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Abstract: *We give an approximation to geometric fractional Brownian motion. The approximation is a simple corollary to a 'teletraffic' functional central limit theorem by Gaigalas and Kaj in [6]. We analyze the central limit theorem of Gaigalas and Kaj from the point of view semimartingale limit theorems to have a better understanding of the arbitrage in the limit model. With this approximation we associate the corresponding pricing model sequence, which has no-arbitrage property and which is complete.*

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1 Introduction

1.1 Geometric fractional Brownian motion

In the classical Black-Scholes pricing model the stock price S is modelled by a geometric Brownian motion: $S_t = e^{W_t - \frac{1}{2}t}$; here W is the standard Brownian motion. This model implies that the one dimensional distributions of the stock prices are log-normal, and the log-returns of the stocks are independent normal random variables. But empirical studies show that log-returns often have so-called long-range dependency property (see [18, Chapter IV]). One way to model this observed long-range dependency is to replace the driving standard Brownian motion by fractional Brownian motion. Then one obtains fractional Black-Scholes model, or geometric fractional Brownian motion, given by the price process $S_t = e^{B_t^H}$; here B^H is a fractional Brownian motion. Fractional Brownian motion (fBm) B^H is a continuous centered Gaussian process. Here $H \in (0,1)$ is the self-similarity index and the covariance of the process B^H is given by

$$E\left(B_s^H B_t^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

A Lévy type of characterization theorem for fBm was recently proved in [15]. The paratemeter H allows to include the standard Brownian motion W to the fBm family: the process $B^{\frac{1}{2}}$ is a standard Brownian motion. The standard Brownian motion is a martingale, but it is well-known that when the parameter $H \neq \frac{1}{2}$, then fBm process B^H is not a semi-martingale.

The parameter H allows us to model dependency in the data, since for $H \in (\frac{1}{2}, 1)$ the increments of the process B^H are positively correlated. However, the fact that the process B^H is not a semimartingale, makes it difficult to use fBm as the source of randomness in Stochastic Finance at least theoretically: the reason is the fact that in the pricing models based on geometric fBm one can also give explicit arbitrage strategies, see [18, p. 659]

However, the possibility to model dependency in the data is an attractive feature of the fBm process B^H, also for the market models in Stochastic

Finance. In spite of the theoretical difficulties related to the arbitrage, there have been several proposals to use it as a model in stochastic finance. One of them is based on the fact that the arbitrage possibilities depend how one defines continuous trading, i.e. stochastic integrals. One can show that the arbitrage possibilities in the fractional Black-Scholes model disappear, if one uses Skorohod integrals to model trading strategies (Hu and Øksendal [9], Elliot and van der Hoek [5]). But a new problem appears: the continuous trading based on Skorohod integrals is difficult to interpret economically (see Sottinen and Valkeila [20], Björk and Hult [2] for more information on this point). On the other hand, if one goes to more realistic market models, and for example includes transaction costs in the market models, then the ideal continuous time trading strategies turn out to be of bounded variation. In this case one can show that geometric fBm models can be economically meaningful (Guasoni [7], Guasoni et.al [8]). It is also well known, that in the case where one cannot use continuous time trading, the pricing models with geometric fBm are to some extent arbitrage free.

1.2 Motivation

The purpose of this note is to study the approximation of geometric fBm $S_t^H = e^{B_t^H}$. The approximation is understood in the sense of weak convergence, more precisely the distributions of the approximating prelimit sequence converge weakly to the distribution of the geometric fBm in the Skorohod space D.

This note has two different motivations. First comes from the fact that there are at least two 'financially' motivated approximations to geometric fBm. The approximation given by Sottinen [19] is based on complicated 'fractional' binomial tree, and as a binary tree this approximation is complete. Surprisingly, this approximation is not arbitrage-free, if the step size in the fractional binomial tree is big enough. Hence there are arbitrage opportunities already in the pre-limit model. Klüppelberg and Kühn [13] proposed an alternative approximation, based on Poisson shot noise processes, to geometric fBm. Their approximation is arbitrage free, but not complete. So one can ask, if there is an approximation to geometric fBm, where the pre-limit sequence is arbitrage-free and complete? In this note we show how to construct such an approximation. As mentioned, the limit has arbitrage opportunities, and our approximation might give some new insight on the arbitrage in the limit.

