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Tuomo Kuusi: *Harnack estimates for supersolutions to a nonlinear degenerate equation*; Helsinki University of Technology, Institute of Mathematics, Research Reports A532 (2007); Monograph.

Abstract: In this work, we prove both global and local weak Harnack estimates for supersolutions to a nonlinear degenerate parabolic partial differential equation using measure–theoretical arguments. The main tools are various estimates for both sub- and supersolutions, expansion of positivity, the comparison principle and the existence result for a Dirichlet problem with zero lateral boundary values and square–integrable initial data.

AMS subject classifications (2000): 35K60

Keywords: Parabolic nonlinear partial differential equations, Harnack estimates, weak solutions, parabolic Moser's method, De Giorgi's estimates, intrinsic time scale, regularity of solutions, Hölder continuity of solutions

Tuomo Kuusi: *Harnackin estimaatit epälineaarisen degeneroituneen osittaisdifferentiaaliyhtälön yläratkaisuille*; Teknillinen korkeakoulu, Matematiikan laitos, tutkimusraportti A532 (2007); monografia.

Tiivistelmä: Todistamme tässä väitöskirjassa sekä globaalin että lokaalin Harnackin estimaatin epälineaarisen degeneroituneen parabolisen osittaisdifferentiaaliyhtälön ratkaisuille käyttäen mittateoreettisia argumentteja. Todistuksen tärkeimmät työkalut koostuvat lukuisista estimaateista sub- ja superratkaisuille, positiivisuuden laajenemisesta, vertailuperiaatteesta ja olemassaolotuloksesta ratkaisulle nolla reuna-arvoilla paikan suhteen ja neliöintegroituvilla alkuarvoilla.

Asiasanat: Paraboliset epälineaariset osittaisdifferentiaaliyhtälöt, Harnackin estimaatit, heikot ratkaisut, parabolinen Moserin menetelmä, De Giorgin estimaatit, luontainen aikaskaala, ratkaisuiden säännöllisyys, ratkaisuiden Hölder-jatkuvuus

Contents

1. Preface

This thesis has been produced during years 2004-2007 at Helsinki University of Technology.

I wish to express my deepest gratitude to my advisor Professor Juha Kinnunen. His example sets the standard for my further research and work. I am grateful to Professor Olavi Nevanlinna for supervising my work. I would like to thank Professors Ugo Gianazza and Juan Louis Vazquez for reading the manuscript and giving valuable comments.

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The people in the groups of nonlinear PDEs and analysis in metric spaces at Helsinki University of Technology has been simply magnificent. I am grateful for the really good spirit.

I would like to thank my family for all the support. They have witnessed my stubbornness in good and bad. My friends have given me so much relief and happiness. I am so lucky to have them. And finally, I really cannot express how grateful I am to my wife Liisa. Together with Eemil, they are everything to me.

Espoo, September 2007

Tuomo Kuusi

2. Introduction

Harnack estimates play a central role in the regularity theory of partial differential equations. In this work, we prove parabolic Harnack estimates for the evolutionary p-Laplace equation using measure theoretical arguments. We study questions similar to those in a recent work [DGV06] by E. DiBenedetto, U. Gianazza and V. Vespri. The problem has a long history in the field of nonlinear degenerate diffusion equations. The question whether the Harnack estimates hold for equations with growth of order p instead of quadratic growth arose after the celebrated result of J. Moser in [Mos64] and [Mos67], see also [Mos71]. E. DiBenedetto and M. Herrero found a partial answer in [DH89]. The major difference from the case $p = 2$ is that the Harnack estimates hold in an intrinsic time scale dictated by the solution itself.

The Harnack estimates have many profound consequences. Amongst others, together with a proper compactness result, they imply the existence of an initial trace. They can also be used in the study of free boundaries and asymptotic behavior. Furthermore, our second main result, the local weak Harnack estimate, can be seen as one of the main tools in the nonlinear parabolic potential theory. Also the Hölder continuity of weak solutions follows from the estimate.

We study weak supersolutions to the degenerate second–order partial differential equation

$$\operatorname{div}\big(\mathcal{A}(x,t,u,\nabla u)\big) = \frac{\partial u}{\partial t} \tag{2.1}$$

in $\mathbb{R}^n \times (0, T_0)$. Function $\mathcal{A}$ is assumed to be a monotone Caratheodory function and satisfy growth conditions similar to the p-Laplace operator with $p > 2$. These conditions are described in detail in Section 3.1. The first main result is that the weak global Harnack principle holds.

Theorem 2.2. *Let u be a nonnegative weak supersolution to (2.1). Then there exists a constant $C = C(n, p, \text{structure of } \mathcal{A})$ such that for almost every $0 < t_0 < T_0$, every $x_0 \in \mathbb{R}^n$, $R > 0$ and $0 < T < T_0 - t_0$ we have*

$$\fint_{B(x_0,R)} u(x,t_0)\,dx \le \Big(\frac{CR^p}{T}\Big)^{1/(p-2)} + C\Big(\frac{T}{R^p}\Big)^{n/p} \operatorname*{ess\,inf}_{Q} u^{\lambda/p},$$

where $\lambda = n(p-2) + p$ and $Q = B(x_0, 2R) \times (t_0 + T/2, t_0 + T)$.

In particular, our theorem applies to the equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{n} a_{ij}(x,t)\,\Big|\frac{\partial u}{\partial x_j}\Big|^{p-2}\,\frac{\partial u}{\partial x_j}\Big) = \frac{\partial u}{\partial t}, \tag{2.3}$$

where $p > 2$, a_{ij} is a bounded measurable function and

$$\mathcal{A}_0|\xi|^2 \le a_{ij}(x,t)\xi_i\xi_j \le \mathcal{A}_1|\xi|^2, \quad 0 < \mathcal{A}_0 < \mathcal{A}_1 < \infty,$$

for almost every (x,t) in $\mathbb{R}^n \times \mathbb{R}$ and every ξ in $\mathbb{R}^n$. The particular case with the identity matrix (a_{ij}) was studied in the monograph [Lio69] by J-L Lions.

The constant C in Theorem 2.2 is stable as $p \to 2$ in the sense that, if $2 < p < p_0$, then it may be chosen so that it depends only on p_0.

The version of the global Harnack estimate we present here is of the same type that D. Aronson and L. Caffarelli proved for the *porous medium equation*

$$\Delta u^m = \frac{\partial u}{\partial t}, \quad m > 1,$$

in [AC83]. The corresponding result for a more general porous medium equation is due to B. Dahlberg and K. Kenig in [DK84]. A good overview of techniques used in these articles can be found in recent monographs [Vaz06a] and [Vaz06b] by J. Vazquez. We also mention a forthcoming monograph [DK] by P. Daskalopoulos and C. Kenig. In [CL98], H. Choe and J. Lee applied the method developed in [DK84] to equation (2.3) with a symmetric matrix (a_{ij}) depending only on x.

For the weak solutions to the homogeneous equation

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = \frac{\partial u}{\partial t}$$

Theorem 2.2 was proved in [DH89]. The methods used in both [AC83] and [DH89] rely on the existence of a self–similar solution. It is not clear how to generalize such a method to more general equations.

In [DGV06], the authors have made a breakthrough by proving Harnack-type estimates for the bounded coefficient case. For continuous solutions, they prove a local intrinsic Harnack estimate

$$u(x,t) \leq C \inf_{y \in B(x,1)} u(y, t + Cu^{2-p}(x,t)),$$

where C is a constant independent of u. They have general growth bounds for $\mathcal{A}$ and they do not need the monotonicity of the operator. They also pay attention to the stability of constants as $p \to 2$ and use neither Hölder continuity of solutions nor the comparison principle. This gives a new proof and generalizes Moser's fundamental results for the case $p > 2$, see also the works of D. Aronson and J. Serrin in [AS67] and N. Trudinger in [Tru68]. In [DGV06], the proof uses extensively De Giorgi's estimates [DG57]. In our proof we have emphasized the roles of super- and subsolutions. This resembles the original idea of Moser. In the proof of the weak Harnack principle for supersolutions, we use the comparison principle and the existence of a weak solution to a Dirichlet problem with L^2-initial data and zero lateral boundary values. For the existence result we refer to [Hun01].

Our proof of the Harnack inequality requires local results proved by Moser's iteration technique. We present these in Section 4. Many of these fairly standard results can be found from [DiB93]. See also

[CL98], where similar techniques were applied in the global setting. In [DiB93], estimates are proved by using intrinsic De Giorgi's estimates. Our contribution here is that we prove the estimates either for sub- or supersolutions. Main ingredients of Section 4 for the proof of our main theorem are estimates for subsolutions with zero lateral boundary values. In particular, we show that, if we have the initial L^1-mass of one, then the diffused mass can be estimated by means of the growth constants of $\mathcal{A}$ – at least for a short time. To see this, we need to estimate the L^{p-1}-norm of the gradient and have a proper estimate for the essential supremum of a subsolution. We also show that a local counterpart for Theorem 2.2 cannot hold in a standard space–time cylinder. If $u \in L^p(0,T;W_0^{1,p}(B(x_0,R)))$ is a nonnegative subsolution with zero lateral boundary values, then there exists a constant C *independent of* u such that

$$\operatorname*{ess\,sup}_{B(x_0,R)\times(T/2,T)} u \le C\Big(\frac{R^p}{T}\Big)^{1/(p-2)}. \tag{2.4}$$

The estimate above is interesting as such.

The crucial step in the proof of Theorem 2.2 is to show that the supersolutions have a property called *expansion of positivity*. This phenomenon is studied in Section 5. Our method to show this is similar to the one used in [DGV06]. That expansion of positivity is the key estimate to prove the Harnack estimate has been known for a long time. The device of the family of expanding cylinders already appears in Krylov's and Safonov's work in [KS81], see also [GV06] and the references therein. We first assume that the initial data of a supersolution has positive values in a set that has positive Lebesgue measure and satisfies a finite-capacity-type constraint. Next, positive values of a supersolution may decay in time. We cancel the decay by simply multiplying the supersolution by the decay factor. It is then easy to see, after a proper change of time variable, that the result is a supersolution. After these steps, we can show that the positivity expands in time. The main real analytical tools for the proof can be found from [DiB93] or [DUV04]. The proof of the expansion of positivity described above uses neither the comparison principle nor the existence result.

In Section 6, we finally prove Theorem 2.2. We use estimates for a solution with zero lateral boundary values, the expansion of positivity, the comparison principle and the existence result to show the following *local* weak Harnack estimate. This is our second main result.

Theorem 2.5. *Let u be a nonnegative weak supersolution in $B(x_0,8R_0)\times(t_0,t_0+T_0)$. Then there exist constants $C_i = C_i(n,p,\text{structure of } \mathcal{A})$, $i=1,2$, such that, for almost every $t_0<t_1<t_0+T_0$, we have*

$$\fint_{B(x_0,R_0)} u(x,t_1)\,dx \le \Big(\frac{C_1R_0^p}{T_0+t_0-t_1}\Big)^{1/(p-2)} + C_2 \operatorname*{ess\,inf}_Q u,$$

where $Q = B(x_0, 4R_0) \times (t_1 + T/2, t_1 + T)$ *and*

$$T = \min\Big(T_0 + t_0 - t_1, C_1 R_0^p \big(\fint_{B(x_0,R_0)} u(x, t_1)\,dx\big)^{2-p}\Big).$$

Constants C_1 *and* C_2 *are stable as* $p \to 2$.

Theorem 2.2 is a consequence of the local result above. As seen by (2.4) the intrinsic time scale is needed in the local Harnack estimate. If the solution has large initial mass and zero lateral boundary values, then the boundary values make the solution decay very rapidly. The correct time scale for the decay is the one introduced in Theorem 2.5. In this sense, the estimate (2.4) can be seen as a counterexample to any better Harnack estimate. In particular, the local version of Theorem 2.2 is not true.

We want to point out that the only part where we need the comparison principle and the existence result is the proof of Theorem 2.5. Indeed, we show that if the supersolution u in $B(0, 6) \times (0, 1)$ has the initial mass of one in a ball $B(0, 1)$, then there exist constants C and T, depending only on the structural constants such that

$$\Big|\{x \in B(0, 2) : u(x, T) > \frac{1}{C}\}\Big| \geq \frac{\big|B(0, 2)\big|}{C}.$$

If one is able to prove this *without* the comparison principle and the existence result, one can generalize for $p > 2$ the weak Harnack principle in [Mos64].

2.1. **Notation.** Our notation is standard. We denote the ball with the radius R and center x as $B(x, R)$. The Lebesgue measure of the set Ω will be denoted as $\big|\Omega\big|$. We use the abbreviation

$$\fint_\Omega f\,d\nu = \frac{1}{\nu(\Omega)} \int_\Omega f\,d\nu$$

for the averaged integral with respect to measure ν. We use a symbol C to denote a constant. We use the notation $C = C(\cdot)$ to describe the arguments of the constant. In the proofs, the constant may vary from line to line, but the arguments are as in the statement of the theorem. By the notation $\Omega' \Subset \Omega$, we mean that the closure of an open bounded set Ω' belongs to Ω. By the Steklov average of the measurable function f, depending on $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we mean

$$f_h(x, t) = \frac{1}{h} \int_t^{t+h} f(x, s)\,ds.$$

We denote by $(f)_+$ the nonnegative part of f, i.e.

$$(f)_+ = \max(f, 0) = \frac{1}{2}(|f| + f).$$

By the parabolic boundary of the set $Q = \Omega \times (\tau_1, \tau_2)$ we mean

$$\partial_p Q = \big(\partial\Omega \times (\tau_1, \tau_2)\big) \cup \big(\bar{\Omega} \times \{\tau_1\}\big).$$

When we have the initial data problem, we denote the initial data of the solution u as u_0.

