $u_B > \lambda$ and then considering separately the balls where u is approximately λ in a large part of the ball and the balls where u is small compared to λ in a large part of the ball. Boxing inequality and the coarea formula give an upper bound for the capacity of the union of the first type of balls, and Poincaré inequality implies that the capacity of the second kind of balls is bounded by the total variation of u from above.

The fact that the 1–capacity and the Hausdorff content of codimension one are quantitatively equivalent plays an essential role in [II], but there are also other equivalent ways to define 1–capacity. One purpose of our paper is to study these characterisations in the spirit of Federer–Ziemer [13].

One characterisation of 1–capacity is through Frostman's lemma, which states that, for every open set $U \subset X$, there exists a Radon measure ν_U such that the restricted fractional maximal operator

$$M^R \nu_U(x) = \sup_{0<r<R} r \frac{\nu_U(B(x,r))}{\mu(B(x,r))}$$

is bounded by 1 from above and $\mathcal{H}^h_{10R}(U) \leq c\, \nu_U(U)$. It follows that

$$\sup\{\nu(U) : \|M^\infty \nu\|_{L^\infty(X)} \leq 1\}$$

is quantitatively equivalent to $\operatorname{cap}_1(U)$ for all open sets $U \subset X$. More information about Frostman's lemma can be found in Mattila [41] and a metric space version is proved in Malý [40].

For compact sets, a characterisation via BV–functions and continuous BV–functions are also equivalent to the standard variational 1–capacity. We define

$$\mathrm{cap}_{BV}(K) = \inf \|Du\|(X),$$

where the infimum is taken over all compactly supported functions $u \in BV(X)$ such that $u \geq 1$ in a neighbourhood of K. The definition of $\mathrm{cap}_{CBV}(K)$ with continuous BV–functions is analogous. From the coarea formula, it follows that the infimum of perimeters of sets containing K,

$$\inf\{P(U, X) \,:\, K \subset U,\, U \text{ is open},\, \mu(U) < \infty\},$$

also gives essentially the same quantity.

If the measure of balls grows too slowly, then the capacity of all compact sets is zero. We say that the space is *p–hyperbolic* if there exists a compact set $K \subset X$ so that $\mathrm{cap}_p(K)$ is positive, see Holopainen [28] for more details. If the space is not p–hyperbolic, it is said to be *p–parabolic*. It turns out that X is 1–hyperbolic if and only if the volume $\mu(B(x_0, R))$ grows at least as fast as R. In this case, the Hausdorff content and Hausdorff measure of codimension one as well as the 1–capacity have the same null sets.

4 Size of the boundary

Paper [III] has been written jointly with Nageswari Shanmugalingam, and it is about the relationship between uniform p–fatness, Hardy's inequality and uniform perfectness, and self–improving phenomena related to them. Rather surprisingly, these analytic, metric and geometric conditions turn out to be equivalent in certain situations.

A set $E \subset X$ is said to be *uniformly p–fat* if there exists a constant $c_0 > 0$ so that, for every point $x \in E$ and for all $0 < r < \infty$,

$$\frac{\mathrm{cap}_p(B(x,r) \cap E, B(x,2r))}{\mathrm{cap}_p(B(x,r), B(x,2r))} \geq c_0. \tag{4.1}$$

Uniform p–fatness is a self–improving phenomenon: if a set is uniformly p–fat, then there exists $q < p$ such that it is also uniformly q–fat. For the proof, we refer to Björn–MacManus–Shanmugalingam [8]. This condition is stronger than the Wiener criterion and it is a capacitary version of a well–known uniform measure thickness condition that has been studied in the metric space setting in, for example, [34].

The set $\Omega \subset X$ satisfies *p–Hardy's inequality*, $1 < p < \infty$, if there exists $0 < c_H < \infty$ such that for all $u \in N_0^{1,p}(\Omega)$,

$$\int_\Omega \left(\frac{|u(x)|}{\mathrm{dist}(x, X \setminus \Omega)} \right)^p \mathrm{d}\mu(x) \leq c_H \int_\Omega g_u(x)^p \,\mathrm{d}\mu(x). \tag{4.2}$$

For more information about Hardy's inequality, see, for example, Davies [10], Hajłasz [19], Lewis [38], and Tidblom [54]. Hardy's inequality provides one way to characterise Sobolev functions with zero boundary values in open sets Ω whose complement is uniformly p–fat. More precisely, Hardy's inequality holds for a function $u \in N^{1,p}(X)$ if and only if $u \in N_0^{1,p}(\Omega)$, see Kinnunen–Martio [36].

Uniform p–fatness of the complement implies p–Hardy's inequality for all $1 \leq p \leq Q$, see Björn–MacManus–Shanmugalingam [8]. Considering a punctured ball

$$\Omega = B(0,1) \setminus \{0\}$$

in $\mathbb{R}^n$ shows that the converse does not hold in general, because Ω satisfies p–Hardy's inequality both when $p > n$ and when $p < n$, but the complement is uniformly p–fat only for $p > n$.

However, when p equals the dimension of the space, p–Hardy's inequality implies uniform p–fatness of the complement. This was first proved by Ancona [4] in the Euclidean plane and later generalised to higher dimensional Euclidean spaces by Lewis [38]. We present in [III] a rather transparent proof for the fact that this result holds also in Ahlfors regular metric spaces.