The second motivation comes from our recent work with Bender and Sottinen [1]. In this work we consider a class of models, where the randomness of the risky asset comes from mixed Brownian - fractional Brownian motion. Take this process to be $\epsilon W + B^H$, where W is a standard Brownian motion, B^H is a fBm with index $H \in (\frac{1}{2}, 1)$, and independent of W. If we take the model of the risky asset S^ϵ to be

$$S_t^\epsilon = \exp\{\epsilon W_t + B_t^H - \frac{1}{2}\epsilon^2 t\},$$

then there is a unique *hedging price* for the standard European type of options, provided that one uses so-called *allowed* (in the terminology [1]) strategies only. But in this model one can let $\epsilon \to 0$, and ask weather the limiting prices make sense? It turns out that we have to following price with an European call with strike K: $(S_0 - K)^+$. We get the same limiting price from our approximation, and we give two different explanations for this. One is based on the path-wise approach given in Dzhaparidze [4], and the other one is based on computing the limit price using the martingale measure of the approximating sequence.

1.3 The structure of the note

First we introduce the 'teletraffic' approximation from [6], discuss its properties from the point of view semimartingale weak limit theorems (see [11]), and prove that the corresponding geometric processes also converge weakly. We then argue that the prelimit sequence defines a sequence of pricing models, which are complete and have the no-arbitrage property. We conclude with a discussion.

2 Approximation of fBm

2.1 Construction of the approximation

We will not prove any new approximation to fBm. Instead, we will the use the 'teletraffic' approximation to fBm, interpret this weak limit theorem as a semimartingale limit theorem of a special kind: the approximating sequence is based on semimartingales, but the limit is not a semimartingale.
We start with an approximation given by Gaigalas and Kaj [6]. This goes as follows: let G be a continuous distribution function of interarrival times for a renewal counting process N. Let $\mu = E\eta_2$. Assume that this distribution has heavy tails:

$$1 - G(t) \sim t^{-(1+\beta)} \tag{1}$$

as $t \to \infty$ with $\beta \in (0,1)$. If η_i are the interarrival times we assume that η_i have the distribution function G for $i \geq 2$; for the first interarrival time η_1 we assume that it has the distribution

$$G_0(t) = \frac{1}{\mu} \int_0^t (1 - G_s) ds, \tag{2}$$

so that the renewal counting process

$$N_t = \sum_{k=1}^{\infty} 1_{\{\tau_k \leq t\}}$$

is stationary, where $\tau_1 = \eta_1$ and $\tau_k := \eta_1 + \cdots + \eta_k$. Take now independent copies $N^{(i)}$ of N, numbers $a_m > 0$, $a_m \to \infty$ such that

$$\frac{m}{a_m^\beta} \to \infty; \tag{3}$$

in the terminology of Gaigalas and Kaj [6] this is the case of *fast connection rate*. Define the workload process W_t^m by

$$W_t^m = \sum_{k=1}^{m} N_t^{(k)}.$$

Note that the process W_t^m is again a counting process, since the interarrival distribution is continuous and the components $N^{(k)}$ are independent, and these facts imply that there are no simultaneous jumps of the components $N^{(k)}$. We have that $EW_t^m = \frac{mt}{\mu}$, since W_t^m is a stationary process. For the following proposition see Gaigalas and Kaj [6]:

Proposition 2.1 *Assume* (1) *and* (3). *Let*

$$Y^m(t) := \mu^{\frac{3}{2}} \sqrt{\frac{\beta(1-\beta)(2-\beta)}{2}} \frac{W_{a_m t}^m - m\mu^{-1} a_m t}{m^{\frac{1}{2}} a_m^{1-\frac{\beta}{2}}}. \tag{4}$$

Then Y^m converges weakly in the Skorohod space D to a fBm B^H, where $H = 1 - \frac{\beta}{2}$.