3. Weak solutions

We are now going to state our assumptions on $\mathcal{A}$, define the weak solutions and prove the comparison principle. We also introduce some interesting examples of weak solutions for different equations.

3.1. **Assumptions on the operator.** Let Ω_T be a domain in $\mathbb{R}^n \times \mathbb{R}$. We assume that $\mathcal{A} : \Omega_T \times \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a Caratheodory function, i.e. that $(x,t) \mapsto \mathcal{A}(x,t,u,F)$ is measurable for every (u,F) in $\mathbb{R} \times \mathbb{R}^n$ and $(u,F) \mapsto \mathcal{A}(x,t,u,F)$ is continuous for almost every $(x,t) \in \Omega_T$.

We assume that the growth conditions

$$\begin{aligned} \mathcal{A}(x,t,u,F) \cdot F &\geq \mathcal{A}_0 |F|^p, \\ |\mathcal{A}(x,t,u,F)| &\leq \mathcal{A}_1 |F|^{p-1} \end{aligned} \tag{3.1}$$

hold for $p > 2$, every $F \in \mathbb{R}^n$ and for almost every $(x,t,u) \in \Omega_T \times \mathbb{R}$. Here $\mathcal{A}_0$ and $\mathcal{A}_1$ are positive constants, which are called the structural constants of $\mathcal{A}$.

Furthermore, we assume that $\mathcal{A}$ is strictly monotone, i.e.

$$(\mathcal{A}(x,t,v,\nabla v) - \mathcal{A}(x,t,u,\nabla u)) \cdot (\nabla v - \nabla u) > 0 \tag{3.2}$$

for every $u, v \in W^{1,p}(\Omega)$ and $\nabla u \neq \nabla v$. The condition is enough to show the existence of the solutions to a Dirichlet problem with zero lateral boundary values and L^2-initial data, and, that the comparison principle holds. For more general monotonicity assumptions, we refer to [Hun01] on the existence of solutions.

3.2. **Parabolic Sobolev spaces.** Suppose that Ω is a domain in $\mathbb{R}^n$. The Sobolev space $W^{1,p}(\Omega)$ is defined to be the space of real-valued functions f such that $f \in L^p(\Omega)$ and the distributional first partial derivatives $\partial f / \partial x_i$, $i = 1, 2, \ldots, n$, exist in Ω and belong to $L^p(\Omega)$. We equip the Sobolev space with the norm

$$\|f\|_{1,p,\Omega} = \Big(\int_\Omega |f|^p \, dx \Big)^{1/p} + \Big(\int_\Omega |\nabla f|^p \, dx \Big)^{1/p}.$$

A function belongs to the local Sobolev space $W^{1,p}_{loc}(\Omega)$ if it belongs to $W^{1,p}(\Omega')$ for every $\Omega' \Subset \Omega$. The Sobolev space with zero boundary values $W^{1,p}_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the Sobolev norm.

We denote by $L^p(t_1, t_2; W^{1,p}(\Omega))$, $t_1 < t_2$, the space of functions such that, for almost every t, $t_1 < t < t_2$, the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$ and

$$\|u\|_{L^p(t_1,t_2;W^{1,p}(\Omega))} = \Big(\int_{t_1}^{t_2}\int_\Omega \big(|u(x,t)|^p + |\nabla u(x,t)|^p\big)\,dx\,dt\Big)^{1/p} < \infty.$$

Notice that the time derivative u_t is deliberately avoided. Definitions of spaces $L^p_{loc}(t_1,t_2;W^{1,p}_{loc}(\Omega))$ and $L^p(t_1,t_2;W^{1,p}_0(\Omega))$ are analogous.

3.3. **Definition of local weak solutions.** Let Ω be an open set in $\mathbb{R}^n$ and $\tau_1 < \tau_2$. A function

$$u \in L^p_{loc}(\tau_1,\tau_2\,;\,W^{1,p}_{loc}(\Omega))$$

is a weak local supersolution (subsolution) of

$$\operatorname{div}\mathcal{A}(x,t,u,\nabla u) = \frac{\partial u}{\partial t}$$

in $\Omega\times(\tau_1,\tau_2)$ if it satisfies the integral inequality

$$\begin{aligned}(3.3)\qquad &\int_{t_1}^{t_2}\int_\Omega \mathcal{A}(x,t,u,\nabla u)\cdot\nabla\eta\,dx\,dt - \int_{t_1}^{t_2}\int_\Omega u\frac{\partial\eta}{\partial t}\,dx\,dt\\ &\quad+\int_\Omega u(x,t_2)\eta(x,t_2)\,dx - \int_\Omega u(x,t_1)\eta(x,t_1)\,dx \ \geq (\leq)\,0\end{aligned}$$

for almost every $\tau_1 < t_1 < t_2 < \tau_2$ and for every nonnegative test function $\eta \in C_0^\infty(\Omega\times(\tau_1,\tau_2))$. Here $\mathcal{A}$ is as in Section 3.1. A function is a local weak solution if it is both a local weak sub- and supersolution. The boundary terms above are taken in the sense of limits

$$\int_\Omega u(x,t_1)\eta(x,t_1)\,dx = \lim_{\sigma\to 0}\frac{1}{\sigma}\int_{t_1}^{t_1+\sigma}\int_\Omega u(x,t)\eta(x,t)\,dx\,dt$$

and

$$\int_\Omega u(x,t_2)\eta(x,t_2)\,dx = \lim_{\sigma\to 0}\frac{1}{\sigma}\int_{t_2-\sigma}^{t_2}\int_\Omega u(x,t)\eta(x,t)\,dx\,dt.$$

3.4. **Initial values.** When we study an initial value problem we always mean L^2-initial values

$$u(\cdot,\tau_1) = u_0 \in L^2(\Omega)$$

unless stated otherwise. We demand that the initial values are attained in the following sense

$$(3.4)\qquad \lim_{h\to 0}\int_\Omega \Big(\frac{1}{h}\int_{\tau_1}^{\tau_1+h} u(x,t)\,dt - u_0(x)\Big)^2\,dx = 0.$$

The same way of obtaining the initial data has been used in, for instance, [DiB93]. We note that almost every $\tau_1 < t < \tau_2$ is a Lebesgue point i.e.

$$\lim_{h\to 0}\int_{\Omega'} \Big(\frac{1}{h}\int_t^{t+h} u(x,s)\,ds - u(x,t)\Big)^2\,dx = 0$$

for every $\Omega' \Subset \Omega$. Therefore, super- and subsolutions attain locally their own initial values for almost every $\tau_1 < t < \tau_2$.

3.5. **The comparison principle.** Under the assumptions on the operator, weak solutions and the initial data we have the following comparison principle.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\tau_1 < \tau_2$. Let u and $-v$ be weak supersolutions in $Q = \Omega \times (\tau_1, \tau_2)$. Suppose that*

$$\max(v-u,0) \in L^p(\tau_1,\tau_2;W_0^{1,p}(\Omega))$$

and

$$v_0 \le u_0 \quad \textit{almost everywhere,}$$

$u_0, v_0 \in L^2(\Omega)$. Then $u \ge v$ almost everywhere in $\Omega \times (\tau_1, \tau_2)$.

Proof. We start with an estimate

$$\begin{aligned}
&\big((v-u)_h(x,t)\big)_+ \\
&\quad=\frac{1}{2h}\Big(\Big|\int_t^{t+h}(v-u)(x,s)\,ds\Big| + \int_t^{t+h}(v-u)(x,s)\,ds\Big) \\
&\quad\le\frac{1}{h}\int_t^{t+h}\frac{1}{2}\big(|v-u|+v-u\big)(x,s)\,ds \\
&\quad=((v-u)_+)_h(x,t).
\end{aligned}$$

We set $w = u - v$. The assumption for $(v-u)_+$, together with the previous estimate, gives

$$(w_h)_+ \in L^p(\tau_1,\tau_2-h;W_0^{1,p}(\Omega)).$$

Now let $\tau_1 < t_1 < t_2 < \tau_2$ and take $h < \tau_2 - t_2$. By the weak formulation, we have for almost every $t_1 < t < t_2$ that

$$0 \le \int_\Omega \frac{\partial u_h}{\partial t}(x,t)\eta(x,t)\,dx + \int_\Omega \big(\mathcal{A}(x,t,u,\nabla u)\big)_h \cdot \nabla\eta(x,t)\,dx$$

and

$$0 \ge \int_\Omega \frac{\partial v_h}{\partial t}(x,t)\eta(x,t)\,dx + \int_\Omega \big(\mathcal{A}(x,t,v,\nabla v)\big)_h \cdot \nabla\eta(x,t)\,dx.$$

We then choose $\eta = (w_h)_+$ as the test function. It is admissible due to the approximation. This implies

$$\begin{aligned}
0 \ge &\int_\Omega \frac{\partial w_h}{\partial t}(w_h)_+\,dx \\
&+\int_\Omega \Big(\big(\mathcal{A}(x,t,v,\nabla v)\big)_h - \big(\mathcal{A}(x,t,u,\nabla u)\big)_h\Big)\cdot\nabla(w_h)_+\,dx
\end{aligned}$$

for almost every $t_1 < t < t_2$. We integrate the inequality over the interval (t_1, t_2). Integration by parts gives

$$\begin{aligned}
\int_{t_1}^{t_2}\int_\Omega \frac{\partial w_h}{\partial t}(w_h)_+\,dx\,dt &= \int_{t_1}^{t_2}\int_\Omega \frac{1}{2}\frac{\partial (w_h)_+^2}{\partial t}\,dx\,dt \\
&=\frac{1}{2}\int_\Omega (w_h)_+^2(x,t_2)\,dx - \frac{1}{2}\int_\Omega (w_h)_+^2(x,t_1)\,dx.
\end{aligned}$$

On the one hand, since $w_0 = 0$ almost everywhere, we have

$$\int_\Omega (w_h)_+^2(x,\tau_1)\,dx = \int_\Omega (w_h(x,\tau_1) - w_0(x))^2\,dx \to 0$$

as $h \to 0$ by the initial condition (3.4). On the other hand, the monotonicity of $\mathcal{A}$ implies

$$\begin{aligned}\lim_{h\to 0}\int_{\tau_1}^{t_2}\int_\Omega \Big(\big(\mathcal{A}(x,t,v,\nabla v)\big)_h - \big(\mathcal{A}(x,t,u,\nabla u)\big)_h\Big)\cdot \nabla (w_h)_+\,dx\,dt \\ = \int_{\tau_1}^{t_2}\int_\Omega \big(\mathcal{A}(x,t,v,\nabla v) - \mathcal{A}(x,t,u,\nabla u)\big)\cdot \nabla (v-u)_+\,dx\,dt \geq 0.\end{aligned}$$

Therefore, we have

$$\int_\Omega (v-u)_+^2(x,t_2)\,dx \leq 0$$

for almost every $\tau_1 < t_2 < \tau_2$. This leads to the result of the theorem.
□

3.6. **Scaling of solutions.** Solutions admit a scaling property. Let u be a local weak supersolution (subsolution) in

$$B(x_0,R_0)\times(t_0,t_0+T_0),$$

where $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$ and $T_0, R_0 > 0$. Then the scaled function

$$v(\xi,\tau) = \Big(\frac{T}{R^p}\Big)^{1/(p-2)} u(x_1 + R\xi, t_1 + T\tau)$$

is a local weak supersolution (subsolution) in

$$B\big(\frac{x_0-x_1}{R},\frac{R_0}{R}\big)\times\big(\frac{t_0-t_1}{T},\frac{t_0-t_1}{T}+\frac{T_0}{T}\big)$$

for every $R > 0$, $T > 0$, $x_1 \in \mathbb{R}^n$ and $t_1 \in \mathbb{R}$. The solution v is not necessarily a supersolution (subsolution) to the same equation as u but to an equation with a similar structure. To see this, we first let $x = x_1 + R\xi$ and $t = t_1 + T\tau$ so that

$$\nabla_\xi = R\nabla_x, \quad \frac{\partial}{\partial\tau} = T\frac{\partial}{\partial t}.$$

We then have

$$\begin{aligned}&|\mathcal{A}(x,t,u(x,t),(\nabla_x u)(x,t))| \\ &=|\mathcal{A}(x_1+R\xi,t_1+T\tau,\gamma v(\xi,\tau),\frac{\gamma}{R}(\nabla_\xi v)(\xi,\tau))| \\ &\leq \mathcal{A}_1\Big(\frac{\gamma}{R}\Big)^{p-1}|\nabla_\xi v|^{p-1},\end{aligned}$$

where

$$\gamma = \Big(\frac{R^p}{T}\Big)^{1/(p-2)}.$$

Similarly

$$\begin{aligned}\mathcal{A}(x_1+R\xi, t_1+T\tau, \gamma v(\xi,\tau), \frac{\gamma}{R}(\nabla_\xi v)(\xi,\tau)) \cdot \nabla_\xi v\\ \geq \mathcal{A}_0\left(\frac{\gamma}{R}\right)^{p-1}|\nabla_\xi v|^p.\end{aligned}$$

