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A PRIORI AND A POSTERIORI ERROR ANALYSIS OF FINITE ELEMENT METHODS FOR PLATE MODELS

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Abstract: *The focus of this dissertation is in the theoretical and computational analysis of the discretization error indused by finite element methods for plate problems. For the Reissner–Mindlin plate model, regularity results with respect to the loading and a priori convergence estimates for the MITC finite elements are presented. The convergence results are valid uniformly with respect to the thickness parameter. In addition, we prove a local superconvergence result for the deflection approximation of MITC elements, and introduce a postprocessing method improving the accuracy of the approximation. The convergence results are confirmed by numerical computations. For the Kirchhoff–Love plate model, a new family of* C^0*-continuous, optimally convergent finite elements is introduced. Furthermore, we derive a reliable and efficient a posteriori error indicator and verify the results by benchmark computations. Another a posteriori error analysis is performed for the Morley plate element.*

AMS subject classifications: 65N30, 74S05, 74K20

Keywords: finite elements, a priori error analysis, a posteriori error analysis, plate models, MITC-elements, adaptivity

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Tiivistelmä: *Väitöskirjassa analysoidaan sekä teoreettisesti että laskennallisesti laattamalleille kehiteltyjen elementtimenetelmien diskretointivirhettä. Reissner–Mindlin-laattamallille esitetään kuormituksen suhteen lausuttuja säännöllisyystuloksia ja MITC-laattaelementtien a priori -virhearvioita. Suppenemistulokset ovat voimassa laatan paksuusparametrin suhteen tasaisesti. Lisäksi MITC-elementtien taipuma-approksimaatiolle todistetaan lokaali superkonvergenssitulos ja esitetään approksimaation tarkkuutta parantava jälkikäsittelymenetelmä. Suppenemistulokset vahvistetaan numeerisilla esimerkeillä. Kirchhoff–Love-laattamallille esitetään uusi* C^0*-jatkuva, optimaalisesti suppeneva elementtiperhe. Näille elementeille johdetaan luotettava ja tehokas a posteriori -virheindikaattori ja tulokset verifioidaan numeerisesti. Lisäksi suoritetaan a posteriori -virheanalyysi Morleyn laattaelementille.*

Asiasanat: elementtimenetelmä, a priori -virheanalyysi, a posteriori -virheanalyysi, laattamallit, MITC-elementit, adaptiivisuus

Preface

The work gathered in this dissertation has been mainly accomplished at Helsinki University of Technology, Institute of Mathematics.

First and foremost, I wish to express my deep gratitude to Professor Rolf Stenberg – the instructor and supervisor of this thesis – for his expert guidance and collaboration with patience. Second, I am grateful to my collaborators and advisors, Docent Mikko Lyly and Ph.D. Lourenço Beirão da Veiga – on the one hand, for their contribution to the articles of this dissertation, on the other hand, for proficient guidance and enjoyable co-operation. Third, I acknowledge Professor Olavi Nevanlinna, the head of the Institute of Mathematics, for scientifically inspiring and appreciative working environment and atmosphere.

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In addition to all this support, many teachers deserve to be acknowledged for their influence and impression on me and my career. I start the glimpses into the past from the secondary school: I learned from Juhani Heinonen, my teacher in mathematics, the importance of clarity in layout and simplification in representation. In high school, my teacher Mirja Kuparinen emphasized that mathematics teaches one to work hard. My first course in mathematics at the university level was lectured by Professor Peter Lindqvist who showed us how fun studying mathematics can be. Later, as an "exchange student" at the University of Helsinki, I attended lectures in topology given by Professor Emeritus Jussi Väisälä, which aroused enthuasiasm in me for abstract mathematics. I esteem Professor Emeritus Gennadi Vainikko, the supervisor of my Master's Thesis, for his humility as well as his clearness in lecturing. I still would like to mention that before my post graduate studies, my superiors in industry have been M.Sc. Hannu Rantala and M.Sc. Eero Torkkeli, both of whom I appreciate for giving me challenging and instructive topics to work on.

Finally, I would like to thank my friends and colleagues at the Institute of Mathematics, in particular, the doctoral students Tomi Tuominen, Antti Niemi, Mika Juntunen and Antti Hannukainen, for discussions and help during my studies – and for healing roars of laughter. In addition, I would like to thank my colleagues in the Laboratory of Structural Mechanics where I have finished this work. Especially, I thank Professor Jukka Aalto, the head of the laboratory, and the post graduate student Timo Manninen, my friend and colleague for years.

I conclude with my deepest gratitude to my wife Anna-Maija, and to my son Taku-Petteri, for their patience, forgiveness and sympathy.

Espoo, October, 2007

Jarkko Niiranen

S.D.G.

List of included publications

[A] Mikko Lyly, Jarkko Niiranen, Rolf Stenberg: A refined error analysis of MITC plate elements; *Mathematical Models and Methods in Applied Sciences*, 16 (2006), 967–977.

[B] Mikko Lyly, Jarkko Niiranen, Rolf Stenberg: Superconvergence and postprocessing of MITC plate elements; *Computer Methods in Applied Mechanics and Engineering*, 196 (2007), 3110–3126.

[C] L. Beirão da Veiga, J. Niiranen, R. Stenberg: A family of C^0 finite elements for Kirchhoff plates I: Error analysis; *SIAM Journal on Numerical Analysis*, 45 (2007), 2047–2071.