2.2 Further properties of the approximation

In order to discuss the application to finance, and to construct an approximation to geometric fBm $S_t^H = e^{B_t^H}$, we will have a look to Proposition 2.1 from the viewpoint of semimartingale limit theorems. We can write the explicit semimartingale decomposition, but only with respect to a big filtration, where we can keep track of the jumps of individual components $N^{(k)}$.
First we recall how one obtains the compensator of a renewal counting process by keeping track on the jump times and using the interarrival distributions; this is due to Jacod (see [14, Theorem 18.2, p.270]). Assume that N is a renewal counting process with interarrival times η_j such that $G_j(t) = P(\eta_j \leq t)$ has the form

$$G_j(t) = \int_0^t g_j(s) ds, \text{where } g_j(s) > 0 \ \forall s \geq 0. \tag{5}$$

Let τ_j be the jump times and define $b_j(t) = \tau_j \wedge t - \tau_{j-1} \wedge t$, $j \geq 1$. Let H_j be the integrated hazard function of G_j :

$$H_j(t) = \int_0^t \frac{g_j(s)}{1 - G_j(s)} ds.$$

Work with the history $\mathbb{F}^N$ of the process N. Then the $(P, \mathbb{F}^N)$- compensator A of N can be written as

$$A_t = \sum_{j=1}^{\infty} H_j(b_j(t)).$$

(see [14, 16]). Note that we have the relation

$$t = \sum_{j=1}^{\infty} b_j(t).$$

Next, consider the workload process W_t^m. Assume that we can keep track of the jumps of the processes $N^{(k)}$, i.e. we work with the filtration $\bar{\mathbb{F}}$, where $\bar{F}_t^m = \sigma\{N_s^{(k)} : s \leq t, k = 1, \ldots, m\}$. Define $b_j^{(k)}(t) = \tau_j^{(k)} \wedge t - \tau_{j-1}^{(k)} \wedge t$, and then by the independence of the processes $N^{(k)}$ the $(\bar{\mathbb{F}}, P)$- compensator of $N^{(k)}$ is

$$A_t^{(k)} = \sum_{j=1}^{\infty} H_j(b_j^{(k)}(t)).$$

Hence we obtain that the $(\bar{\mathbb{F}}^m, P)$- compensator A^m of the workload process W^m is

$$A_t^m = \sum_{k=1}^{m} A_t^{(k)};$$

note also that

$$mt = \sum_{k=1}^{m} \sum_{j=1}^{\infty} b_j^{(k)}(t).$$

The process Y^m given by (4) is a semimartingale, since it has bounded variation on compacts. Let us now write the semimartingale decomposition of the process Y^m with respect to the big filtration $\mathbb{F}^m$, where $F_t^m = \bar{F}_{a_m t}$ and probability measure P, associated to the interarrival times given by (1) and (2). To simplify notation put

$$c(\mu, \beta) := \mu^{\frac{3}{2}} \sqrt{\frac{\beta(1-\beta)(2-\beta)}{2}},$$

$c_m := m^{\frac{1}{2}} a_m^{1-\frac{\beta}{2}}$ and $\Lambda_t^m := \frac{m\mu^{-1} a_m t}{c_m}$.
Since the process Y^m is a semimartingale, it has a semimartingale decomposition

$$Y^m = M^m + L^m; \tag{6}$$

here $L_t^m = c(\mu, \beta) \frac{A_{a_m t}^m}{c_m} - \Lambda_t^m$.
The martingale part of the semimartingale Y^m is

$$M_t^m := c(\mu, \beta) \frac{W_{a_m t}^m - A_{a_m t}^m}{c_m}.$$

Note that the compensator of W^m with respect to the big filtration $\bar{\mathbb{F}}$ is continuous, and hence the process L^m is a continuous process with bounded variation.
The square bracket of the martingale part M^m of the semimartingale Y^m is

$$[M^m, M^m]_t = (c(\mu, \beta))^2 \frac{W_{a_m t}^m}{c_m^2}.$$

But our assumption (3) implies

$$E[M^m, M^m]_t = (c(\mu,\beta))^2 \mu t a_m^{\beta-1} \to 0,$$

as $m \to \infty$. With the Doob inequality we obtain that $\sup_{s\le t} |M_s^m| \xrightarrow{P} 0$.
Denote uniform on compacts convergence in probability by $\xrightarrow{ucp}$.
We obtain the following semimartingale interpretation of the Proposition 2.1:

Proposition 2.2 *Assume* (1) *and* (3). *Let*

$$Y^m = M^m + L^m$$

be the semimartingale decomposition of the process Y^m given by (3). *Then the sequence L^m of continuous bounded variation processes converges weakly in the Skorohod space D to the fBm B^H with $H = 1 - \frac{\beta}{2}$, and $M^m \xrightarrow{ucp} 0$, as $m \to \infty$.*