Thus, if we define

$$\widetilde{\mathcal{A}}(\xi,\tau,v,\nabla v) = \left(\frac{R}{\gamma}\right)^{p-1}\mathcal{A}(x_1+R\xi, t_1+T\tau, \gamma v(\xi,\tau), \frac{\gamma}{R}(\nabla_\xi v)(\xi,\tau)),$$

then $\widetilde{\mathcal{A}}$ has the same structural constants as $\mathcal{A}$. We change variables in the weak formulation and get

$$\begin{aligned}0 \geq (\leq) \int_{\tau_1}^{\tau_2}\int_\Omega \left(\frac{\gamma}{R}\right)^{p-1}\widetilde{\mathcal{A}}(\xi,\tau,v,\nabla v)\cdot\frac{1}{R}\nabla\eta\, TR^n\,d\xi\,d\tau\\ -\int_{\tau_1}^{\tau_2}\int_\Omega \gamma v\frac{1}{T}\frac{\partial\eta}{\partial\tau}\,TR^n\,d\xi\,d\tau\end{aligned}$$

where

$$\Omega\times(\tau_1,\tau_2) = B\big(\frac{R_0}{R},\frac{x_0-x_1}{R}\big)\times\big(\frac{t_0-t_1}{T},\frac{t_0-t_1}{T}+\frac{T_0}{T}\big)$$

and $\eta\in C_0^\infty(\Omega\times(\tau_1,\tau_2))$. We divide the inequality above by γ/T, use the definition of γ and obtain that v is indeed a weak supersolution (subsolution) in $\Omega\times(\tau_1,\tau_2)$.

3.7. **Examples of weak solutions.** We recall that some fascinating weak solutions are known. The following functions are classical solutions to the partial differential equation in the set

$$\mathbb{R}^n\times(0,\infty)\backslash\partial\{u(x,t)>0\}.$$

As initial data, they all have Dirac's delta function with a certain mass depending on n, p and the positive constant C given in the formulations. Initial values are attained in the sense of distributions.

The first example is the homogeneous equation

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = \frac{\partial u}{\partial t}.$$

It was found in [Bar52] that the Barenblatt solution

$$B_p(x,t) = t^{-n/\lambda}\Big(\frac{p-2}{p}\lambda^{\frac{1}{p-1}}\Big(C^{\frac{p}{p-1}}-\Big(\frac{|x|}{t^{1/\lambda}}\Big)^{\frac{p}{p-1}}\Big)\Big)_+^{\frac{p-1}{p-2}},$$

$\lambda = n(p-2)+p$, is a weak solution in $\mathbb{R}^n\times(0,\infty)$. The solution was used to describe the propagation of the heat after an explosion of a hydrogen bomb in the atmosphere.

The second equation introduced in [Lio69] is

$$\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2}\frac{\partial u}{\partial x_i}\Big) = \frac{\partial u}{\partial t}.$$

This time we have the weak solution

$$u(x,t) = t^{-n/\lambda}\Big(\frac{p-2}{p}\lambda^{\frac{1}{p-1}}\Big(C^{\frac{p}{p-1}} - \sum_{i=1}^{n}\Big(\frac{|x_i|}{t^{1/\lambda}}\Big)^{\frac{p}{p-1}}\Big)\Big)_+^{\frac{p-1}{p-2}},$$

$\lambda = n(p-2)+p$. As can seen, the free boundary $\partial\{u(x,t)>0\}$ is a ball with respect to $p/(p-1)$ norm of $\mathbb{R}^n$. Note that in the stationary case the equation is separable.

The third example is the equation

$$\operatorname{div}\big(|\nabla u|^{p-2}B(x)\nabla u\big) = \frac{\partial u}{\partial t},$$

where

$$B(x) = \Big(\frac{|Kx|}{|K^TKx|}\Big)^{p-2}(K^TK)^{-1}$$

and K is a positive (or negative) definite constant matrix. The equation has the weak solution

$$u(x,t) = t^{-n/\lambda}\Big(\frac{p-2}{p}\lambda^{\frac{1}{p-1}}\Big(C^{\frac{p}{p-1}} - \Big(\frac{|Kx|}{t^{1/\lambda}}\Big)^{\frac{p}{p-1}}\Big)\Big)_+^{\frac{p-1}{p-2}}.$$

The support of u is now an ellipsoid.

These solutions are sometimes referred to as self-similar solutions. The reason for this is that the scaled function

$$v(x,t) = t^{n/\lambda}u(t^{1/\lambda}x,t)$$

depends only on the spatial variable x. Generally speaking, it is an easy exercise to show that, if u is a weak solution in $\mathbb{R}^n\times(0,\infty)$ to the general equation, then v defined above is a weak solution to

$$\operatorname{div}\Big(\tilde{\mathcal{A}}(x,t,v,\nabla v) + \frac{1}{\lambda}vx\Big) = t\frac{\partial v}{\partial t}$$

in $\mathbb{R}^n\times(0,\infty)$, where

$$\tilde{\mathcal{A}}(x,t,v,\nabla v) = t^{(p-1)(n+1)/\lambda}\mathcal{A}(t^{1/\lambda}x,t,t^{-n/\lambda}v,t^{-(n+1)/\lambda}\nabla v)$$

satisfies the assumptions of Section 3.1.

4. Estimates for sub- and supersolutions

This section is devoted to technical results. We wish to present all the calculations in detail for possible future reference. In fact, one can find similar results from several articles (see for instance [BC04], [CL98]), but usually the results are said to hold only for solutions. We wish to make a clear distinction between super- and subsolutions.

We will first prove a Caccioppoli type of estimate in Lemma 4.1. We will then use it together with a parabolic Sobolev's estimate. This is the starting point of Moser's iteration technique. As a result, we obtain reverse Hölder inequalities. We then iterate these and obtain estimates for the essential supremum of the subsolution. The method was used for the first time in the elliptic case in [Mos61]. The full power of it for both elliptic and parabolic cases was later exploited in several papers, see, for instance, [AS67] and [Tru68].

Besides boundedness, we give a bound to the growth rate of the set $\{u(\cdot,t) > 0\} \subset \mathbb{R}^n$ when u is a subsolution. The result follows from estimates to subsolutions with zero initial data. Furthermore, we study subsolutions with zero lateral boundary values. As a result we obtain an estimate that shows our second main result, Theorem 2.5, cannot hold in the standard space–time cylinder $B(x_0,R)\times(t_0,t_0+R^p)$. Moreover, we prove estimates for the L^{p-1}-norm of the gradient.

For the sake of completeness, we also give an integrability estimate for supersolutions.

We have tried to take some extra care about the geometry. Bookkeeping of constants in proofs would be easier if we proved all lemmas in the space–time cylinder $B(0,1)\times(0,1)$. We could then scale results back to the cylinder $B(x_0,R)\times(t_0,t_0+T)$. We may, however, want to generalize the following results to equations like

$$\operatorname{div}\left(\mathcal{A}(x,t,u,\nabla u)\right) = \frac{\partial u}{\partial t} + \mathcal{B}(x,t,u,\nabla u)$$

with more general growth conditions. Consequently, the solutions, in general, do not have the scaling property anymore.

4.1. **Caccioppoli estimate.** A result stated in the following lemma is essentially a consequence of a substitution of a suitable test function in equation (3.3). More precisely, the test function depends on u. It is clear that the test function chosen this way does not necessarily belong to the correct test function space. The time derivative of u is, in general, only a generalized function. Nevertheless, we may regularize the solution by truncating it, and then use either Friedrich's mollifiers, Steklov averages, or some other suitable method. Together with the approximation argument this, justifies the choice of such a test function. The rigorous treatment can be found in, for instance, [DiB93].

Lemma 4.1. *Let $\varepsilon \in \mathbb{R}\backslash\{-1,0\}$ and $\delta > 0$. Suppose that $u \geq \delta$ is a subsolution (if $\varepsilon > 0$) or a supersolution (if $\varepsilon < 0$) in $\Omega\times(\tau_1,\tau_2)$. Then we have*

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_{\Omega}|\nabla u|^p u^{-1+\varepsilon}\varphi^p\,dx\,dt &+ \frac{p^2}{2\mathcal{A}_0|\varepsilon(1+\varepsilon)|}\operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_{\Omega}u^{1+\varepsilon}\varphi^p\,dx\\ &\leq\Big(\frac{\mathcal{A}_1 p}{\mathcal{A}_0|\varepsilon|}\Big)^p\int_{\tau_1}^{\tau_2}\int_{\Omega}u^{p-1+\varepsilon}|\nabla\varphi|^p\,dx\,dt\\ &\quad+\frac{p^2}{\mathcal{A}_0|\varepsilon(1+\varepsilon)|}\int_{\tau_1}^{\tau_2}\int_{\Omega}u^{1+\varepsilon}\Big|\frac{\partial\varphi}{\partial t}\Big|\varphi^{p-1}\,dx\,dt,\end{aligned}$$

where $\varphi \in C_0^\infty(\Omega\times(\tau_1,\tau_2))$.

Proof. From the weak formulation we have that supersolutions (subsolutions) satisfy the following regularized integral inequality

$$0 \le (\ge) \int_\Omega \frac{\partial u_h}{\partial t}(x,t)\eta(x,t)\,dx + \int_\Omega \big(\mathcal{A}(x,t,u,\nabla u)\big)_h \cdot \nabla\eta(x,t)\,dx$$

for almost every $\tau_1 < t < \tau_2 - h$. We choose formally the test function

$$\eta_h = u_h^\varepsilon \varphi^p,$$

where φ belongs to $C_0^\infty(\Omega \times (\tau_1,\tau_2))$. We denote $\eta_0 = u^\varepsilon\varphi^p$. We then choose $\tau_1 < t_1 < t_2 < \tau_2 - h$ and integrate the regularized equation. It follows from the properties of Steklov averages that

$$\begin{aligned}(4.2)\qquad &\int_{t_1}^{t_2}\int_\Omega \big(\mathcal{A}(x,t,u,\nabla u)\big)_h \cdot \nabla\eta_h(x,t)\,dx\,dt \\ &\qquad \to \int_{t_1}^{t_2}\int_\Omega \mathcal{A}(x,t,u,\nabla u)\cdot\nabla\eta_0(x,t)\,dx\,dt\end{aligned}$$

as $h \to 0$. Integration by parts gives

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \frac{\partial u_h}{\partial t}u_h^\varepsilon\varphi^p\,dx\,dt &= \frac{1}{1+\varepsilon}\int_{t_1}^{t_2}\int_\Omega \frac{\partial u_h^{1+\varepsilon}}{\partial t}\varphi^p\,dx\,dt \\ = -&\frac{1}{1+\varepsilon}\Big(\int_{t_1}^{t_2}\int_\Omega u_h^{1+\varepsilon}\frac{\partial\varphi^p}{\partial t}\,dx\,dt \\ &+ \int_\Omega u_h^{1+\varepsilon}(x,t_2)\varphi^p(x,t_2)\,dx - \int_\Omega u_h^{1+\varepsilon}(x,t_1)\varphi^p(x,t_1)\,dx\Big).\end{aligned}$$

Thus, for almost every $\tau_1 < t_1 < t_2 < \tau_2$ we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \frac{\partial u_h}{\partial t}u_h^\varepsilon\varphi^p\,dx\,dt &\to -\frac{1}{1+\varepsilon}\Big(\int_{t_1}^{t_2}\int_\Omega u^{1+\varepsilon}\frac{\partial\varphi^p}{\partial t}\,dx\,dt \\ &+ \int_\Omega u^{1+\varepsilon}(x,t_2)\varphi^p(x,t_2)\,dx - \int_\Omega u^{1+\varepsilon}(x,t_1)\varphi^p(x,t_1)\,dx\Big).\end{aligned}$$

as $h \to 0$. Furthermore, we have that

$$\nabla\eta_0 = \varepsilon u^{\varepsilon-1}\varphi^p\nabla u + pu^\varepsilon\varphi^{p-1}\nabla\varphi.$$

By substituting η_0 in (4.2), collecting terms and dividing the result by ε we obtain

$$\begin{aligned}(4.3)\qquad 0 \ge &\int_{t_1}^{t_2}\int_\Omega u^{-1+\varepsilon}\varphi^p\mathcal{A}(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt \\ &+ \frac{p}{\varepsilon}\int_{t_1}^{t_2}\int_\Omega u^\varepsilon\varphi^{p-1}\mathcal{A}(x,t,u,\nabla u)\cdot\nabla\varphi\,dx\,dt \\ &- \frac{1}{\varepsilon(1+\varepsilon)}\Big(p\int_{t_1}^{t_2}\int_\Omega u^{1+\varepsilon}\varphi^{p-1}\frac{\partial\varphi}{\partial t}\,dx\,dt \\ &+ \int_\Omega u^{1+\varepsilon}(x,t_2)\varphi^p(x,t_2)\,dx - \int_\Omega u^{1+\varepsilon}(x,t_1)\varphi^p(x,t_1)\,dx\Big).\end{aligned}$$