[D] Lourenço Beirão da Veiga, Jarkko Niiranen, Rolf Stenberg: A family of C^0 finite elements for Kirchhoff plates II: Numerical results; Helsinki University of Technology, Institute of Mathematics, Research Reports, A526 (2007), 29 pages; accepted for publication in *Computer Methods in Applied Mechanics and Engineering*.

[E] L. Beirão da Veiga, J. Niiranen, R. Stenberg: A posteriori error estimates for the Morley plate bending element; *Numerische Mathematik*, 106 (2007), 165–179.

Author's contribution: In [A], the author has contributed partly to the analysis and writing of Section 3, and to minor parts in Section 2.

The author is partly responsible for the proofs and writing of the theoretical analysis in [B]. In addition, the substantial parts of the programming design and implementation as well as the presentation and analysis of the numerical results have been accomplished by the author.

The derivation, analysis and writing of the papers [C] and [E] have been accomplished by the author and the co-authors in a very close collaboration with each other.

The author has the main responsibility for the paper [D], i.e., the main parts of the writing and programming as well as the presentation and analysis of the numerical results have been performed by the author.

The main results of the papers as well as some further applications have been presented in seminars and international conferences by the author [25, 41, 40] and the co-authors [24, 26, 42].

Contents

1 Introduction

The main targets in engineering design are safety and durability – although the importance of cost and environmental aspects in design has significantly grown during the last decade. Even if cars, bridges, aircraft and nuclear power plants, for instance, have to fullfil the expectations of global markets, they still have to meet certain carefylly prescribed minimum requirements in resisting mechanical loadings. For this twofold challenge, a variety of efficient instruments are provided by modern computer-aided design methods.

Computational methods of mathematical modeling can be seen as the hard core of an engineering decision-making process. In the following, we briefly describe such a process for which a schematic representation is shown in Table 1, cf. [6][Ch. 1]. In the beginning of a process, the physical problem and the design criteria are prescribed. Next, the problem is formulated as a general mathematical model, an idealized – possibly strongly imperfect – representation of reality. In general, problems described by complex mathematical models can not be solved exactly, and thus computational methods and approximate solutions become necessary. Depending on the availability and cost of computational resources, the general mathematical model can be simplified by utilizing engineering experience and intuition. Finally, the problem based on the simplified mathematical model is solved approximately by numerical methods and the solution obtained is then used for engineering decisions: the geometry or the material of the model can be changed, the simplified mathematical model can be refined etc. In principle, the whole process from the first beginning to the final approval should be an adaptive feedback process [6].

The key issue in each step of the process is error. In particular, in this work, we mainly consider the *discretization error*, i.e., the difference between the exact solution of the simplified mathematical model and its numerical approximation. In addition, we slightly touch the gap between the general and simplified mathematical models, the *modeling error*. In the following, we do not discuss the gap between the general mathematical model and the real physical problem, the idealization error, nor other possible error components present in the process, i.e., errors in manufacturing stages or errors in interpretating the results of the analysis, for example.

This dissertation consists of an overview and five publications, referred as [A]–[E], which can be considered as a collection of building blocks for engineering design in structural mechanics. These blocks comprise of theoretical results and methods for computer implementation as well as numerical evaluations benchmarking the methods proposed. In order to link the concepts of design processes to the present work, we give the following identification: the theory of three-dimensional elasticity is our mathematical model; our simplified models are the Reissner–Mindlin and Kirchhoff–Love (or Kirchhoff–Germain) plate models; the numerical methods in our case are finite element methods; the discretization error we analyze by deriving a priori and a posteriori error estimates for the finite element methods.

Table 1: Error sources in engineering decision-making processes.

General level (Structural mechanics level)	Error component
Physical problem and Design criteria (Structural design)	
$\updownarrow$	$\Rightarrow$ Idealization error
General mathematical model (Three-dimensional elasticity)	
$\updownarrow$	$\Rightarrow$ Modeling error
Simplified mathematical model (Elastic plate model)	
$\updownarrow$	$\Rightarrow$ Discretization error
Numerical method (Plate finite element method)	
$\uparrow$	
$\downarrow$	
Engineering decisions and Feedback (Changes in the structure, model, element)	
$\updownarrow$	
Review and Approval	

In the next Section, we briefly discuss the framework of the classical plate models of Reissner–Mindlin and Kirchhoff–Love in the sense of modeling error. Then, in Section 3, we focus on the mathematical error analysis, the a priori and a posteriori analysis for the finite element methods considered. Finally, in Section 4, we summarize the main results of the publications.

2 On the accuracy of plate models

As described above, estimating the modeling error should be a part of a design process, while the error depends on the level of idealization. In particular, a structural model based on the theory of three-dimensional elasticity can be simplified and remodeled by using dimensionally reduced structure models of different levels: one-dimensional rods and beams, two-dimensional membranes and plates [48, 22], or shallow shells and general shells [48, 22, 23]. All these models are closely related to each other, in particular, the shell problem decouples into the plate and membrane problems when the curvature of the shell approaches zero [19][Ch. 4].

In the following two subsections, we shortly derive and discuss the dimensionally reduced plate models of Reissner–Mindlin and Kirchhoff–Love. Our aim is to collect here the main principles and assumptions behind the models in order to describe the physical backround of the mathematical finite element error analysis for plates.

2.1 Dimension reduction

For plate structures, the most commonly used models are the Reissner–Mindlin and Kirchhoff–Love plates [22, 48, 30], the former being the lowest-order member of a more general class, the hierarchical plate models [6, 1, 53, 51]. In the following, we briefly describe the physical setting of these two plate models and define the physical quantities and material parameters of the problems. We first shortly recall the path of the dimension reduction from the three-dimensional continuum model to the two-dimensional plate models – from the kinematical assumptions to the equilibrium equations and boundary conditions in terms of the stress resultants. Finally, the constitutive assumptions of linear elasticity give us the formulations applied in the finite element error analysis.