The proof consists of the following steps. We first observe that p–Hardy's inequality implies uniform perfectness of the complement. A set $E \subset X$ is uniformly perfect with constant $c_{UP} \geq 1$, if, for every $x \in X$ and $r > 0$, the set

$$E \cap (B(x, c_{UP} r) \setminus B(x,r))$$

is nonempty whenever $E \setminus B(x,r)$ is nonempty. This can be shown by a construction of a suitable test function and it follows that c_{UP} depends only on c_H and the constants related to Ahlfors regularity. Then we show that uniform perfectness of the complement implies a quantitative estimate for the Hausdorff s–content of the complement when $s > 0$ is sufficiently small. This is achieved by a covering argument and then iteratively replacing balls that are sufficiently near each other by a single ball in such a way that the Hausdorff content does not increase. Using uniform perfectness, we are able to get a lower bound for the diameter of the central ball in the final cover. Finally, the Poincaré inequality implies uniform p–fatness of the complement for every $p > Q - s$. This connection to uniform perfectness was first studied by Sugawa [53] in the Euclidean plane. It thus follows that uniform p–fatness, p–Hardy's inequality and uniform perfectness are all equivalent in Ahlfors Q–regular spaces when $p = Q$.

We also show in [III] that Hardy's inequality holds if and only if

$$\int_K \frac{1}{\operatorname{dist}(x, X \setminus \Omega)^p} \, \mathrm{d}\mu \leq c \operatorname{cap}_p(K, \Omega)$$

for every compact $K \subset \Omega$. This characterisation is a special case of results in [IV], which are discussed more in the next section.

5 Sobolev inequalities

Important inequalities in Sobolev space theory include *Sobolev embedding theorems*. In $\mathbb{R}^n$, for every $u \in W^{1,p}(\mathbb{R}^n)$, we have

$$\|u\|_{\frac{np}{n-p}} \leq C(n,p)\|\nabla u\|_p, \tag{5.1}$$

when $1 \leq p < n$, and

$$|u(x) - u(y)| \leq C(n,p)|x-y|^{1-n/p}\|\nabla u\|_p,$$

for every $x, y \in R^n$, when $p > n$. Moreover, for $u \in W_0^{1,n}(\Omega)$, we have Trudinger's inequality

$$\int_\Omega \exp\left(\varepsilon\left(\frac{|u|}{\|\nabla u\|_n}\right)^{n/(n-1)}\right) \mathrm{d}x \leq C|\Omega|,$$

for some $\varepsilon > 0$ and $C > 0$ depending only on n. The Poincaré inequality is a strong enough condition to guarantee Sobolev-type embeddings also in the metric space setting.

In article [IV], which has been written jointly with Juha Kinnunen, we study the Sobolev inequality

$$\left(\int_\Omega |u|^q \,\mathrm{d}\nu\right)^{\frac{1}{q}} \leq c_S \left(\int_\Omega g_u^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}, \tag{5.2}$$

in the metric space setting. It is well known that the Sobolev inequality (5.1) with $p = 1$ can be deduced by the coarea formula from the isoperimetric inequality

$$|E|^{(n-1)/n} \leq c(n)\mathcal{H}^{n-1}(\partial E),$$

where E is a smooth enough subset of $\mathbb{R}^n$, $|E|$ is the Lebesgue measure of E, and $\mathcal{H}^{n-1}$ is the $(n-1)$–dimensional Hausdorff measure, and vice versa. This observation originates with Federer–Fleming [12] and Maz′ya [42]. When $p > 1$, the isoperimetric inequality needs to be replaced by isocapacitary inequalities, but, also in this situation, weak-type estimates imply strong estimates. This approach to Sobolev inequalities was first proposed by Maz′ya, see [44] and the references therein. In [IV], we extend these results to a general metric space context. We consider two ranges of indices separately. When $1 \leq q \leq p$, the isocapacitary condition takes the form

$$\nu(K)^{p/q} \leq c\,\mathrm{cap}_p(K,\Omega), \tag{5.3}$$

and the condition is required to hold for every compact subset K of Ω. When $q > p$, the isocapacitary characterisation assumes a more complicated form involving sums or an integral representation. The key estimates in the proofs include a Cavalieri-type inequality and the following strong-type inequality for the capacity

$$\int_0^\infty \lambda^{p-1}\,\mathrm{cap}_p(\{u > \lambda\})\,\mathrm{d}\lambda \leq 2^{2p-1}\int_\Omega g_u^p\,\mathrm{d}\mu,$$

whose proof is based on a general truncation argument attributable to Maz′ya. These estimates lead to the observation that some weak-type inequalities imply strong-type inequalities. See also [6], [20] and [23].

Taking $\mathrm{d}\nu(x) = \mathrm{dist}(x, X \setminus \Omega)^{-p}\,\mathrm{d}\mu(x)$ and $p = q$, we observe that (5.3) gives also a characterisation of p–Hardy's inequality and it is thus a generalisation of results in [III].

When $p = 1$ and $\Omega = X$, we are able to use results from [II] concerning the equivalence of 1–capacity and Hausdorff content of codimension one to show that the condition

$$\nu(B)^{p/q} \leq c\,\mathrm{cap}_p(B, \Omega), \tag{5.4}$$

with all balls $B \Subset \Omega$ instead of all compact sets $K \Subset \Omega$ is enough to guarantee the Sobolev inequality.

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