Remark 2.1 *We gave the semimartingale decomposition of the process Y^m with respect to the filtration $\mathbb{F}$. Since the process Y^m is adapted to its own filtration $\mathbb{F}^{Y^m} = \mathbb{F}^{W^m}$, it has a different semimartingale decomposition with respect to $(\mathbb{F}^{Y^m}, P)$, and the new compensator is $\tilde{L}^m$. The process $\tilde{L}^m$ is the $(\mathbb{F}^{Y^m}, P)$ dual predictable projection of the process Y^m. To compute the process $\tilde{L}^m$ explicitly is apparently difficult, because the interarrival times are not identically distributed.*

2.3 Approximation to geometric fBm

Consider the solution to the equation

$$dS_t^m = S_{t-}^m dY_s^m, \text{ with } S_0^m = S_0, \tag{7}$$

where $Y^m = c(\mu,\beta)\frac{W^m}{c_m} - \Lambda^m$, and

$$\Delta Y_t^m = c(\mu,\beta)\frac{\Delta W_t^m}{c_m} = \Delta M_t^m.$$

It is known that the equation (7) has a unique solution of the form

$$S_t^m = S_0 e^{-\Lambda_t^m} \prod_{s\le t} (1 + \Delta Y^m(s)) =: \mathcal{E}(Y^m)_t. \tag{8}$$

Proposition 2.3 *Assume* (1) *and* (3) *and let Y^m be as in* (4). *Then the solution to* (7), *given by* (8), *converges weakly to the geometric fBm $S_t = S_0 e^{B_t^H}$ in the Skorohod space D.*

Proof We have that

$$S_0 e^{Y^m(t) - \frac{1}{2}\frac{W^m_{a_m t}}{c_m^2}} \le S_t^m \le S_0 e^{Y^m(t)};$$

but we already know that $\frac{W^m_{a_m t}}{c_m^2} \to 0$ in $L^1(P)$, as $m \to \infty$. Hence the claim follows by the continuous mapping theorem. □
With the notation of proposition 2.2 we have

Corollary 2.1 *The sequence of continuous bounded variation processes e^{L^n} converges weakly to the geometric fBm e^{B^H} in the Skorohod space D.*

3 Some properties of the approximation

3.1 Set-up

Assume that we have (1) and (2), the process Y^m is defined by (4), and S^m is defined by (7). We interpret the prelimit approximation S^m as a stock price. To simplify the discussion we assume that the interest rate for the *bank account* is equal to 0, and that there is no drift on the stock price. So we have a sequence of pricing models

$$(S^m, \mathbb{F}^m, P^m) \overset{w(P^m)}{\longrightarrow} (S_0 e^{B^H}, \mathbb{F}^{B^H}, P), \tag{9}$$

where B^H is a fBm with $H = 1 - \frac{\beta}{2} \in (\frac{1}{2}, 1)$.
We will show that the prelimit market model with S^m and bank account is complete and arbitrage-free model.

3.2 Prelimit market models are complete

We consider the following market model, so-called *Poisson market* according to the terminology of Dzhaparidze [4]. We follow the arguments of Dzhaparidze and show that the prelimit market is complete. Note that the argument given below is pathwise.
Let N be a counting process, $\alpha > 0$ and $\gamma > 0$ are constants, and consider the pathwise solution S to the following linear equation

$$dS_t = S_{t-}\left(\alpha dN_t - \gamma dt\right) \text{ with } S_0 = s;$$

then the unique solution to this is

$$S_t = s e^{-\gamma t} \prod_{s \le t} (1 + \alpha \Delta N_s) = s e^{-\gamma t} (1 + \alpha)^{N_t}.$$