The growth conditions (3.1) imply

$$u^{\varepsilon-1}\varphi^p\mathcal{A}(x,t,u,\nabla u)\cdot\nabla u \ge \mathcal{A}_0|\nabla u|^p u^{\varepsilon-1}\varphi^p.$$

By Young's inequality we conclude

$$\begin{aligned}
&\frac{p}{\varepsilon} u^{\varepsilon} \varphi^{p-1} \mathcal{A}(x,t,u,\nabla u)\cdot \nabla\varphi \geq -\frac{\mathcal{A}_1 p}{|\varepsilon|}|\nabla u|^{p-1}\varphi^{p-1}|\nabla\varphi| u^{\varepsilon}\\
&\geq -\mathcal{A}_0\Big(|\nabla u| u^{(-1+\varepsilon)/p}\varphi\Big)^{p-1}\Big(\frac{\mathcal{A}_1 p}{\mathcal{A}_0|\varepsilon|}u^{(p-1+\varepsilon)/p}|\nabla\varphi|\Big)\\
&\geq -\frac{(p-1)}{p}\mathcal{A}_0|\nabla u|^p u^{-1+\varepsilon}\varphi^p - \frac{\mathcal{A}_0}{p}\Big(\frac{\mathcal{A}_1 p}{\mathcal{A}_0|\varepsilon|}\Big)^p u^{p-1+\varepsilon}\varphi^p.
\end{aligned}$$

Substituting these intermediate results in (4.3) we get

$$\begin{aligned}
0 \geq &\int_{t_1}^{t_2}\int_\Omega |\nabla u|^p u^{-1+\varepsilon}\varphi^p\,dx\,dt\\
&-\Big(\frac{\mathcal{A}_1 p}{\mathcal{A}_0|\varepsilon|}\Big)^p \int_{t_1}^{t_2}\int_\Omega u^{p-1+\varepsilon}|\nabla\varphi|^p\,dx\,dt\\
&-\frac{p}{\mathcal{A}_0\varepsilon(1+\varepsilon)}\Big(p\int_{t_1}^{t_2}\int_\Omega u^{1+\varepsilon}\varphi^{p-1}\frac{\partial\varphi}{\partial t}\,dx\,dt\\
&+\int_\Omega u^{1+\varepsilon}(x,t_2)\varphi^p(x,t_2)\,dx - \int_\Omega u^{1+\varepsilon}(x,t_1)\varphi^p(x,t_1)\,dx\Big).
\end{aligned} \tag{4.4}$$

We can now choose t_i such that

$$\int_\Omega u^{1+\varepsilon}(x,t_i)\varphi^p(x,t_i)\,dx\,dt \geq \frac{1}{2}\operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_\Omega u^{1+\varepsilon}\varphi^p dx,$$

$i=1,2$. If $\varepsilon(1+\varepsilon)>0$, we choose t_2 and let $t_1\to\tau_1$, and if $\varepsilon(1+\varepsilon)<0$, we choose t_1 and let $t_2\to\tau_2$. In both cases, we have the result of lemma. □

Remark 1. In the cases $\varepsilon<-1$ and $\varepsilon>0$, we only need to assume that the test function φ belongs to the space

$$W^{1,p}(\tau_1,\tau_2;W_0^{1,p}(\Omega))$$

and $\varphi(\cdot,\tau_1)=0$. Moreover, if $\varepsilon\geq 1$, the assumption $u\geq\delta$ may be replaced with the condition $u\geq 0$.

Remark 2. Suppose that the initial data u_0 at the time τ_1 is zero for almost every $x\in\Omega$. Let $\varepsilon=1$. Then the result of the lemma continues to hold with test functions $\varphi\in C_0^\infty(\Omega)$ since

$$\lim_{h\to 0}\int_\Omega u_h^2(x,\tau_1)\varphi^p(x)\,dx = 0.$$

This is because of the L^2-continuity defined in Section 3.4.

Remark 3. Suppose that

$$u\in L^p_{loc}(\tau_1,\tau_2;W_0^{1,p}(\Omega)).$$

is a nonnegative subsolution. We have shown in the proof of the comparison principle that

$$u_h\in W^{1,p}(t_1,t_2-h;W_0^{1,p}(\Omega)).$$

By an approximation and a truncation argument we may choose the test function $\eta = u_h^{\varepsilon}\varphi$ with $\varepsilon \geq 1$, where φ depends only on time. We then proceed as in the proof of Lemma 4.1. The Caccioppoli estimate becomes

$$\varepsilon(1+\varepsilon)\int_{\tau_1}^{\tau_2}\int_{\Omega}|\nabla v|^p u^{-1+\varepsilon}\varphi^p\,dx\,dt + \frac{p^2}{2\mathcal{A}_0}\operatorname*{ess\,sup}_{\tau_1<t<\tau_2}\int_{\Omega}u^{1+\varepsilon}\varphi^p\,dx \leq \frac{p^2}{\mathcal{A}_0}\int_{\tau_1}^{\tau_2}\int_{\Omega}u^{1+\varepsilon}\Big|\frac{\partial\varphi}{\partial t}\Big|\varphi^{p-1}\,dx\,dt,$$

where $\varphi \in C^\infty(t_1,t_2)\cap C([t_1,t_2])$, $\varphi \geq 0$ and $\varphi(\cdot,t_1)=0$.

4.2. **Parabolic Sobolev estimate.** We use the following Sobolev's imbedding theorem.

Theorem 4.5. *Let $1 < p, \kappa < \infty$ and suppose that*

$$u \in L^p\big(t_1,t_2;W_0^{1,p}(B(x_0,R))\big).$$

Then there exists a constant C=C(n,p) such that

$$\int_{t_1}^{t_2}\int_{B(x_0,R)}|u|^{\kappa p}\,dx\,dt \leq C\int_{t_1}^{t_2}\int_{B(x_0,R)}|\nabla u|^p\,dx\,dt\Big(\operatorname*{ess\,sup}_{t_1<t<t_2}\int_{B(x_0,R)}|u|^{(\kappa-1)n}\,dx\Big)^{p/n}.$$

For the proof we refer to [DiB93].

4.3. **Results for subsolutions.** We first show that subsolutions are bounded. The technical assumption

$$u \geq \Big(\frac{R_0^p}{T_0}\Big)^{1/(p-2)} > 0$$

is used in the sequel. The condition could be easily replaced with

$$u \geq \Big(\frac{\rho R_0^p}{T_0}\Big)^{1/(p-2)} > 0$$

with some $\rho > 0$. Then all the constants would depend also on ρ. We note that, for the heat equation ($p=2$), this condition reduces to $T_0 \simeq R_0^2$.

Lemma 4.6. *Let*

$$u \geq \Big(\frac{R_0^p}{T_0}\Big)^{1/(p-2)} > 0$$

be a subsolution in $B(x_0,R_0)\times(t_0-T_0,t_0)$, and let $\delta_0>0$. Then there exists a constant $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1,\delta_0)$ such that

$$\operatorname*{ess\,sup}_{B(x_0,\sigma R_0)\times(t_0-\sigma^p T_0,t_0)} u \leq \Big(\frac{T_0}{R_0^p}\frac{C}{(1-\sigma)^{n+p}}\fint_{t_0-T_0}^{t_0}\fint_{B(x_0,R_0)}u^{p-2+\delta}\,dx\,dt\Big)^{1/\delta}$$

for every $\delta \geq \delta_0$ and $0<\sigma<1$.

Proof. Let $\sigma R_0 \le s < S < R_0$. We set

$$r_0 = S, \quad r_j = (S - (S-s)(1-2^{-j})), \quad j = 0,1,2,\dots,$$

and denote

$$U_j = B_j \times \Gamma_j = B(x_0, r_j) \times (t_0 - (r_j/R_0)^p T_0, t_0),$$
$$U(S) = B(x_0, S) \times (t_0 - (S/R_0)^p T_0, t_0).$$

We choose test functions $\varphi_j \in C^\infty(U_j) \cap C(\overline{U}_j)$, $j = 0,1,2,\dots$, such that

$$0 \le \varphi_j \le 1, \qquad \varphi_j = 1 \quad \text{in} \quad U_{j+1}, \qquad \varphi_j = 0 \quad \text{on} \quad \partial_p U_j$$

and

$$|\nabla \varphi_j| \le \frac{C}{S-s} 2^j, \quad \Big|\frac{\partial \varphi_j}{\partial t}\Big| \le \frac{R_0^p}{T_0} \frac{C}{(S-s)^p} 2^{pj}.$$

The first step in the proof is to apply parabolic Sobolev's inequality, Theorem 4.5. With described test functions it implies

(4.7)
$$\begin{aligned}\int_{U_{j+1}} u^{\kappa\alpha}\,dx\,dt &\le C \int_{U_j} (u^{\alpha/p} \varphi_j^{\beta/p})^{\kappa p}\,dx\,dt \\ &\le \int_{U_j} |\nabla (u^{\alpha/p}\varphi_j^{\beta/p})|^p\,dx\,dt \Big(\operatorname*{ess\,sup}_{\Gamma_j} \int_{B_j} (u^{\alpha/p}\varphi_j^{\beta/p})^{(\kappa-1)n}\,dx\Big)^{p/n}\end{aligned}$$

for some $\alpha \in \mathbb{R}$, $\beta \ge p$ and $\kappa > 1$. We choose

$$\alpha = p - 1 + \varepsilon, \quad \kappa = 1 + \frac{p(1+\varepsilon)}{n(p-1+\varepsilon)}, \quad \beta = \frac{p(p-1+\varepsilon)}{1+\varepsilon},$$

where $\varepsilon \ge 1$. We then use Lemma 4.1 to estimate terms on the right hand side. First, we have

$$\begin{aligned}&\operatorname*{ess\,sup}_{\Gamma_j} \int_{B_j} (u^{\alpha/p}\varphi_j^{\beta/p})^{(\kappa-1)n}\,dx = \operatorname*{ess\,sup}_{\Gamma_j} \int_{B_j} u^{1+\varepsilon}\varphi_j^p\,dx \\ \le& C\Big(\frac{1+\varepsilon}{\varepsilon^{p-1}} \int_{U_j} u^{p-1+\varepsilon} |\nabla\varphi_j|^p\,dx\,dt + \int_{U_j} u^{1+\varepsilon}\Big|\frac{\partial\varphi_j}{\partial t}\Big|\varphi_j^{p-1}\,dx\,dt\Big).\end{aligned}$$

Similarly, we obtain the estimate

$$\begin{aligned}&\int_{U_j} |\nabla(u^{\alpha/p}\varphi_j^{\beta/p})|^p\,dx\,dt \\ \le& C\Big((p-1+\varepsilon)^p \int_{U_j} |\nabla u|^p u^{-1+\varepsilon}\varphi_j^p\,dx\,dt + \int_{U_j} |\nabla\varphi_j|^p u^{p-1+\varepsilon}\,dx\,dt\Big) \\ \le& C\big(1 + \frac{(p-1+\varepsilon)^p}{\varepsilon^p}\big) \int_{U_j} |\nabla\varphi_j|^p u^{p-1+\varepsilon}\,dx\,dt \\ &+ C\frac{(p-1+\varepsilon)^p}{\varepsilon(1+\varepsilon)} \int_{U_j} u^{1+\varepsilon}\Big|\frac{\partial\varphi_j}{\partial t}\Big|\varphi_j^{p-1}.\,dx\,dt\end{aligned}$$

Moreover, the assumption $u \ge (R_0^p/T_0)^{1/(p-2)}$ implies

$$u^{1+\varepsilon} \le \frac{T_0}{R_0^p} u^{p-1+\varepsilon}$$

and consequently we deduce from (4.7) that

$$\begin{aligned}
&\int_{U_{j+1}} u^{p-1+p/n+\gamma\varepsilon}\,dx\,dt\\
&\le\Big(C\varepsilon^{p-2}\int_{U_j}|\nabla\varphi_j|^p u^{p-1+\varepsilon}+u^{1+\varepsilon}\Big|\frac{\partial\varphi_j}{\partial t}\Big|\,dx\,dt\Big)^{\gamma}\\
&\le\Big(C\varepsilon^{p-2}\int_{U_j}|\nabla\varphi_j|^p u^{p-1+\varepsilon}+u^{p-1+\varepsilon}\frac{T_0}{R_0^p}\Big|\frac{\partial\varphi_j}{\partial t}\Big|\,dx\,dt\Big)^{\gamma}\\
&\le\Big(\frac{C2^{jp}\varepsilon^p}{(S-s)^p}\int_{U_j}u^{p-1+\varepsilon}\,dx\,dt\Big)^{\gamma},
\end{aligned}$$

where $\gamma=1+p/n$.