A plate structure is assumed to occupy a three-dimensional set

$$\mathcal{P} = \Omega \times (-\frac{t}{2}, \frac{t}{2}) \,, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ denotes the midsurface of the plate and $t \ll \mathrm{diam}(\Omega)$ denotes the thickness of the plate.

The displacement field of the plate is denoted by $\boldsymbol{u} = (u_x, u_y, u_z)$, with the global Cartesian coordinates x, y, z and $u_i = u_i(x, y, z)$, $i = x, y, z$. In the Reissner–Mindlin plate theory, the following kinematical assumptions are imposed:

(K1) The fibres normal to the midsurface remain straight during the deformation.

(K2) The fibres normal to the midsurface do not stretch.

(K3) The points in the midsurface deform only in the z-direction.

Under these conditions, the Kantorovič's method [48][Ch. 10] gives the displacement field in the form

$$u_x = -z\beta_x(x, y) \,, \quad u_y = -z\beta_y(x, y) \,, \quad u_z = w(x, y) \,, \tag{2.2}$$

where w denotes the deflection of the midsurface, while β_x and β_y, respectively, denote the rotations around the y and x axis, with orientations following the slope of the deflection. Regarding the kinematical assumptions, it should be pointed out that additional terms giving the z^2 and t^2-dependence for the displacement component u_z will be discussed further below in Section 2.2.

The corresponding deformation is defined by the linear strain tensor

$$\boldsymbol{e}(\boldsymbol{u}) = \frac{1}{2}\big(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\big) \,, \tag{2.3}$$

where the tensor gradient $\boldsymbol{\nabla}$ is defined as usual, cf. the Appendix of [E]. In the component form, we have

$$e_{xx} = -z\frac{\partial \beta_x}{\partial x}\,, \quad e_{yy} = -z\frac{\partial \beta_y}{\partial y}\,, \quad e_{zz} = 0\,, \tag{2.4}$$

$$e_{xy} = -\frac{z}{2}(\frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x})\,, \quad e_{xz} = \frac{1}{2}(\frac{\partial w}{\partial x} - \beta_x)\,, \quad e_{yz} = \frac{1}{2}(\frac{\partial w}{\partial y} - \beta_y)\,. \tag{2.5}$$

In the Kirchhoff–Love plate theory, an additional kinematical assumption is imposed:

(K4) The fibres normal to the midsurface remain normals during the deformation.

This condition couples the deflection w and the rotation $\boldsymbol{\beta} = (\beta_x, \beta_y)$ as

$$\nabla w - \boldsymbol{\beta} = \mathbf{0}\,, \tag{2.6}$$

and implies that the displacement field takes the form

$$u_x = -z\frac{\partial w(x,y)}{\partial x}\,, \quad u_y = -z\frac{\partial w(x,y)}{\partial y}\,, \quad u_z = w(x,y)\,. \tag{2.7}$$

In this case as well, the z^2-dependence for the displacement component u_z will be discussed further below. The corresponding deformations may now be written as

$$e_{xx} = -z\frac{\partial^2 w}{\partial x^2}\,, \quad e_{yy} = -z\frac{\partial^2 w}{\partial y^2}\,, \quad e_{zz} = 0\,, \tag{2.8}$$

$$e_{xy} = -z\frac{\partial^2 w}{\partial x \partial y}\,, \quad e_{xz} = 0\,, \quad e_{yz} = 0\,. \tag{2.9}$$

Here we note that the transverse shear deformations e_{xz} and e_{yz} vanish due to the assumption (2.6).

Next, we define the stress resultants, i.e., respectively, the moments and the shear forces:

$$\boldsymbol{M} = \begin{pmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} \quad \text{with} \quad M_{ij} = -\int_{-t/2}^{t/2} z\,\sigma_{ij}\,dz\,,\; i,j = x,y\,, \tag{2.10}$$

$$\boldsymbol{Q} = \begin{pmatrix} Q_x \\ Q_y \end{pmatrix} \quad \text{with} \quad Q_i = \int_{-t/2}^{t/2} \sigma_{iz}\,dz\,,\; i = x,y\,, \tag{2.11}$$

where the stress tensor is assumed to be symmetric, $\sigma_{ij} = \sigma_{ji}$, $i,j = x,y,z$.

In order to apply the principle of virtual work, we define the linear strain tensor

$$\boldsymbol{\varepsilon}(\boldsymbol{\beta}) = \frac{1}{2}\big(\boldsymbol{\nabla}\boldsymbol{\beta} + (\boldsymbol{\nabla}\boldsymbol{\beta})^T\big)\,, \tag{2.12}$$

which is usually referred as the curvature tensor. When comparing the definitions (2.3) and (2.12) we note that it holds $e_{ij} = -z\varepsilon_{ij}$, $i,j = x,y$. With the virtual deflection δw and the virtual rotation $\delta\boldsymbol{\beta}$, this relation gives the energy balance in the form

$$\int_\Omega \boldsymbol{M} : \boldsymbol{\varepsilon}(\delta\boldsymbol{\beta})\, dx\, dy + \int_\Omega \boldsymbol{Q} \cdot (\nabla\delta w - \delta\boldsymbol{\beta})\, dx\, dy = \int_\Omega F\, \delta w\, dx\, dy\,, \qquad (2.13)$$

where we have assumed that no given boundary stress resultants are present nor any nonzero boundary diplacements are prescribed. The load resultant F includes the surface loads in the z-direction as well as a possible body force resultant in the z-direction defined as

$$F_b = \int_{-t/2}^{t/2} F_z\, dz\,, \qquad (2.14)$$

where the body force density $F_z = F_z(x, y, z)$ is usually assumed to be independent of z, or at least even in z [57]. The effect of possible surface loadings in the x and y-directions will be discussed below in Section 2.2.