Denote the jump times N by τ_k, $k = 1, 2, \ldots$. Fix $T > 0$ and assume that there is no jump at time T. Let $M \ge 0$ be such that $\tau_M < T < \tau_{M+1}$. Define $s_k(t)$ by

$$s_k(t) = s(1+\alpha)^k e^{-\gamma t} 1_{[\tau_k, \tau_{k+1})}(t) = s(1+\alpha)^k e^{-\alpha\lambda t} 1_{[\tau_k, \tau_{k+1})}(t);$$

with $\lambda = \frac{\gamma}{\alpha}$. The functions s_k describe the states of the price process S_t. Obviously

$$S_t = \sum_{k=0}^{M} s_k(t) 1_{[\tau_k, \tau_{k+1})}(t).$$

We can write the left-hand limit process S_{t-} as follows

$$S_{t-} = s_0(t) 1_{[0, \tau_1]}(t) + \sum_{k=1}^{M} s_k(t) 1_{(\tau_k, \tau_{k+1}]}(t).$$

Let S_{t-} be in the state $s_k(t)$. Then, at time t the stock price either stays in this state or jumps to the state $s_{k+1}(t)$. Define the difference operator D in the state space of S as follows: if S_{t-} is in state $s_k(t)$ then DS_t is in the state

$$D_{k+1}(S_t) = s_{k+1}(t) - s_k(t). \tag{10}$$

We have then the following

Proposition 3.1 *The states of the stock price process satisfy the following differential equations*

$$\frac{ds_k(t)}{dt} = -\lambda D_{k+1}(S_t), \text{ when } t \in (\tau_k, \tau_{k+1}]. \tag{11}$$

Proof See [4, Proposition 4.4.1.] □
The Poisson probabilities $p_j(\lambda)$ are defined by $p_j(\lambda) = \frac{\lambda^j}{j!}e^{-\lambda}$ for $\lambda > 0$. One can include the value $\lambda = 0$ by defining $p_j(0) = \delta_{j0}$, where

$$\delta_{j0} = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if otherwise} \end{cases}$$

is the Kronecker delta.
Consider the following system of differential-difference equations

$$\frac{dx_k(t)}{dt} = -\lambda\left(x_{k+1}(t) - x_k(t)\right), \ k = 0, 1, \ldots \tag{12}$$

subject to boundary conditions

$$x_k(T) = w_k(T) \ k = 0, 1, \ldots \tag{13}$$

Proposition 3.2 *A solution to the system* (12) *with boundary conditions* (13) *is given by*

$$x_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))w_{k+j}(T), \tag{14}$$

if the numbers allow $w_k(T)$ allow differentiation under the summation sign.

Proof See [4, Proposition 4.2.3.] □.
Next we indicate how to apply the above results a hedging problem in finance. Let W_T be a functional of the price process path S_t, $0 \leq t \leq T$. If the price process S has a state $s_k(T)$, then the value functional W_T has a state $w_k(T)$. Recall that a market model is *complete*, if we can find a self-financing strategy π such that

$$W_T = V_T^\pi = v + \int_0^T \pi_s dS_s.$$

It follows from [4, Proposition 4.4.5.] that if we define a value process V^π by the formula

$$V_t^\pi = v + \int_0^t \pi_s dS_s, \tag{15}$$

where

$$v = \sum_{j=0}^{\infty} p_j(\lambda T) w_j(T) \tag{16}$$

and when $s \in (\tau_k, \tau_{k+1}]$ we define

$$\pi_s = \sum_{j=0}^{\infty} p_j\left(\lambda\left(T - s\right)\right) \frac{(1+\alpha)\, w_{j+k}(T) - w_{j+k+1}(T)}{\alpha}, \tag{17}$$

then we obtain self-financing strategy π, which replicates the claim $W(T)$.

Remark 3.1 *The probabilistic interpretation of the equations* (15), (16) *and* (17) *that the process N is a Poisson process with intensity λ. Note that the results were obtained in a pathwise way, without any probability. In the subsection 3.3 we discuss the probabilistic interpretation of this result.*

We end this subsection by giving an option pricing formula [for European call only, but of course all is valid for a more much bigger class of options]. Recall some properties of Poisson probabilities. Put

$$F(j_0; \lambda) = \sum_{j > j_0} p_j(\lambda). \tag{18}$$

We have the following connection between $F(j_0; \lambda)$ in Gamma integrals

$$\Gamma(c, x) = \int_x^{\infty} e^{-t} t^{c-1} dt, \text{ where } c, x > 0:$$

$$F(j_0; \lambda) = 1 - \frac{\Gamma(j_0 + 1, \lambda)}{j_0!}. \tag{19}$$

Proposition 3.3 *Consider the pricing of an European option $(S_T - K)^+$ in the Poisson market model. Then the fair price C^E of this option is given by*

$$C^E = S_0 F(j_0; (1+\alpha)\lambda T) - K F(j_0; \lambda T), \tag{20}$$

where

$$j_0 = \lfloor \frac{\log\left(\frac{K}{S_0} + \gamma T\right)}{\log(1+\alpha)} \rfloor \tag{21}$$