We then choose

$$\varepsilon_j=(1+\rho)\gamma^j-1,\quad \rho\ge 1,\quad \alpha_j=p-2+(1+\rho)\gamma^j,$$

$j=0,1,2,\ldots$, so that $p-1+p/n+\gamma\varepsilon_j=p-1+\varepsilon_{j+1}$. With this notation we have

$$\text{(4.8)}\qquad \fint_{U_{j+1}} u^{\alpha_{j+1}}\,dx\,dt\le\Big(\frac{|U_j|}{|U_{j+1}|^{1/\gamma}}\frac{C}{(S-s)^p}(2\gamma)^{jp}\fint_{U_j}u^{\alpha_j}\,dx\,dt\Big)^{\gamma}.$$

Next, a direct calculation gives

$$\prod_{k=0}^{j}(2\gamma)^{p(j-k)\gamma^k}=\Big(\prod_{k=0}^{j}(2\gamma)^{pk\gamma^{-k}}\Big)^{\gamma^j}\le\Big((2\gamma)^{p/(\gamma-1)^2}\Big)^{\gamma^{j+1}}$$

and

$$\sum_{k=1}^{j+1}\gamma^k=\frac{\gamma}{\gamma-1}\big(\gamma^{j+1}-1\big).$$

The calculation shows that the constants will stay bounded in the iteration below. We repeatedly use (4.8) and get

$$\begin{aligned}
&\Big(\fint_{U_{j+1}}u^{\alpha_{j+1}}\,dx\,dt\Big)^{1/\alpha_{j+1}}\\
&\le\Big(\frac{\big|U_j\big|}{\big|U_{j+1}\big|^{1/\gamma}}\frac{C(2\gamma)^{jp}}{(S-s)^p}\fint_{U_j}u^{\alpha_j}\,dx\,dt\Big)^{\gamma/\alpha_{j+1}}\\
&\le\Big(\frac{\big|U_j\big|}{\big|U_{j+1}\big|^{1/\gamma}}\frac{C(2\gamma)^{jp}}{(S-s)^p}\Big)^{\gamma/\alpha_{j+1}}\Big(\frac{\big|U_{j-1}\big|}{\big|U_j\big|^{1/\gamma}}\frac{C(2\gamma)^{(j-1)p}}{(S-s)^p}\Big)^{\gamma^2/\alpha_{j+1}}\\
&\quad\times\Big(\fint_{U_{j-1}}u^{\alpha_{j-1}}\,dx\,dt\Big)^{\gamma^2/\alpha_{j+1}}\\
&\le\big|U_{j+1}\big|^{-1/\alpha_{j+1}}\Big(\Big(\frac{C}{(S-s)^p}\Big)^{\gamma/(\gamma-1)}(2\gamma)^{p/(\gamma-1)^2}\big|U_0\big|\Big)^{\gamma^{j+1}/\alpha_{j+1}}\\
&\quad\times\Big(\fint_{U_0}u^{\alpha_0}\,dx\,dt\Big)^{\gamma^{j+1}/\alpha_{j+1}}.
\end{aligned}$$

Since

$$\frac{\gamma^j}{\alpha_j} = \frac{1}{\gamma^{-j}(p-1)+1+\rho} \to \frac{1}{1+\rho}$$

as $j \to \infty$, we get

$$\operatorname*{ess\,sup}_{U(s)} u \le \Big(\frac{C}{(S-s)^{n+p}} \int_{U(S)} u^{p-1+\rho}\,dx\,dt\Big)^{1/(1+\rho)}.$$

This proves the result for $\delta \ge 2$. We then choose $\rho = 1$. By Young's inequality, we obtain for every $2 > \delta \ge \min\{\delta_0, 1\}$ that

$$\begin{aligned}\operatorname*{ess\,sup}_{U(s)} u \le& \Big(\operatorname*{ess\,sup}_{U(S)} u^{2-\delta} \frac{C}{(S-s)^{n+p}} \int_{U(S)} u^{p-2+\delta}\,dx\,dt\Big)^{1/2}\\ \le& \frac{1}{2}\operatorname*{ess\,sup}_{U(S)} u + \Big(\frac{C}{(S-s)^{n+p}} \int_{U(R_0)} u^{p-2+\delta}\,dx\,dt\Big)^{1/\delta}.\end{aligned}$$

A standard iteration argument (see, for example, [Gia93] Lemma 5.1) implies the assertion of the lemma. □

We use the previous result to show the following lemma.

Lemma 4.9. *Let*

$$u \ge \Big(\frac{R_0^p}{T_0}\Big)^{1/(p-2)} > 0$$

be a subsolution in $B(x_0, R_0) \times (t_0 - T_0, t_0)$. *Then there exists a constant* $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ *such that*

$$\operatorname*{ess\,sup}_{B(x_0,R_0)\times(t_0-\sigma^p T_0,t_0)} u \le \frac{T_0}{R_0^p}\frac{C}{(1-\sigma)^{p(n+1)}}\Big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0} \fint_{B(x_0,R_0)} u\,dx\Big)^{p-1}.$$

for every $0 < \sigma < 1$.

Proof. Let $(1+\sigma)R_0/2 \le s < S < R_0$. We set

$$U(S) = B(x_0, S) \times (t_0 - (S/R_0)^p T_0, t_0).$$

We choose the test function $\varphi \in C^\infty(U(S)) \cap C(\overline{U}(S))$, such that

$$0 \le \varphi \le 1, \qquad \varphi_j = 1 \quad \text{in} \quad U(s), \qquad \varphi_j = 0 \quad \text{on} \quad \partial_p U(S)$$

and

$$|\nabla\varphi| \le \frac{C}{(S-s)}, \qquad \Big|\frac{\partial\varphi}{\partial t}\Big| \le \frac{R_0^p}{T_0}\frac{C}{(S-s)^p}.$$

We choose $\varepsilon = 1$, use (4.7) with

$$\alpha = p, \quad \kappa = 1 + \frac{1}{n}, \quad \beta = p$$

and obtain

$$\begin{aligned}&\int_{U(s)} u^{p+p/n}\,dx\,dt\\ &\quad\le \int_{U(S)} |\nabla(u\varphi)|^p\,dx\,dt\Big(\operatorname*{ess\,sup}_{t_0-(S/R_0)^p T_0<t<t_0} \int_{B(x_0,S)} u\varphi\,dx\Big)^{p/n}.\end{aligned}$$

Lemma 4.1 now implies that

$$\begin{aligned}\int_{U(S)} |\nabla(u\varphi)|^p \, dx\, dt \le & C \int_{U(S)} |\nabla\varphi|^p u^p + u^2 \Big|\frac{\partial\varphi}{\partial t}\Big| \, dx\, dt \\ \le & \frac{C}{(S-s)^p} \int_{U(S)} u^p \, dx\, dt.\end{aligned}$$

Here we have also used the assumption $u^{2-p} \le T_0/R_0^p$. Thus

$$\begin{aligned}&\fint_{U(s)} u^{p+p/n} \, dx\, dt \\ &\quad\le \frac{CR_0^p}{(S-s)^p} \Big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0} \fint_{B(x_0,R_0)} u \, dx\Big)^{p/n} \fint_{U(S)} u^p \, dx\, dt.\end{aligned}$$

Furthermore, by Hölder's and Young's inequalities, we get

$$\begin{aligned}\fint_{U(s)} u^p \, dx\, dt \le & \Big(\fint_{U(s)} u^{p+p/n} \, dx\, dt\Big)^{n/(n+1)} \\ \le & \Big(\frac{CR_0^p}{(S-s)^p} \big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0} \fint_{B(x_0,R_0)} u \, dx\big)^{p/n} \fint_{U(S)} u^p \, dx\, dt\Big)^{n/(n+1)} \\ \le & \frac{1}{2} \fint_{U(S)} u^p \, dx\, dt + \frac{CR_0^{np}}{(S-s)^{np}} \big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0} \fint_{B(x_0,R_0)} u \, dx\big)^p.\end{aligned}$$

Again, the same argument as in the end of the proof of Lemma 4.6 gives

$$\fint_{U((1+\sigma)R_0/2)} u^p \, dx\, dt \le \frac{C}{(1-\sigma)^{np}} \Big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0} \fint_{B(x_0,R_0)} u \, dx\Big)^p.$$

We now use Lemma 4.6 together with Hölder's inequality and arrive at

$$\begin{aligned}\operatorname*{ess\,sup}_{U(\sigma R_0)} u \le & \frac{T_0}{R_0^p} \frac{C}{(1-\sigma)^{n+p}} \fint_{U((1+\sigma)R_0/2)} u^{p-1} \, dx\, dt \\ \le & \frac{T_0}{R_0^p} \frac{C}{(1-\sigma)^{n+p}} \Big(\fint_{U((1+\sigma)R/2)} u^p \, dx\, dt\Big)^{(p-1)/p} \\ \le & \frac{T_0}{R_0^p} \frac{C}{(1-\sigma)^{p(n+1)}} \Big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0} \fint_{B(x_0,R_0)} u \, dx\Big)^{p-1},\end{aligned}$$

which proves the result. □

We are ready to prove the following theorem.

Theorem 4.10. *Let u be a nonnegative subsolution in $B(x_0,R_0)\times(t_0-T_0,t_0)$. Then there exists a constant $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ such that*

$$\operatorname*{ess\,sup}_{Q} u \le C\Big(\frac{R_0^p}{T_0}\Big)^{1/(p-2)} + C\frac{T_0}{R_0^p} \Big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0} \fint_{B(x_0,R_0)} u \, dx\Big)^{p-1},$$

where $Q = B(x_0,R_0/2) \times (t_0 - T_0/2, t_0)$.

Proof. If u is a nonnegative subsolution in $B(x_0,R_0) \times (t_0-T_0,t_0)$, then an application of the previous lemma to the subsolution $v = (R_0^p/T_0)^{1/(p-2)} + u$ gives the result. □

Furthermore, Theorem 4.10 implies

Corollary 4.11. *Let u be a nonnegative subsolution in $B(x_0, R_0) \times (t_0 - T_0, t_0)$. Suppose further that*

$$R = \Big(\operatorname*{ess\,sup}_{t_0 - T_0 < t < t_0} \int_{B(x_0,R_0)} u\,dx \Big)^{(p-2)/\lambda} T_0^{1/\lambda} \le R_0, \quad \lambda = n(p-2) + p.$$

Then there exists a constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ such that

$$\operatorname*{ess\,sup}_{Q} u \le C T_0^{-n/\lambda} \Big(\operatorname*{ess\,sup}_{t_0 - T_0 < t < t_0} \int_{B(x_0,R_0)} u\,dx \Big)^{p/\lambda},$$

where $Q = B(x_0, R/2) \times (t_0 - T_0/2, t_0)$.

Proof. With the indicated choice of R we apply Theorem 4.10 and obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_{Q} u \le & C \Big(\Big(\frac{R^p}{T_0}\Big)^{1/(p-2)} + \frac{T_0}{R^p} \Big(\operatorname*{ess\,sup}_{t_0 - T_0 < t < t_0} \fint_{B(x_0,R)} u\,dx \Big)^{p-1} \Big), \\
\le & C \Big(\Big(\frac{R^p}{T_0}\Big)^{1/(p-2)} + \frac{T_0}{R^{p+n(p-1)}} \Big(\operatorname*{ess\,sup}_{t_0 - T < t < t_0} \int_{B(x_0,R_0)} u\,dx \Big)^{p-1} \Big) \\
= & C T_0^{-n/\lambda} \Big(\operatorname*{ess\,sup}_{t_0 - T_0 < t < t_0} \int_{B(x_0,R_0)} u\,dx \Big)^{p/\lambda}
\end{aligned}$$

as the straightforward calculation shows. □

4.4. **Zero initial data.** We next prove results when some additional information is known about a subsolution. The first lemmas deal with the case when the subsolution has zero initial data.

Lemma 4.12. *Let u be a nonnegative subsolution in $B(0, 1) \times (0, 1)$. Let u have zero initial data, i.e.*

$$u_0 = 0 \quad \textit{almost everywhere in} \quad B(0, 1).$$

Then there exists a constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ such that, if

$$\operatorname*{ess\,sup}_{0<t<1} \int_{B(0,1)} u\,dx \le C,$$

then

$$u = 0 \quad \textit{almost everywhere in} \quad B(0, 1/2) \times (0, 1).$$

Remark. As the proof shows, the constant tends to zero as $p \to 2$. This is of course correct, because in the case of the heat equation the diffusion is infinitely fast and the positivity expands immediately to the whole ball $B(0, 1)$.