Now, we see that the plate problems have been reduced into a two-dimensional formalism such that the unknowns depend on $(x, y) \in \Omega$ only, not on z explicitly. Furthermore, integration by parts gives for (2.13) the following equilibrium equations in Ω. For the Reissner–Mindlin model, it holds that

$$-\mathrm{div}\, \boldsymbol{Q} = F\,, \qquad (2.15)$$

$$\mathbf{div}\, \boldsymbol{M} + \boldsymbol{Q} = \mathbf{0}\,, \qquad (2.16)$$

where **div** stands for the tensor divergence defined in the Appendix of [E]. For the Kirchhoff–Love model, we simply get

$$\mathrm{div}\, \mathbf{div}\, \boldsymbol{M} = F\,. \qquad (2.17)$$

In this case as well, in order to satisfy the force equilibrium (2.15), the shear force is defined by the moment equilibrium equation (2.16).

Regarding the boundary conditions of the Reissner–Mindlin model, the plate is considered to be hard clamped on the part $\Gamma_{\mathrm{C_H}}$ of its boundary $\partial\Omega$, soft clamped on the part $\Gamma_{\mathrm{C_S}} \subset \partial\Omega$, hard simply supported on the part $\Gamma_{\mathrm{S_H}} \subset \partial\Omega$, soft simply supported on $\Gamma_{\mathrm{S_S}} \subset \partial\Omega$ and free on $\Gamma_{\mathrm{F}} \subset \partial\Omega$. Then the boundary conditions are of the form

$$\begin{array}{ll} w = 0\,, \quad \boldsymbol{\beta} = \mathbf{0} & \text{on } \Gamma_{\mathrm{C_H}}\,, \\ w = 0\,, \quad \boldsymbol{\beta}\cdot\boldsymbol{n} = \mathbf{0}\,, \quad \boldsymbol{s}\cdot\boldsymbol{M}\boldsymbol{n} = 0 & \text{on } \Gamma_{\mathrm{C_S}}\,, \\ w = 0\,, \quad \boldsymbol{\beta}\cdot\boldsymbol{s} = 0\,, \quad \boldsymbol{n}\cdot\boldsymbol{M}\boldsymbol{n} = 0 & \text{on } \Gamma_{\mathrm{S_H}}\,, \\ w = 0\,, \quad \boldsymbol{n}\cdot\boldsymbol{M}\boldsymbol{n} = 0\,, \quad \boldsymbol{s}\cdot\boldsymbol{M}\boldsymbol{n} = 0 & \text{on } \Gamma_{\mathrm{S_S}}\,, \\ \boldsymbol{n}\cdot\boldsymbol{M}\boldsymbol{n} = 0\,, \quad \boldsymbol{s}\cdot\boldsymbol{M}\boldsymbol{n} = 0\,, \quad \boldsymbol{Q}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma_{\mathrm{F}}\,, \end{array} \qquad (2.18)$$

where $\boldsymbol{n}$ and $\boldsymbol{s}$, respectively, denote the unit outward normal and the unit counterclockwise tangent to the boundary, respectively, while $\boldsymbol{n}\cdot\boldsymbol{M}\boldsymbol{n}$ denotes the bending moment and the twisting moment is denoted by $\boldsymbol{s}\cdot\boldsymbol{M}\boldsymbol{n}$.

The Kirchhoff–Love model distinguishes neither between the hard and soft clamped nor between the hard and soft simply supported boundaries:

$$\begin{aligned}
&w = 0\,, \quad \nabla w \cdot \boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{C}}\,,\\
&w = 0\,, \quad \boldsymbol{n} \cdot \boldsymbol{M}\boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{S}}\,,\\
&\boldsymbol{n} \cdot \boldsymbol{M}\boldsymbol{n} = 0\,, \quad \tfrac{\partial}{\partial \boldsymbol{s}}(\boldsymbol{s} \cdot \boldsymbol{M}\boldsymbol{n}) + \boldsymbol{n} \cdot \mathbf{div}\,\boldsymbol{M} = 0 && \text{on } \Gamma_{\mathrm{F}}\,,\\
&(\boldsymbol{s}_1 \cdot \boldsymbol{M}\boldsymbol{n}_1)(c) = (\boldsymbol{s}_2 \cdot \boldsymbol{M}\boldsymbol{n}_2)(c) && \forall c \in \mathcal{V}\,,
\end{aligned} \tag{2.19}$$

where the indices 1 and 2 refer to the sides of the boundary angle at a corner point c on the free boundary Γ_{F}.

For linearly elastic plates, in addition to the kinematical assumptions above, the following standard constitutive assumptions are possessed: the material of the plate is assumed to be

(C1) linearly elastic, i.e., defined by the generalized Hooke's law,

(C2) homogeneous, i.e., independent of the coordinates x, y, z,

(C3) isotropic, i.e., independent of the coordinate system.

Furthermore, it is assumed that

(C4) the transverse normal stress vanishes, i.e., $\sigma_{zz} = 0$.

We remark that the assumption (C4), which implies a plane stress state for the Kirchhoff–Love model, contradicts with the condition $\varepsilon_{zz} = 0$ which would rather imply the plane strain state. However, the asymptotical analysis for the modeling error referred below in the next subsection justifies the last assumption.