We can now apply the above to the approximating model S^m. Now from (4) we obtain

$$\begin{aligned} \alpha &= \alpha^m = c(\mu, \beta) \frac{a_m^{\frac{\beta}{2}}}{m^{\frac{1}{2}}}, \\ \gamma &= \gamma^m = c(\mu, \beta)\mu \frac{a_m^{\frac{\beta}{2}}}{m^{\frac{1}{2}}} \qquad \text{and} \\ \lambda &= \lambda^m = \frac{\gamma^m}{\alpha^m} = \mu. \end{aligned}$$

From (3) we obtain that $\alpha^m \to 0$, $\gamma^m \to 0$, and if $K > S_0$, then $j_0 \to \infty$, and if $K < S_0$, then $j_0 \to -\infty$. Put this in (20) and we obtain that the limiting price is $(S_0 - K)^+$.

3.3 Prelimit market models are arbitrage-free

The basic randomness of the approximating pricing model sequence S^m comes from the workload process W^m. We shall show that there exists a probability measure Q^m such that W^m is a Poisson process with intensity $\frac{m}{\mu}$. We work first with the single component of the workload process.
Assume that N is a renewal counting process with first interarrival time distribution given by (2) and all the rest interarrival times have distribution given by (1). We assume that with respect to the measure our counting process N is a renewal counting process. Fix $T > 0$. First we shall show that there exists a probability measure Q such that $Q_T \sim P_T$, where $Q_T = Q|F_T^N$, $P_T = P|F_T^N$, and with respect to the measure Q the counting process N is a Poisson process with intensity μ^{-1}. Put $G_i = G$ when $i \geq 1$ and define

$$\kappa(s, N) := \frac{g_{N_{s-}}(s)}{1 - G_{N_{s-}}(s)}.$$

Define the density between the measures Q and P by

$$\frac{dQ}{dP}|F_t^N = e^{\int_0^t \left(\kappa(s,N) - \frac{1}{\mu}\right) ds + \int_0^t \log \frac{1}{\mu\kappa(s,N)} dN_s} \tag{22}$$

Obviously we have

$$\frac{dP}{dQ}|F_t^N = e^{\int_0^t \left(\frac{1}{\mu} - \kappa(s,N)\right) ds + \int_0^t \log(\mu\kappa(s,N)) dN_s} \tag{23}$$

The Hellinger process between the measures P and Q is then given by

$$h(P, Q)_t = \frac{1}{2} \int_0^t \left(\sqrt{h(s, N)} - \sqrt{\frac{1}{\mu}} \right)^2 ds.$$

Under our assumptions $h(P, Q)_t \leq A_t + \nu t < \infty$ $(P + Q)$-a.s.; we can now use [11, Theorem IV.2.1] and conclude that the measures P_T and Q_T are equivalent. We have shown the following:

Lemma 3.1 *Assume that X is a counting process. With respect to measure Q it is a Poisson process with intensity $\frac{1}{\mu}$ and with respect to a measure P is it a renewal counting process with first interarrival time distribution given by (2) and all the rest interarrival times have distribution given by (1). Moreover, the laws P_T and Q_T are equivalent.*

The next step is to show that the law of the process

$$W_t^m = \sum_{k=1}^m N_t^{(k)}$$

is equivalent to the law of Poisson process with intensity $\frac{m}{\mu}$. Note that the process W_t^m is not any more a renewal counting process. We show that the prelimit pricing models driven by the processes Y^m have the no-arbitrage property. For this it is sufficient to show that the original probability measure is equivalent to a probability measure Q such that the process W_t^m is a Poisson process with intensity $\frac{m}{\mu}$. The proof is not very difficult, and will follow from the Lemmas 3.1 and 3.2.