Proof. Recall that, because of the second remark after the proof of Lemma 4.1, we may choose the test function $\eta = u\varphi^p$, where $\varphi \in$

$C_0^\infty(B(0,1))$, $0 \le \varphi \le 1$. The Caccioppoli estimate Lemma 4.1 becomes

$$\int_0^1 \int_{B(0,1)} |\nabla u|^p \varphi^p \, dx \, dt + \frac{p^2}{2\mathcal{A}_0} \operatorname*{ess\,sup}_{0<t<1} \int_\Omega u^2 \varphi^p \, dx \le \Big(\frac{\mathcal{A}_1 p}{\mathcal{A}_0}\Big)^p \int_0^1 \int_\Omega u^p |\nabla \varphi|^p \, dx \, dt.$$

We first use the estimate 4.7 as at the proof of Lemma 4.6 and obtain

$$\int_0^1 \int_{B(0,1)} u^{p+2p/n} \varphi^p \, dx \, dt \le \Big(C \int_0^1 \int_{B(0,1)} u^p |\nabla \varphi|^p \, dx \, dt\Big)^{1+p/n}. \tag{4.13}$$

Notice that in the calculation of this we do not need the assumption $u \ge 1$, since φ does not depend on time. Similarly, we follow the proof of Lemma 4.9 and get

$$\int_0^1 \int_{B(0,1)} u^{p+p/n} \varphi^p \, dx \, dt \le C \Big(\operatorname*{ess\,sup}_{0<t<1} \int_{B(0,1)} u \, dx\Big)^{p/n} \int_0^1 \int_{B(0,1)} |\nabla \varphi|^p u^p \, dx \, dt.$$

We test this with cut-off functions $\varphi \in C_0^\infty(B(0,S))$ such that

$$\varphi = 1 \quad \text{in} \quad B(0,s) \quad \text{and} \quad |\nabla \varphi| \le \frac{C}{S-s},$$

$3/4 \le s < S < 1$. As a result we obtain

$$\int_0^1 \int_{B(0,s)} u^{p+p/n} \, dx \, dt \le \frac{C}{(S-s)^p} \Big(\operatorname*{ess\,sup}_{0<t<1} \int_{B(0,1)} u \, dx\Big)^{p/n} \int_0^1 \int_{B(0,S)} u^p \, dx \, dt.$$

We apply Young's inequality and get

$$\int_0^1 \int_{B(0,s)} u^{p+p/n} \, dx \, dt \le \frac{1}{2} \int_0^1 \int_{B(0,S)} u^{p+p/n} \, dx \, dt + \frac{C}{(S-s)^{(n+1)p}} \Big(\operatorname*{ess\,sup}_{0<t<1} \int_{B(0,1)} u \, dx\Big)^{p+p/n}.$$

Therefore, an iteration gives

$$\int_0^1 \int_{B(0,3/4)} u^{p+p/n} \, dx \, dt \le C \Big(\operatorname*{ess\,sup}_{0<t<1} \int_{B(0,1)} u \, dx\Big)^{p+p/n}. \tag{4.14}$$

We now set $R_j = 1/2 + 2^{-2-j}$ and take $\varphi_j \in C_0^\infty(B(0,R_j))$ such that

$$\varphi_j = 1 \quad \text{in} \quad B(0,R_{j+1}), \quad |\nabla \varphi_j| \le C 2^j.$$

We denote

$$M_j = \int_0^1 \fint_{B(0,R_j)} u^{p+p/n} \, dx \, dt.$$

We apply (4.13) together with Hölder's inequality and obtain

$$\begin{aligned} M_j \leq& \Big(\int_0^1 \int_{B(0,1)} u^{p+2p/n} \varphi_j^p \, dx \, dt \Big)^{(n+1)/(n+2)} \\ \leq& C^j \Big(\int_0^1 \fint_{B(0,R_{j-1})} u^p \, dx \, dt \Big)^{(1+p/n)(n+1)/(n+2)} \leq C^j M_{j-1}^{(1+p/n)n/(n+2)}. \end{aligned}$$

We rewrite this as

$$M_j \leq C^j M_{j-1}^{1+(p-2)/(n+2)}.$$

It is then a standard iteration argument (see, for example, [DiB93]) that states $M_j \to 0$ as $j \to \infty$ if

$$M_0 \leq C^{-((n+2)/(p-2))^2}.$$

By (4.14), this is the case, provided that C is small enough in the assumptions. □

Lemma 4.15. *Let u be a nonnegative subsolution in $B(x_0, 2R_0) \times (t_0, t_0 + T_0)$. Suppose further that*

$$u_0 = 0 \quad \text{almost everywhere in} \quad B(x_0, 2R_0).$$

Then there exists a constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ such that

$$u(x,t) = 0$$

for almost every $x \in B(x_0, R_0)$ and

$$t_0 < t < \min\big(T_0, R_0^p\big(C \operatorname*{ess\,sup}_{t_0<t<T_0} \fint_{B(x_0,2R_0)} u \, dx\big)^{2-p}\big).$$

Proof. We denote

$$T = \min\big(T_0, R_0^p\big(C \operatorname*{ess\,sup}_{t_0<t<T_0} \fint_{B(x_0,2R_0)} u \, dx\big)^{2-p}\big)$$

and define the scaled solution

$$v(x,t) = \frac{T^{1/(p-2)}}{R_0^p} u(x_1 + R_0 x, t_0 + Tt),$$

where $x_1 \in B(x_0, R_0)$. It is clear that

$$\operatorname*{ess\,sup}_{0<t<1} \fint_{B(0,1)} v \, dx = \frac{T^{1/(p-2)}}{R_0^p} \operatorname*{ess\,sup}_{0<t<T} \fint_{B(x_1,R_0)} u \, dx \leq \frac{1}{C}.$$

Therefore, for C large enough, v satisfies the assumptions of Lemma 4.12. Hence,

$$u(x,t) = 0$$

for almost every $(x,t) \in B(x_0, R_0) \times (t_0, t_0 + T)$. □

It immediately follows that, if subsolution's initial data is zero, then it has a representative that attains its initial values continuously.

Corollary 4.16. *Let u be a nonnegative subsolution in $B(x_0, R_0) \times (t_0, t_0 + T_0)$. Suppose further that*

$$u_0 = 0 \quad \textit{almost everywhere in} \quad B(x_0, R_0).$$

Then

$$\operatorname*{ess\,sup}_{x \in B(x_0,R)\times(t_0,t)} u \to 0 \quad \textit{as} \quad t \to t_0.$$

for every $R < R_0$.

Remark. Using the result above one can show that, if the initial data is continuous, then there is a representative of the weak solution that attains its initial values continuously. Moreover, on the initial boundary, the refined solution has the same modulus of continuity as the initial data.

We may now use the full power of Moser's iteration technique and obtain the following lemma.

Lemma 4.17. *Let u be a nonnegative subsolution in $B(x_0, R_0)\times(t_0, t_0+T_0)$. Let $\delta_0 > 0$. Suppose further that*

$$u_0 = 0 \quad \textit{in} \quad B(x_0, R_0).$$

Then there exists a constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1, \delta_0)$ such that

$$\operatorname*{ess\,sup}_{Q} u \le \Big(\frac{C}{((1-\sigma)R_0)^{n+p}} \int_{t_0}^{t_0+T_0} \int_{B(x_0,R_0)} u^{p-2+\delta}\, dx\, dt\Big)^{1/\delta},$$

$\delta \ge \delta_0$, and

$$\operatorname*{ess\,sup}_{Q} u \le \frac{T_0}{R_0^p} \frac{C}{(1-\sigma)^{p(n+1)}} \Big(\operatorname*{ess\,sup}_{t_0<t<t_0+T_0} \fint_{B(x_0,R_0)} u\, dx\Big)^{p-1}$$

for every $1/2 \le \sigma < 1$, where $Q = B(x_0, \sigma R_0) \times (t_0, t_0 + T_0)$.

Proof. Both results are essentially consequences of the fact that we can, in the light of Corollary 4.16, take formally test functions

$$\eta = u^\varepsilon \varphi,$$

where $\varepsilon \ge 1$ and $\varphi \in C_0^\infty(\Omega)$. We can then repeat the proof of Caccioppoli's estimate Lemma 4.1 with $\partial\varphi/\partial t = 0$. We set

$$r_0 = S, \quad r_j = S - (S-s)(1-2^{-j}), \quad j = 0, 1, 2, \ldots$$

and denote

$$U_j = B_j \times \Gamma = B(x_0, r_j) \times (t_0, t_0 + T_0).$$

We choose test functions $\varphi_j \in C_0^\infty(B_j)$, $j = 0, 1, 2, \ldots$, such that

$$0 \le \varphi_j \le 1, \quad \varphi_j = 1 \quad \text{in} \quad B_{j+1}$$

and

$$|\nabla \varphi_j| \le \frac{C 2^j}{S - s}.$$

The first of the results follows then by repeating the proof of Lemma 4.6. The second result follows analogously. □

4.5. **Zero lateral boundary values.** We now prove results for the Dirichlet problem with zero boundary values on the lateral boundary.

Lemma 4.18. *Let*

$$u \in L^p(t_0 - T_0, t_0; W_0^{1,p}(B(x_0, R_0))).$$

be a nonnegative subsolution in $B(x_0,R_0) \times (t_0 - T_0, t_0)$. *Let* $\delta_0 > 0$. *Then there exists a constant* $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1, \delta_0)$ *such that*

$$\begin{aligned}&\operatorname*{ess\,sup}_{B(x_0,R_0)\times(t_0-\sigma T_0,t_0)} u \\ &\quad\le \Big(\big(\frac{C}{(1-\sigma)T_0}\big)^{(n+p)/p} \int_{t_0-T_0}^{t_0} \int_{B(x_0,R_0)} u^{1-\lambda/p+\delta}\, dx\, dt \Big)^{1/\delta},\end{aligned}$$

$\lambda = n(p-2)+p$, *for every* $\delta \ge -1 + \lambda/p + \delta_0$ *and* $0 < \sigma < 1$.

Proof. We have remarked after Lemma 4.1 that, because of the zero lateral boundary values, we may choose a test function φ that depends only on time when $\varepsilon \ge 1$. We set

$$\sigma_j = \sigma - (\sigma - \sigma')(1 - 2^{-j})$$

and

$$U_j = B(x_0, R_0) \times \Gamma_j = B(x_0, R_0) \times (t_0 - \sigma_j T_0, t_0),$$

$$U(\sigma) = B(x_0, R_0) \times (t_0 - \sigma T_0, t_0).$$

We choose $\varphi_j \in C^\infty(\bar{\Gamma}_j)$ such that

$$0 \le \varphi_j \le 1, \quad \varphi_j = 1 \quad \text{in} \quad \Gamma_{j+1}, \quad \nabla \varphi_j = 0, \quad \Big|\frac{\partial \varphi_j}{\partial t}\Big| \le \frac{1}{T_0}\frac{C2^j}{1-\sigma}.$$

We proceed as in the proof of Lemma 4.6 and get

$$\begin{aligned}&\int_{U_{j+1}} u^{p-1+p/n+\gamma\varepsilon}\, dx\, dt \\ &\quad\le \Big(C\varepsilon^{p-2} \int_{U_j} |\nabla \varphi_j|^p u^{p-1+\varepsilon} + u^{1+\varepsilon} \Big|\frac{\partial \varphi_j}{\partial t}\Big|\, dx\, dt \Big)^{\gamma} \\ &\quad\le \Big(\frac{1}{T_0}\frac{C2^j \varepsilon^{p-2}}{1-\sigma} \int_{U_j} u^{1+\varepsilon}\, dx\, dt \Big)^{\gamma},\end{aligned}$$

where $\gamma = 1 + p/n$. This time we choose

$$\varepsilon_j = \Big(1 + \frac{p-2+p/n}{p/n} + \rho\Big)\gamma^j - \frac{p-2+p/n}{p/n} = \Big(1 + \frac{\lambda}{p} + \rho\Big)\gamma^j - \frac{\lambda}{p},$$

where $\lambda = n(p-2)+p$ and $\rho \ge 0$. Let $\alpha_j = 1 + \varepsilon_j$. With this notation we have

$$⨍_{U_{j+1}} u^{\alpha_{j+1}}\, dx\, dt \le \Big(\frac{|U_j|}{|U_{j+1}|^{1/\gamma}} \frac{1}{T_0} \frac{C}{1-\sigma} (2\gamma)^{jp} ⨍_{U_j} u^{\alpha_j}\, dx\, dt \Big)^{\gamma}.$$

Iterating this estimate as in the proof of Lemma 4.6, we get

$$\Big(\fint_{U_{j+1}} u^{\alpha_{j+1}}\,dx\,dt\Big)^{1/\alpha_{j+1}} \le \big|U_{j+1}\big|^{1/\alpha_{j+1}}\Big(\big(\frac{C}{T_0(1-\sigma)}\big)^{\gamma/(\gamma-1)} \times (2\gamma)^{p\gamma/(\gamma-1)^2}\big|U_0\big|\fint_{U_0} u^{\alpha_0}\,dx\,dt\Big)^{\gamma^{j+1}/\alpha_{j+1}}.$$

Since

$$\frac{\gamma^j}{\alpha_j} = \frac{1}{\gamma^{-j}(1-\lambda/p)+1+\lambda/p+\rho} \to \frac{1}{1+\lambda/p+\rho}$$

as $j\to\infty$, we have

$$\operatorname*{ess\,sup}_{U(\sigma')} u \le \Big(\big(\frac{C}{(\sigma-\sigma')T_0}\big)^{(n+p)/p}\int_{U(\sigma)} u^{2+\rho}\,dx\,dt\Big)^{1/(1+\lambda/p+\rho)}.$$

This proves the result for $\delta \ge 1+\lambda/p$. Let then $\rho = 0$. By Young's inequality, we get for every $1+\lambda/p > \delta \ge -1+\lambda/p+\min\{\delta_0,1\}$ that

$$\begin{aligned}\operatorname*{ess\,sup}_{U(\sigma')} u \le& \Big(\operatorname*{ess\,sup}_{U(\sigma)} u^{1+\lambda/p-\delta} \\ &\times \big(\frac{C}{(\sigma-\sigma')T_0}\big)^{(n+p)/p}\int_{U(\sigma)} u^{1-\lambda/p+\delta}\,dx\,dt\Big)^{1/(1+\lambda/p)} \\ \le& \frac{1}{2}\operatorname*{ess\,sup}_{U(\sigma)} u \\ &+ \Big(\big(\frac{C}{(\sigma-\sigma')T_0}\big)^{(n+p)/p}\int_{U(\sigma)} u^{1-\lambda/p+\delta}\,dx\,dt\Big)^{1/\delta}\end{aligned}$$

and the result follows by the iteration. □

By choosing $\delta = \lambda/p$, we conclude that the result of Corollary 4.11 holds up to the boundary.