Under these assumptions, the constitutive equations are now of the following form:

$$\sigma_{xx} = \frac{E}{1-\nu^2}(e_{xx} + \nu e_{yy})\,, \tag{2.20}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}(e_{yy} + \nu e_{xx})\,, \tag{2.21}$$

$$\sigma_{xy} = 2Ge_{xy}\,, \quad \sigma_{xz} = 2Ge_{xz}\,, \quad \sigma_{yz} = 2Ge_{yz}\,, \tag{2.22}$$

with the material constants, Young modulus E, the Poisson ratio ν and the shear modulus

$$G = \frac{E}{2(1+\nu)}\,. \tag{2.23}$$

With $\boldsymbol{I}$ denoting the identity tensor, the moment tensor is now of the form

$$\boldsymbol{M} = D\big((1-\nu)\boldsymbol{\varepsilon}(\boldsymbol{\beta}) + \nu\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{\beta})\boldsymbol{I}\big) \quad \text{with} \quad D = \frac{Et^3}{12(1-\nu^2)}\,. \tag{2.24}$$

For the Reissner–Mindlin model, we get the nonzero transverse shear stresses and the shear force is defined as

$$\boldsymbol{Q} = Gt(\nabla w - \boldsymbol{\beta})\,. \tag{2.25}$$

For the Kirchhoff–Love model, the constitutive relations imply the formulation of the well known biharmonic problem,

$$D\Delta^2 w = F\,. \tag{2.26}$$

Finally, we note that for mathematical analysis, in order to get a nontrivial solution to the Reissner–Mindlin problem in the limit $t \rightarrow 0$, i.e., the Kirchhoff–Love solution, the loading is scaled such that $F = Ct^r f$ with a constant C, a proper exponent r and a loading f independent of t [22, 5, 57]. As usual, we adopt the scaling

$$f = \frac{F}{Gt^3}, \tag{2.27}$$

see [14][Th. 3.1] as well. The corresponding scaled moment and shear force, respectively, are then defined as

$$\boldsymbol{m}(\boldsymbol{\beta}) = \frac{\boldsymbol{M}(\boldsymbol{\beta})}{Gt^3} \quad \text{and} \quad \boldsymbol{q}(w,\boldsymbol{\beta}) = \frac{\boldsymbol{Q}(w,\boldsymbol{\beta})}{Gt^3}\,. \tag{2.28}$$

This notation will be applied in the papers [A]–[E] below when defining the bilinear forms of the variational formulations for the plate problems.

2.2 Mathematical justification

It has been generally agreed that the Reissner–Mindlin theory is more accurate than the Kirchhoff–Love theory, especially for moderately thin plates [22, 5]. However, this claim is rarely argued with mathematical analysis. Here we briefly consider this issue by refering to some recent studies [5, 57, 27] on the modeling of plate structures. The results are mainly based on asymptotic analysis: the accuracy of a plate model is measured in an asymptotic sense, i.e., the model is justified if the solution converges to the solution of the three-dimensional elasticity theory when the thickness of the plate converges to zero.

The range of applicability of the plate models has been analyzed by Arnold, Madureira and Zhang [5] for a clamped plate under a quite general loading; in addition to a body force f_z constant in the z-direction, there exist possible surface tractions g_i, $i = x, y, z$, on the upper and lower surfaces of the plate, in the x, y and z-directions, respectively. For these surface tractions, the following assumptions are made:

$$g_z(x,y,-\frac{t}{2}) = g_z(x,y,\frac{t}{2}) = g_z(x,y)\,, \tag{2.29}$$

$$g_i(x,y,-\frac{t}{2}) = -g_i(x,y,\frac{t}{2}) = g_i(x,y)\,, i = x,y\,. \tag{2.30}$$

In physical terms, these symmetry and antisymmetry conditions imply that the plate is rather under bending type than streching type loadings.

Next, we only briefly recall the main results of the analysis in [5]. First, the total loading, with proper regularity assumptions, classifies the problem as follows: either the condition

$$\operatorname{div} \boldsymbol{g} + \frac{1}{t} g_z + f_z \neq 0 \,, \tag{2.31}$$

holds or, in contrast,

$$\operatorname{div} \boldsymbol{g} + \frac{1}{t} g_z + f_z = 0 \,, \tag{2.32}$$

where we have used the notation $\boldsymbol{g} = (g_x, g_y)$. The left-hand side in the two conditions above follows from the loading functional in the right-hand side of (2.13) written in its general form for the Kirchhoff–Love model, with all the components of the surface load included, cf. [57] and the references therein. We note that a plate under a transverse loading only, i.e., with $\boldsymbol{g} = \boldsymbol{0}$, $g_z + tf_z \neq 0$, as assumed in (2.13), clearly satisfies the condition (2.31).

In order to proceed, we introduce the following notation: $\boldsymbol{U}^*$ with the superscript *, with $\boldsymbol{U}^* = (u_x^*, u_y^*)$ as an example, refers to the solution of the three-dimensional problem of elasticity, whereas $\boldsymbol{U}^M$ denotes the corresponding solution of the plate model. Here M refers either to the Reissner–Mindlin model or to the Kirchhoff–Love model. The former model corresponds to the displacement field (2.2), with $\boldsymbol{U}^M = (u_x, u_y)$ as an example, and the latter one, respectively, follows the equation (2.7).