Lemma 3.2 *Let X^k, $k = 1, \ldots, m$ we a sequence of counting processes. Assume that with respect to the measure Q they are independent Poisson processes with intensity $\frac{1}{\mu}$, and with respect to the measure P they are independent renewal counting processes, and their interarrival times satisfy* (5). *Then the sum process $W^m = \sum_{k=1}^m X^k$ is a counting process with respect to the measures P and Q, with respect to the measure Q it is a Poisson process with intensity $\frac{m}{\mu}$, and the Q-law of W^m, Q^m is equivalent to the P- law, P^m of W^m on $[0, T]$. Here the filtration is the big filtration $F_t^m := \vee_{k=1}^m F_t^{X^k}$.*

Proof Since the processes X^k are stochastically continuous and independent with respect to the measures Q and P, we have that $P(\Delta X_s^k = 1, \Delta X_s^l = 1) = 0$ for $k \neq l$ for all $s \geq 0$, and similarly with respect to the measure Q. Hence the aggregated process W^m is a counting process.
Obviously the sum of independent Poisson processes is again a Poisson process, not only in the big filtration $\mathbb{F}^m$, but in the filtration $\mathbb{F}^{W^m}$, too. If the $(P, \mathbb{F}^{X^k})$ compensator of X^k is A^k, and because the sum on independent martingales is a martingale again, we have that the $(P, \mathbb{F}^m)$ compensator of W^m is $\sum_{k=1}^m A^k$. We can now repeat the argument given to obtain Lemma 3.1 and conclude that the measures Q_T^m and P_T^m are equivalent in the filtration $\mathbb{F}^m$. □

Remark 3.2 *If we consider the measures P^m and Q^m restricted to the filtration $\mathbb{F}^{W^m}$ they are also equivalent on $[0, T]$, since $\mathbb{F}^{W^m} \subset \mathbb{F}^m$. But it is difficult to write the $(P^m, \mathbb{F}^{W^m})$- compensator of W^m explicitly (see Remark 2.1).*

Let us now return to the model driven by (4). We have that the aggregated process W^m is a Poisson process with intensity $\frac{m}{\mu}$. We can interpret that it has the law Q^m described in the Lemma 3.1 and the Lemma 3.2. With respect to the original measure P, which corresponds to the renewal counting process model with interarrival times given (1) for interarrivals after the first jump and by (2) for the first interarrival. We have that the measures are equivalent on the interval $[0, a_m T]$. This means that the approximation process Y^m is a martingale with respect to $(Q^m, \mathbb{F}^m)$, or with respect to $(Q^m, \mathbb{F}^{Y^m})$, too.
What happens with the approximation? Recall that $S^m = \mathcal{E}(Y^m)$. But $Y^m = M^m + L^m$, where L^m is a continuous process, and hence $[M^m, L^m] = 0$. So using Yor's formula for stochastic exponents we can write the approximating sequence as

$$S_t^m = S_0 e^{L_t^m} \mathcal{E}(M^m)_t,$$

where $\mathcal{E}(M^m) \overset{ucp}{\to} 1$ with respect to the measure P^m. We know that the approximation $(S^m, \mathbb{F}^m, P^m)$ weakly converges to the geometric fBm. On the other hand, with respect to the martingale measure Q^m the sequence Y^m is a martingale sequence, $Y^m \xrightarrow{ucp} 0$ with respect to Q^m, and $S^m \xrightarrow{ucp} S_0$ with respect to Q^m. So in the price $(S_0 - K)^+$ has a limit

$$(S_0 - K)^+ = \lim_m E_{Q^m}(S_T^m - K)^+$$

for the European call.

4 Discussion and conclusion

Consider the market model of the following type. The stock price S is driven by a process $X = \epsilon W + B^H$; here W is a standard Brownian motion, B^H is a fBm with Hurst index $H > \frac{1}{2}$, independent of W; the linear stochastic differential equation defining the stock price is

$$dS_t^\epsilon = S_t^\epsilon dX_t, \text{ with } S_0 \tag{24}$$

as the initial value. One can show that the solution to (24) is

$$S_t^\epsilon = S_0 e^{\epsilon W_t + B_t^H - \frac{1}{2}\epsilon^2 t}.$$

It was shown in [17] that the hedging price for standard European type of options is the same as in the model, where we do not have the fBm component B^H at all. Recently we in Bender et. al. [1] have extended this argument to a bigger class of options, and also discussed arbitrage possibilities in this kind of models. So the price of an European call $(S_T^\epsilon - K)^+$ is given by the classical Black & Scholes pricing formula

$$S_0 \Phi\left(\frac{\log\frac{S_0}{K} + \frac{1}{2}\epsilon^2 T}{\epsilon\sqrt{T}}\right) - K\Phi\left(\frac{\log\frac{S_0}{K} - \frac{1}{2}\epsilon^2 T}{\epsilon\sqrt{T}}\right). \tag{25}$$