Corollary 4.19. *Under the assumptions of Lemma 4.18, we have*

$$\operatorname*{ess\,sup}_{B(x_0,R_0)\times(t_0-\sigma T_0,t_0)} u \le \Big(\frac{C}{(1-\sigma)}\Big)^{(n+p)/\lambda}\Big(\frac{R_0^p}{T_0}\Big)^{n/\lambda}\Big(\operatorname*{ess\,sup}_{t_0-T_0<t<t_0}\fint_{B(x_0,R_0)} u\,dx\Big)^{p/\lambda}.$$

Moreover, the proof of Lemma 4.18 gives more. We may indeed pick $\delta = \lambda/p - 1 > 0$. Then the constant diverges as $p \to 2$, but the right hand side does not depend on u. We state the result as a theorem. It gives, for the subsolutions with zero lateral boundary values, an upper bound for the decay of the essential supremum.

Theorem 4.20. *Let*

$$u \in L^p(t_0,t_0+T_0;W_0^{1,p}(B(x_0,R_0))).$$

be a nonnegative subsolution in $B(x_0,R_0)\times(t_0,t_0+T_0)$*. Then there exists a constant* $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ *such that*

$$\fint_{B(x_0,R_0)} u(x,t_0+T)\,dx \le C\Big(\frac{R_0^p}{T}\Big)^{1/(p-2)}$$

and

$$\operatorname*{ess\,sup}_{B(x_0,R_0)\times(t_0+T/2,t_0+T)} u \leq C\Big(\frac{R_0^p}{T}\Big)^{1/(p-2)}$$

for almost every $0 < T < T_0$.

The theorem above gives a uniform integrability estimate for subsolutions with zero lateral boundary values. If the subsolution exists up to the time T_0, then the L^δ-norm with small δ is bounded with a constant that depends only on T_0. In particular, the constant does not depend on the initial data.

Corollary 4.21. *Let* $0 < \delta < p-2$ *and let*

$$u \in L^p(t_0, t_0+T_0; W_0^{1,p}(B(x_0,R_0)))$$

be a nonnegative subsolution in $B(x_0,R_0)\times(t_0,t_0+T_0)$. *Then there is a constant* $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1,\delta,T_0)$ *such that*

$$\int_{t_0}^{t_0+T_0}\int_{B(x_0,R_0)} u^\delta\,dx\,dt \leq C.$$

The next lemma gives an estimate to a subsolution with zero lateral boundary values and zero initial data in an annulus near the boundary. It is needed in the proof of Theorem 2.5. We give only the outline of the proof since we have already introduced all the technical steps needed.

Lemma 4.22. *Let*

$$u \in L^p(t_0, t_0+T_0; W_0^{1,p}(B(x_0,R_0))).$$

be a nonnegative subsolution in $B(x_0,R_0)\times(t_0,t_0+T_0)$. *Suppose further that*

$$u_0 = 0 \quad \textit{almost everywhere in} \quad B(x_0,R_0)\backslash B(x_0,R_1),$$

$R_1 < R_0$. *Then there exists a constant* $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ *such that*

$$\operatorname*{ess\,sup}_{Q_\sigma} u \leq \frac{T}{(R_0-R_1)^p}\frac{C}{\sigma^{p(n+1)}}\Big(\operatorname*{ess\,sup}_{t_0<t<t_0+T_0} \fint_{B(x_0,R_0)\backslash B(x_0,R_1)} u\,dx\Big)^{p-1},$$

where $0 < T < T_0$, $0 < \sigma < 1$ *and*

$$Q_\sigma = B(x_0,R_0)\backslash B(x_0,R_1+\sigma(R_0-R_1))\times(t_0,t_0+T).$$

Proof. The proof is based on the fact that we may take test functions of the type

$$\eta = u\max(u^{\varepsilon-1},k)\varphi^p, \quad \varepsilon \geq 1, \quad k > 0$$

where $\varphi \in C^\infty(B(x_0,R_0)\backslash B(x_0,R_1))$ and φ vanishes on the boundary of $B(x_0,R_1)$ and is equal to 1 in $B(x_0,R_1+\sigma(R_0-R_1))$. By Corollary 4.16, a representative of u attains its initial values continuously in the support of φ. It is then clear that

$$\eta \in L^p(t_0,t_0+T_0;W_0^{1,p}(B(x_0,R_0)))$$

and $\eta(\cdot,t_0) = 0$ almost everywhere. The result now follows as in the proof of Lemma 4.17. $\square$

4.6. **Gradient estimates.** We next prove estimates for the gradient. Similar proof for solutions can be found in [DiB93] and in the global setting in [CL98]. The main ingredient of the proof is Corollary 4.11.

Lemma 4.23. *Let u be a nonnegative subsolution in $B(x_0, 3R_0) \times (0, T_0)$ and let $\varepsilon > 0$. We define*

$$N = \operatorname*{ess\,sup}_{0<t<T_0} \int_{B(x_0,3R_0)} u \, dx$$

and suppose that

$$T_0 < R_0^\lambda N^{2-p}, \quad \lambda = n(p-2) + p.$$

Then there exists a positive constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1, \varepsilon)$ such that for every $0 < \tau < T_0$

$$\int_0^\tau \int_{B(x_0,R_0)} |\nabla u|^{p-1} \, dx \, dt \le C\big(N^{p-2}\tau\big)^{1/\lambda} N,$$

where $p > 2 + \varepsilon$.

Proof. First, we use Hölder's inequality and get

$$\begin{aligned}
&\int_0^\tau \int_{B(x_0,R_0)} |\nabla u|^{p-1} \, dx \, dt \\
&= \int_0^\tau \int_{B(x_0,R_0)} |\nabla u|^{p-1} u^{1/2} u^{-1/2} t^{1/(2p)} t^{-1/(2p)} \, dx \, dt \\
&\le \Big(\int_0^\tau \int_{B(x_0,R_0)} u^{p/2} \, t^{-1/2} \, dx \, dt \Big)^{1/p} \\
&\quad \times \Big(\int_0^\tau \int_{B(x_0,R_0)} u^{-p/(2p-2)} t^{1/(2p-2)} |\nabla u|^p \, dx \, dt \Big)^{(p-1)/p}.
\end{aligned}$$

From Corollary 4.11 we have that

$$\operatorname*{ess\,sup}_{B(y,R_0/2)\times(t/2,t)} u \le C t^{-n/\lambda} N^{p/\lambda}$$

for every $0 < t < \tau$ and $y \in B(x_0, 2R_0)$ since

$$T_0 < R_0^\lambda N^{2-p}$$

by the assumptions. This implies

$$\begin{aligned}
\Psi_1(\tau) &= \int_0^\tau \int_{B(x_0,R_0)} u^{p/2} t^{-1/2} \, dx \, dt \\
&\le \int_0^\tau t^{-1/2} \|u\|_{L^\infty}^{p/2-1} \int_{B(x_0,R_0)} u \, dx \, dt \\
&\le C N^{1+p(p-2)/(2\lambda)} \int_0^\tau t^{-1/2-n(p-2)/(2\lambda)} \, dt = C N^{1+p(p-2)/(2\lambda)} \tau^{p/(2\lambda)}.
\end{aligned}$$

Next, we choose the test function

$$\eta = u^{(p-2)/(2p-2)} t^{1/(2p-2)} \varphi^p,$$

where $\varphi \in C_0^\infty(B(x_0, 2R_0))$ depends only on the spatial variable and has the properties $\varphi = 1$ in $B(x_0, R_0)$ and $|\nabla\varphi| \le C/R_0$. We substitute the chosen test function in the weak formulation and, in a way similar to that of the derivation of (4.4) in the proof of Lemma 4.1, obtain

$$\begin{aligned}\Psi_2(\tau) =& \int_0^\tau \int_{B(x_0,R_0)} |\nabla v|^p u^{-1+(p-2)/(2p-2)} t^{1/(2p-2)}\,dx\,dt\\ \le& C\Big(\frac{p-1}{p-2}\Big)^p \int_0^\tau \int_{B(x_0,2R_0)} u^{p-1+(p-2)/(2p-2)} t^{1/(2p-2)} |\nabla\varphi|^p\,dx\,dt\\ &+ C\frac{p-1}{p-2}\int_0^\tau \int_{B(x_0,2R_0)} u^{1+(p-2)/(2p-2)} t^{1/(2p-2)}\,dx\,dt\\ &- C\frac{p-1}{p-2}\int_{B(x_0,2R_0)} u^{1+(p-2)/(2p-2)}(x,\tau)\tau^{1/(2p-2)}\varphi^p(x)\,dx\end{aligned}$$

and finally

$$\begin{aligned}\Psi_2(\tau) \le& \frac{C(\varepsilon)}{R_0^p}\int_0^\tau \int_{B(x_0,2R_0)} u^{p-1+(p-2)/(2p-2)} t^{1/(2p-2)}\,dx\,dt\\ &+ C(\varepsilon)\int_0^\tau \int_{B(x_0,2R_0)} u^{1+(p-2)/(2p-2)} t^{-1+1/(2p-2)}\,dx\,dt.\end{aligned}$$

In the previous calculation we have noted the singularity of the constant as $p \to 2$. We treat the terms separately. For the first term we have

$$\begin{aligned}&\int_0^\tau \int_{B(x_0,2R_0)} u^{p-1+(p-2)/(2p-2)} t^{1/(2p-2)}\,dx\,dt\\ &\le \int_0^\tau \|u\|_{L^\infty}^{(p-2)(1+1/(2p-2))} t^{1/(2p-2)} \int_{B(x_0,2R_0)} u\,dx\,dt\\ &\le CN^{1+(p/\lambda)(p-2)(1+1/(2p-2))}\int_0^\tau t^{-(n(p-2)/\lambda)(1+1/(2p-2))+1/(2p-2)}\,dt\\ &= CN^{1+(p/\lambda)(p-2)(1+1/(2p-2))}\tau^{p(2p-1)/(\lambda(2p-2))}\\ &= C\big(N^{p-2}\tau\big)^{p/\lambda} N^{1+(p/\lambda)(p-2)/(2p-2)}\tau^{p/(\lambda(2p-2))}\\ &\le CR_0^p N^{1+(p/\lambda)(p-2)/(2p-2)}\tau^{p/(\lambda(2p-2))}\end{aligned}$$

by the assumptions. The second term can be estimated similarly

$$\begin{aligned}&\int_0^\tau \int_{B(x_0,2R_0)} u^{1+(p-2)/(2p-2)} t^{-1+1/(2p-2)}\,dx\,dt\\ &\le CN^{1+(p/\lambda)(p-2)/(2p-2)}\int_0^\tau t^{-(n/\lambda)(p-2)/(2p-2)-1+1/(2p-2)}\,dt\\ &= CN^{1+(p/\lambda)(p-2)/(2p-2)}\tau^{p/(\lambda(2p-2))}.\end{aligned}$$

Altogether, we have the estimate

$$\Psi_2(\tau) \le CN^{1+(p/\lambda)(p-2)/(2p-2)}\tau^{p/(\lambda(2p-2))}.$$

Combining the results for Ψ_1 and Ψ_2 we get

$$\int_0^\tau \int_{B(x_0,R_0)} |\nabla u|^{p-1}\,dx\,dt \le \Psi_1(\tau)^{1/p}\Psi_2(\tau)^{1-1/p} \le C\big(N^{p-2}\tau\big)^{1/\lambda}N,$$

which proves the result. $\square$

We still need a "stable" version of the previous lemma as $p \to 2$. We have not been able to modify the previous proof to achieve this. Nevertheless, we prove a weaker version of the result, which will be enough in the proof of Theorem 2.5.

Lemma 4.24. *Let u be a nonnegative subsolution in $B(x_0, 3R_0) \times (0, T_0)$. We define*

$$N = \operatorname*{ess\,sup}_{0<t<T_0} \int_{B(x_0,3R_0)} u\, dx$$

and suppose that

$$N/R_0^n \geq 1, \quad \textit{and} \quad T_0 < R_0^\lambda N^{2-p}, \quad \lambda = n(p-2)+p.$$

Then there exists a positive constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ such that for every $0 < \tau < T_0$ holds

$$\int_0^\tau \int_{B(x_0,R_0)} |\nabla u|^{p-1}\, dx\, dt \leq C M^{p-1} \tau^{(1-1/M)/\lambda} N^{1+(p-2+p/(nM))/\lambda}$$

for every $M \geq 3$. Moreover, the constant C is stable as $p \to 2$.