Now, if the condition (2.31) holds, the both models converge with identical rates of convergence as follows:

$$\frac{||\boldsymbol{U}_* - \boldsymbol{U}_M||}{||\boldsymbol{U}_*||} \leq Ct^p \,, \tag{2.33}$$

where the variable $\boldsymbol{U}$, the corresponding norm $||\cdot||$ and the convergence rate are listed in Table 2. The norm $\|\boldsymbol{u}\|_{E(\mathcal{P})}$ denotes the energy norm corresponding to the energy equilibrium (2.13).

It is worth noting that for the Kirchhoff–Love model the z^2-dependence for the displacement component u_z is needed for the last two rows in Table 2. For the Reissner–Mindlin model, additional z^2- and t^2-dependent terms are required. In this situation, the Reissner–Mindlin model would represent the hierarchic $(1,1,2)$-model rather than the $(1,1,0)$-model presented above in (2.2). Here the first two hierarch indices 1 refer to the z-dependence of the displacement components u_x and u_y, respectively, while the last index (2 or 0) refers to the z-dependence of u_z.

If, in contrast, the condition (2.32) holds, the Reissner–Mindlin solution converges in the relative energy norm with the order $\mathcal{O}(t^{1/2})$, whereas the Kirchhoff–Love solution is simply zero. For a more detailed discussion, we refer to [5] and the references therein.

Table 2: Convergence rates for the relative errors of the form (2.33) for the Reissner–Mindlin and Kirchhoff–Love plate models satisfying (2.31) [5].

Variable and Norm $\|\boldsymbol{U}\|$	Convergence rate
$\|(u_x, u_y)\|_{L^2(\mathcal{P})}$	$\mathcal{O}(t)$
$\|u_z\|_{L^2(\mathcal{P})}$	$\mathcal{O}(t)$
$\|\boldsymbol{\nabla}(u_x, u_y)\|_{L^2(\mathcal{P})}$	$\mathcal{O}(t^{1/2})$
$\|\frac{\partial(u_x,u_y)}{\partial z}\|_{L^2(\mathcal{P})}$	$\mathcal{O}(t)$
$\|\nabla u_z\|_{L^2(\mathcal{P})}$	$\mathcal{O}(t)$
$\|\frac{\partial u_z}{\partial z}\|_{L^2(\mathcal{P})}$	$\mathcal{O}(t^{1/2})$
$\|\boldsymbol{u}\|_{E(\mathcal{P})}$	$\mathcal{O}(t^{1/2})$

Zhang [57] has accomplished a similar analysis for the case including stress boundary conditions. First, the conditions (2.31) and (2.32) are augmented with certain conditions in terms of given boundary stress resultants. Then the problem is classified as a bending-dominated, shear-dominated or intermediate case. The condition (2.31) corresponds to the bending-dominated case, while the condition (2.32) corresponds to the shear-dominated case. It has been remarked that the intermediate case is excluded if the boundary is free from given boundary stress resultants.

Altogether, as noted above, a plate under a transverse loading only falls into the bending-dominated category. This implies that the asymptotic behaviour of the Reissner–Mindlin model for transversal loadings is qualitatively identical with the asymptotic behaviour of the Kirchhoff–Love model. However, for a clamped plate, Destuynder [27] provides for both models a different constant C in (2.33) bounding the asymptotical convergence of the relative error in a strain energy norm. Finally, we note that quantitative differences between the plate models, with respect to boundary conditions and the shear force distribution, for instance, have been analyzed in [6][Ch. 17] by Szabó and Babuška.

3 Finite element error analysis for plates

In this section, we list and briefly describe some paths of the mathematical error analysis for finite element methods of Reissner–Mindlin and Kirchhoff–Love plates. Our aim is not to provide a complete, comprehensive or chronological literature review, we rather point out the main routes leading to our contributions and consider the methods parallel to our approach.

3.1 A priori and a posteriori error analysis

The mathematical error analysis of finite element methods falls into two main categories [2, 12, 14, 21, 6, 56]: *A priori error* estimates can be seen mainly as qualitative measures, whereas *a posteriori error* estimates provide both qualitative and quantitative information. Moreover, the a priori error analysis is usually global in nature; the a posteriori error analysis, instead, frequently gives both local and global information.

The main difference between these two branches is the data needed for the error estimation. In a priori estimates, the error is measured in terms of the loading given for the problem or in terms of the mathematical regularity of the exact solution which is unknown, in general. Therefore, the analysis mainly theoretically qualifies the method considered. In a posteriori error estimates, only the approximate solution itself is needed for measuring the error. Furthermore, the residual based a posteriori estimates considered in this work, in particular, are composed of local error indicators which give enough information to adaptively refine the finite element mesh and thus to efficiently decrease the error. Altogether, the a posteriori error analysis serves practical purposes, while the a priori error analysis is mainly oriented for theoretical qualification. Together these approaches provide a broad view on the reliability of the approximation method considered as well as a practical tool box for powerful computations.

The mathematical regularity of the exact solution is the key issue both in theoretical error analysis and in numerical computations. Namely, the error of the approximation mainly concentrates in the same regions as the low regularity. Particularly, singularities in the solution imply high peaks in the error. In theory, this phenomenon is addressed by the a priori error analysis, while, in practice, it can be handled by adaptive solution algorithms based on a posteriori error indicators. A priori error analysis is able to predict that in order to decrease the error the mesh size h should be diminished in those parts of the domain in which the exact solution is irregular. In practice, however, the behaviour of the solution is unknown. In an adaptive solution algorithm, a posteriori error indicators utilize the approximate solution for assigning and quantizing the distribution of the error, especially the error peaks. In h-adaptivity treated in this work, remeshing is accomplished such that mesh refinements are directed towards the error peaks. On the contrary, in those areas in which the solution is regular and the error concentration is low, the mesh can be coarsened. In a more general methodology, the *hp*-

adaptivity [6, 52], coarsening goes along with raising the polynomial order p, whereas refining follows lowering the order of elements.