Take now $\epsilon_n \to 0$ and define S^{ϵ_n} by

$$S_t^{\epsilon_n} = S_0 e^{\epsilon_n W_t + B_t^H - \frac{1}{2}\epsilon_n^2 t}.$$

We have that $S_t^{\epsilon_n} \xrightarrow{w} S_0 e^{B_t^H}$, as $n \to \infty$, and we have again an approximation to geometric fBm. It is easy to check that the limit, as $\epsilon_n \to 0$, of the price in (25) is given by

$$(S_0 - K)^+ . \tag{26}$$

Recall that se obtained the same limit for European call as a limit of hedging prices in subsection 3.2 and as a limit risk neutral prices in subsection 3.3. Note that if a price process S is continuous and has bounded variation, then we have the following

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T 1_{\{S_s \geq K\}} dS_s; \tag{27}$$

hence a candidate for the hedging price would be $(S_T - K)^+$. But this makes no sense, since this kind of pricing model has arbitrage opportunities, unless $S_t = S_0$ for all $t \leq T$.
We have shown that in our approximation:

- The prelimit sequence $S^m = \mathcal{E}(Y^m)$ is a P^m semimartingale and Q^m martingale.
- The weak limit along P^m is geometric fBm, which is not a semimartingale, and along Q^m is the constant S_0, which is a martingale.

We can formulate yet another property of our approximation:

- The sequence of measures P^m and Q are entirely separated: there exists events C_m such that $P^m(C_m) \to 1$ and $Q^m(C_m) \to 0$, as $m \to \infty$.

In [12] Kabanov and Kramkov discuss so-called asymptotic arbitrage, which is related to the notions of contiguity and entire separation.
This means the following: let π^m be a sequence of self-financing strategies and S^m a vector valued price process such that

$$(\pi^m \cdot S^m)_t := \sum_{k=1}^{m} \int_0^t \pi_s^{m,k} dS_s^{m,k} \geq -1. \tag{28}$$

They defined the following three types of asymptotic arbitrage, but we mention only one:

- If in addition to (28), we have $\limsup_m P^m\left((\pi^m \cdot S^m) \geq C\right) = 1$ as $m \to \infty$ for any $C > 0$, then π^m realizes *strong asymptotic arbitrage.*

We refer to Kabanov and Kramkov [12] for more information how asymptotic arbitrage is related to contiguity and entire separation. We mention only that entire separation implies some kind of asymptotic arbitrage.
We end our discussion by reformulating our approximation in the spirit of large financial markets. Define the price process of the i^{th} asset $S^{(i)}$ by

$$dS_t^{(i)} = S_{t-}^{(i)} c(\mu, \beta) \frac{1}{c_m} d\left(N_t^{(i)} - A_t^{(i)}\right), \tag{29}$$

where $N^{(i)}$ is the renewal counting process, $A^{(i)}$ is the compensator of $N^{(i)}$ with respect to the filtration $\mathbb{F}^{N^{(i)}}$, and where $c(\mu, \beta) := \mu^{\frac{3}{2}} \sqrt{\frac{\beta(1-\beta)(2-\beta)}{2}}$, and $c_m := m^{\frac{1}{2}} a_m^{1-\frac{\beta}{2}}$, as before. Let $S_0^{(i)} = \frac{S_0}{m}$. Then the model in (4) can be considered as the sum $S^m = \sum_{i=1}^m S^{(i)}$, and as we know already, the martingale measures Q^m and the 'historical' measures P^m are entire separated and the market model $(\tilde{S}^m, \mathbb{F}^m, P^m)$ with $\tilde{S}^m = (S^{(1)}, \ldots, S^{(m)})$ admits asymptotic arbitrage.

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