3.2 Error analysis for Reissner–Mindlin plates

For Reissner–Mindlin plates, the numerical locking is the main difficulty in developing and analyzing the finite element methods. The locking phenomenon can be seen as an inability of the numerical approximation to capture the asymptotical behaviour of the parameter dependent plate model. In other words, the finite element spaces are not rich enough for an approximate solution to satisfy the so called Kirchhoff constraint (2.6) which is satisfied by the exact solution of the problem when the thickness parameter approaches zero.

In the famous MITC family [7, 15], and in other reduced constraint elements as well, instead of the exact fullfilment of (2.6) in the limit for the approximative solution $(w_h, \boldsymbol{\beta}_h)$, a reduction operator $\boldsymbol{R}_h$ is introduced and a modified limit condition is applied,

$$\boldsymbol{R}_h(\nabla w_h - \boldsymbol{\beta}_h) = \boldsymbol{0}\,. \tag{3.1}$$

Furthermore, in order to achieve the stability for the method, the MITC finite element spaces for the rotation need to be augmented by bubble degrees of freedom.

Mathematical error analysis for locking-free Reissner–Mindlin plate elements has been accomplished for several single elements and finite element families. For MITC type elements, the a priori error analysis is usually based on the stability theory of saddle point problems [13], and it was first carried out for the limiting case $t = 0$ by Bathe, Brezzi and Fortin [7], and then, for the general case, by Brezzi, Fortin and Stenberg [15] as well as Peisker and Braess [49]. The error estimates obtained are valid for all variables uniformly with respect to the thickness parameter t. Similar results for other elements can be found in [4, 32], for instance. In the saddle point approach, the problem is written in mixed form with the shear force as a new unknown which can be interpreted as a Lagrange multiplier associated to the modified Kirchhoff constraint. Then, by means of a Helmholtz decomposition, the mixed formulation is shown to be equivalent to a system of two Poisson problems and a Stokes-like system. Finally, a set of conditions is given involving the reduction operator and the finite element spaces for the deflection, rotation and the shear force. This route has been utilized for analyzing hp-methods in [55, 3].

A different approach has been proposed by Pitkäranta and Suri [50]. Instead of resting on the stability theory of saddle point problems, they split the error into the consistency error and the approximation error, both parts including boundary layer terms. A set of conditions is given in this method as well involving the reduction operator and the finite element spaces for the deflection and the rotation. A drawback of this approach is that it does not directly yield uniform estimates for the shear force. However, it provides optimal convergence for curved elements. Moreover, it explicitly takes into account boundary layers – analysis of [50] has been accomplished for free boundaries. This approach has been later used for shell problems in [35, 36]. In [A], we have analyzed MITC elements by mixing this approach with the previous one and taking into account the boundary effects for clamped, convex, polygonal plates. However, our analysis still rests on the Helmholtz decomposition and the mixed theory of [49] and [15]. In [B], we have applied the convergence results of [A] for the superconvergence and postprocessing estimates in the clamped case.

Stabilized formulations form another class of elements in which the conditions originating from Brezzi's theory can be mainly avoided. A general formulation and analysis for these methods have been proposed by Hughes and Franca [37]. Stabilized versions of the well-known nonconforming Arnold-Falk element [4] have been proposed by Durán and Ghioldi [31] and by Franca and Stenberg [34]. Two families of stabilized plate elements have been introduced later by Lyly and Stenberg: the first one with the unequal order interpolated deflection and rotation together with standard finite elements spaces [54, 43]; the second one with equal order interpolated variables combined with a reduced bilinear form [43]. Our new family of C^0-continuous Kirchhoff elements in [C, D] originates from the former family of unequal interpolation.

The first steps in a posteriori error analysis for Reissner–Mindlin plate elements have been taken by Liberman [38], Carstensen and Weinberg [18, 16] and recently by Carstensen and Schöberl [17] as well as Lovadina and Stenberg [39].

3.3 Error analysis for Kirchhoff–Love plates

For designing finite element methods for Kirchhoff–Love plates, the fundamental difficulty originates from the corresponding variational formulation: the natural variational space for the biharmonic problem is the second-order Sobolev space H^2. Thus, a conforming finite element approximation of the Kirchhoff problem requires globally C^1-continuous elements which imply a high polynomial order. However, the error analysis for these methods follows the classical finite element theory [21].

In order to avoid using high-order polynomial spaces – for example the fifth order polynomials of the well known Argyris triangle [21, 12] – there exists a variety of non-standard finite elements as the Hsieh–Clough–Tocher triangle, [8, 21] Discrete Kirchhoff Triangle [8, 9] and the Morley triangle [45, 21, 44] which does not even satisfy the conditions of C^0-continuity.

One modern approach of using C^0-continuous approximations is based on continuous–discontinuous Galerkin methods and stabilization techniques [33]. Another natural alternative is to write the Kirchhoff–Love problem as a limit of the Reissner–Mindlin problem written in mixed form. In the presence of free boundary conditions, however, this leads to a method which is not consistent, which deteriorates the convergence rate of the method.

Destuynder and Nevers [29, 28] have obtained a remedy to the boundary inconsistency by adding a term penalizing the Kirchhoff condition along the free boundaries, cf. [10, 11] as well. In [C, D], we use a similar approach and present a stabilized family of C^0-continuous elements which does not suffer from the boundary inconsistency.

The variety of a posteriori error analysis for Kirchhoff–Love plate elements is still quite limited. Charbonneau, Dossou and Pierre [20] have analyzed the Ciarlet–Raviart formulation for clamped boundaries while Neittaanmäki and Repin [46] have considered problems related to the biharmonic operator by deriving functional a posteriori error estimates [47] for any conforming approximations with clamped boundaries. In addition to these two papers, there seems to occur no other contributions than our analysis in [E] for the Morley element in the clamped case and the one in [C, D] for our C^0-continuous family with clamped, simply supported or free boundaries.

4 Conclusions

We have now briefly described the framework of the work from both the physical and mathematical point of views. The focus of this dissertation is in the theoretical and computational analysis of the discretization error indused by finite element methods for plate problems. In addition, the mathematical regularity of the exact solution, which is the key issue in the analysis, has a role in the work as well. Furthermore, the comparison of the plate models has naturally raised some questions on the modeling error, and this issue we have lightened above as well.

Finally, the main results of the publications we summarize as follows: In [A], the interior and boundary regularity of a hard clamped Reissner–Mindlin plate with a convex, polygonal domain is analyzed in Theorem 2.1. The regularity of the solution is given with respect to the loading. This regularity result is utilized in Theorem 3.1. and Remark 3.1 in which a priori convergence estimates for the MITC finite elements are presented. The results are valid uniformly with respect to the thickness parameter.

In [B], a local superconvergence result in the H^1-(semi)norm for the deflection approximation of the MITC finite elements is proved in Theorem 4.1. This result gives an indication that the vertex values obtained by the MITC methods are superconvergent, which is confirmed by the numerical computations of Section 6.2. By utilizing the superconvergence property, Postprocessing scheme 5.1 improves the accuracy of the deflection approximation. This is proved in Theorem 5.1 for the local H^1-seminorm, and in Theorem 5.2 for a hard clamped, convex, polygonal plate in the H^1-norm. These convergence estimates are verified by numerical computations in Section 6 for plates with hard clamped, hard simply supported and free boundaries discretized with uniform and non-uniform meshes.

In [C], a family of C^0-continuous finite elements for the Kirchhoff–Love plate model is introduced as Method 3.1. The consistency and stability of the method are proved, respectively, in Theorems 3.2 and 4.3. The corresponding a priori convergence results are given in Theorem 4.4 and Lemma 4.6. The convergence is optimal with respect to both the polynomial degree and the regularity of the solution. Furthermore, a local a posteriori error indicator is introduced, and in Theorems 5.4 and 5.6, respectively, it is shown to be both reliable and efficient. A special feature of this family is that it adopts the same approach as typically used for Reissner–Mindlin plates: the variables are the scalar deflection and the rotation vector. On the one hand, compared to a typical Kirchhoff element, the number of degrees of freedom increases due to the additional rotation components. On the other hand, however, the presence of rotation gives an option for naturally mixing these elements with other elements having the rotation degrees of freedom.

In [D], the computational part of the analysis for the family of [C] is accomplished. First, a constructive motivation of the method is presented in Section 3. Second, in Section 6, some computational aspects of the method are discussed and results from a wide range of benchmark computations are presented. The a priori error estimate of the method, particularly for a free boundary case, is verified by the numerical results of Section 6.2. The a posteriori error estimates are confirmed in Section 6.3 with tests including both convex and non-convex domains – the latter implying corner singularities of different orders.

In [E], an a posteriori error analysis is performed for the Morley plate element, a classical nonconforming finite element for the Kirchhoff–Love plate problem. A local a posteriori error indicator is first introduced and then, in Theorems 1 and 2, respectively, it is shown to be both reliable and efficient. In particular, the reliability proof relies on the Helmholtz decomposition of Lemma 1.

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Errata

The following misprints and errors have been noticed in the publications.

Paper A

Page 972, equation (3.1): $w_{|K}$ should be $v_{|K}$.
Page 975, equation (3.36): Π_h should be $\mathbf{\Pi}_h$, as defined in the right hand side of the equation (3.18).

Paper C

Page 4, equation (3.1): w should be v and $w_{|K}$ should be $v_{|K}$.

Paper D

Page 14, in the beginning of the second paragraph: *For the error analysis* should be replaced by *For the proof of the upper bound.*
Page 14, before **Theorem 5.2**: *reliability result* should be *efficiency result.*
Page 24: Figure 3 should be replaced by Figure 1 below. This replacement does not imply any qualitative revision to the corresponding comments on the effectivity index on page 19.

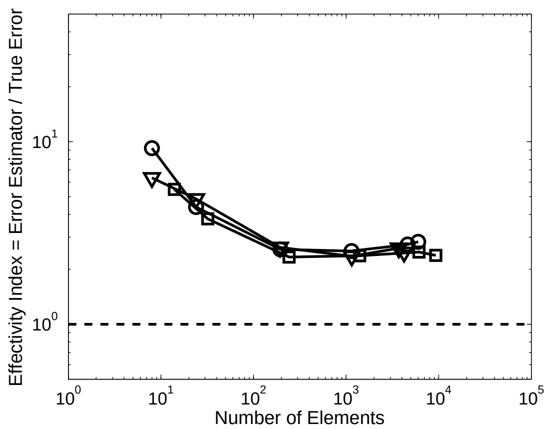

Figure 1: Effectivity index for the adaptive refinements: Clamped (squares), simply supported (circles), simply supported and free (triangles) boundaries.

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