Proof. As in the proof of Lemma 4.23, we obtain by Corollary 4.11 that

$$\operatorname*{ess\,sup}_{B(y,R_0/2)\times(t/2,t)} u \leq C t^{-n/\lambda} N^{p/\lambda}$$

for any $0 < t < T_0$ and $y \in B(x_0, 2R_0)$. By the assumptions

$$t^{-n/\lambda} N^{p/\lambda} > T_0^{-n/\lambda} N^{p/\lambda} > N/R_0^n \geq 1.$$

Thus we have

$$\operatorname*{ess\,sup}_{B(y,R_0/2)\times(t/2,t)} (u+1) \leq C t^{-n/\lambda} N^{p/\lambda}$$

for any $0 < t < T_0$ and $y \in B(x_0, 2R_0)$. Now denote $v = u+1$. We then use Hölder's inequality and get

$$\begin{aligned}
&\int_0^\tau \int_{B(x_0,R_0)} |\nabla v|^{p-1}\, dx\, dt \\
&\leq \int_0^\tau \int_{B(x_0,R_0)} |\nabla v|^{p-1} v^{(\varepsilon-1)(p-1)/p} v^{(p-1)/p} t^{\delta(p-1)/p} t^{-\delta(p-1)/p}\, dx\, dt \\
&\leq \Big(\int_0^\tau \int_{B(x_0,R_0)} v^{p-1}\, t^{-(p-1)\delta}\, dx\, dt \Big)^{1/p} \\
&\quad \times \Big(\int_0^\tau \int_{B(x_0,R)} v^{-1+\varepsilon} t^\delta |\nabla v|^p\, dx\, dt \Big)^{(p-1)/p}
\end{aligned}$$

for some $\delta, \varepsilon > 0$. Here we used the fact that $v \geq 1$. We estimate the first term as

$$\begin{aligned}\Psi_1(\tau) =& \int_0^\tau \int_{B(x_0,R_0)} v^{p-1} t^{-(p-1)\delta}\, dx\, dt \\ \leq& \int_0^\tau t^{-(p-1)\delta} \|v\|_{L^\infty}^{p-2} \int_{B(x_0,2R_0)} v\, dx\, dt \\ \leq& CN^{1+p(p-2)/\lambda} \int_0^\tau t^{-(p-1)\delta - n(p-2)/\lambda}\, dt \\ =& C(\delta)\tau^{p/\lambda-(p-1)\delta} N^{1+p(p-2)/\lambda}\end{aligned}$$

provided that $\delta < p/(p-1)\lambda$. Next, we choose the test function $\eta = v^\varepsilon t^\delta \varphi^p$, where $\varphi \in C_0^\infty(B(x_0, 2R_0))$ depends only on the spatial variable and has properties $\varphi = 1$ in $B(x_0, R_0)$ and $|\nabla\varphi| \leq C/R_0$. We insert the chosen test function in the weak formulation and follow the proof of Lemma 4.1. We conclude

$$\begin{aligned}\Psi_2(\tau) =& \int_0^\tau \int_{B(x_0,R_0)} |\nabla v|^p v^{-1+\varepsilon} t^\delta\, dx\, dt \\ \leq& \frac{C}{\varepsilon^p} \int_0^\tau \int_{B(x_0,2R_0)} v^{p-1+\varepsilon} t^\delta |\nabla\varphi|^p\, dx\, dt \\ &+ \frac{C\delta}{\varepsilon} \int_0^\tau \int_{B(x_0,2R_0)} v^{1+\varepsilon} t^{\delta-1} \varphi^p\, dx\, dt \\ &- \frac{C}{\varepsilon} \int_{B(x_0,2R_0)} v^{1+\varepsilon}(x,\tau)\tau^\delta \varphi^p\, dx\end{aligned}$$

and consequently

$$\begin{aligned}\Psi_2(\tau) \leq& \frac{C(\varepsilon)}{R_0^p} \int_0^\tau \int_{B(x_0,2R_0)} v^{p-1+\varepsilon} t^\delta\, dx\, dt \\ &+ C(\varepsilon) \int_0^\tau \int_{B(x_0,2R_0)} v^{1+\varepsilon} t^{-1+\delta}\, dx\, dt.\end{aligned}$$

We require that $\delta - (n/\lambda)\varepsilon > 0$ and estimate

$$\begin{aligned}&\int_0^\tau \int_{B(x_0,2R_0)} v^{1+\varepsilon} t^{-1+\delta}\, dx\, dt \\ &\leq CN^{\varepsilon p/\lambda} \int_0^\tau t^{-(n/\lambda)\varepsilon-1+\delta} \int_{B(x_0,2R_0)} v\, dx\, dt = C\tau^{\delta-(n/\lambda)\varepsilon} N^{1+\varepsilon p/\lambda}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^\tau \int_{B(x_0,2R_0)} v^{p-1+\varepsilon} t^\delta\, dx\, dt \\ \leq& CN^{(p-2+\varepsilon)p/\lambda} \int_0^\tau t^{(n/\lambda)(p-2+\varepsilon)} t^\delta \int_{B(x_0,2R_0)} v\, dx\, dt \\ \leq& CN^{1+(p-2+\varepsilon)p/\lambda} \int_0^\tau t^{-(n/\lambda)(p-2+\varepsilon)+\delta}\, dt \\ =& C\tau^{p/\lambda+\delta-(n/\lambda)\varepsilon} N^{1+(p-2+\varepsilon)p/\lambda}\end{aligned}$$

since $p/\lambda + \delta - (n/\lambda)\varepsilon > 0$. Altogether, we have the estimate

$$\begin{aligned}\Psi_2(\tau) \le & C\tau^{\delta-(n/\lambda)\varepsilon} N^{1+\varepsilon p/\lambda}\Big(1 + \big(\tau N^{p-2} R_0^{-\lambda}\big)^{p/\lambda}\Big) \\ \le & C\tau^{\delta-(n/\lambda)\varepsilon} N^{1+\varepsilon p/\lambda}.\end{aligned}$$

Combining the results for Ψ_1 and Ψ_2 we get

$$\int_0^\tau \int_{B(x_0,R_0)} |\nabla v|^{p-1}\,dx\,dt \ \le\ \Psi_1(\tau)^{1/p}\Psi_2(\tau)^{1-1/p} \le C\tau^{(1-n(p-1)\varepsilon/p)/\lambda}.$$

We choose

$$\delta = \frac{p}{2\lambda(p-1)} \quad \text{and} \quad \varepsilon = \frac{p}{n(p-1)}\frac{1}{M},$$

$M \ge 3$, and the result follows. □

Remark. The condition $N/R_0^n \ge 1$ can easily be replaced with the condition $N/R_0^n \ge \mu = \mu(n, p, \mathcal{A}_0, \mathcal{A}_1) > 0$.

4.7. **Results for supersolutions.** We prove a counterpart of Lemma 4.9 for supersolutions.

Lemma 4.25. *Let*

$$u \ge \Big(\frac{R_0^p}{T_0}\Big)^{1/(p-2)} > 0$$

be a supersolution in $B(x_0, R_0)\times(t_0, t_0+T_0)$ *and let* $0 < \varepsilon < 1/2$. *Then there exists a constant* $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ *such that*

$$\Big(⨍_{t_0}^{t_0+\sigma^p T_0} ⨍_{B(x_0,\sigma R_0)} u^q\,dx\,dt\Big)^{1/q} \le \frac{C}{\varepsilon^n(1-\sigma)^n} \operatorname*{ess\,sup}_{t_0<t<t_0+T_0} ⨍_{B(x_0,R_0)} u\,dx,$$

where $q \le p - 1 + p/n - \varepsilon$ *and* $0 < \sigma < 1$.

Proof. We set

$$U(S) = B(x_0, S) \times (t_0, t_0 + (S/R_0)^p T_0).$$

for $S < R_0$. We choose test functions $\varphi \in C_0^\infty(U(S))$ such that

$$0 \le \varphi \le 1, \quad \varphi = 1 \quad \text{in} \quad U(s), \quad \sigma R_0 \le s < S$$

and

$$|\nabla\varphi| \le \frac{C}{S-s}, \quad \Big|\frac{\partial\varphi}{\partial t}\Big| \le \frac{R_0^p}{T_0}\frac{C}{(S-s)^p}.$$

We then choose for $0 < \varepsilon < 1/2$

$$\alpha = p - 1 - \varepsilon, \quad \kappa = 1 + \frac{p}{n(p-1-\varepsilon)}, \quad \beta = p$$

so that (4.7) becomes

$$\begin{aligned}\int_{U(s)} u^{p-1+p/n-\varepsilon}\,dx\,dt \le & C\int_{U(S)} |\nabla(u^{(p-1-\varepsilon)/p}\varphi)|^p\,dx\,dt \\ & \times\Big(\operatorname*{ess\,sup}_{t_0<t<t_0+T_0} \int_{B(x_0,R_0)} u\,dx\Big)^{p/n}.\end{aligned}$$

Lemma 4.1 implies that

$$\begin{aligned}
&\int_{U(S)} |\nabla(u^{(p-1-\varepsilon)/p}\varphi)|^p \, dx\, dt \\
&\le \frac{C}{\varepsilon^p} \int_{U(S)} |\nabla\varphi|^p u^{p-1-\varepsilon} \, dx\, dt \\
&\quad + \frac{C}{(1-\varepsilon)\varepsilon} \int_{U(S)} u^{1-\varepsilon} \Big| \frac{\partial \varphi^p}{\partial t} \Big| \, dx\, dt \\
&\le \frac{C}{\varepsilon^p} \int_{U(S)} \Big(|\nabla\varphi|^p + \frac{T_0}{R_0^p} \Big| \frac{\partial\varphi}{\partial t} \Big| \varphi^{p-1} \Big) u^{p-1+\varepsilon} \, dx\, dt,
\end{aligned}$$

where we have used the assumption $u^{2-p} \le T_0/R_0^p$. Hence, we obtain

$$\begin{aligned}
\int_{U(s)} u^{p-1+p/n-\varepsilon} \, dx\, dt \le & \Big(\operatorname*{ess\,sup}_{t_0<t<t_0+T_0} \int_{B(x_0,R_0)} u \, dx \Big)^{p/n} \\
& \times \frac{C}{(S-s)^p} \frac{1}{|U(s)|\varepsilon^p} \int_{U(S)} u^{p-1+\varepsilon} \, dx\, dt.
\end{aligned}$$

We apply Young's inequality and get

$$\begin{aligned}
⨍_{U(s)} u^{p-1+p/n-\varepsilon} \, dx\, dt &\le \frac{1}{2} ⨍_{U(S)} u^{p-1+p/n-\varepsilon} \, dx\, dt \\
&+ \Big(\frac{C R_0^p}{(S-s)^p} \frac{1}{\varepsilon^p} \big(\operatorname*{ess\,sup}_{t_0<t<t_0+T_0} ⨍_{B(x_0,R_0)} u \, dx \big)^{p/n} \Big)^{(p-1+p/n-\varepsilon)n/p}.
\end{aligned}$$

An iteration argument together with Hölder's inequality now implies the result. □

5. Expansion of positivity

A fundamental property of a solution to a diffusion equation is that the information, or, in other words, positivity, spreads as time evolves. Showing this is an essential step in proving Harnack's inequality for solutions to evolutionary p-Laplace type of equations. The simple description of the phenomenon is the following. Let us start with positive initial data supported in a ball of radius one. We then let time evolve and after some time the values of the supersolution are positive in a ball of radius two. We show that expansion of positivity occurs for supersolutions.

We consider a Dirichlet problem with the nonnegative initial data $u_0 \in L^2(B(0,R))$. We suppose that there exists a weak nonnegative supersolution u to

$$\begin{cases} \operatorname{div}\big(\mathbf{A}(x,t,u,\nabla u)\big) = \dfrac{\partial u}{\partial t}, & \text{in} \quad B(0,R) \times (0,\infty), \\ u(\cdot,0) = u_0 \end{cases}, \tag{5.1}$$

where $R \geq 2$. Here $\mathcal{A}$ is as in Section 3.1. We assume that the initial data u_0 has the following property: The set

$$V = \{x \in B(0,2) : u_0(x) \geq N\}, \quad N > 0,$$

contains a set $U \subset B(0,1)$ such that

$$|U| \geq \nu_U |B(0,1)|, \quad U \subset V,$$

for some $\nu_U > 0$. Moreover, we assume that there is a Sobolev function $w \in W_0^{1,p}(B(0,2))$ such that $w \geq 1$ for almost every $x \in U$ and $w = 0$ for almost every $x \in B(0,2)\backslash V$. We call

$$C_{\text{cap}} = \int_{B(0,2)} |\nabla w|^p \, dx.$$

The main result of this section is the following theorem. We have presented its proof so that the use of the comparison principle has been avoided. In the spirit of [DGV06], we pay some extra attention to the stability of constants as $p \to 2$.

Theorem 5.2. *Let u be a weak supersolution to (5.1) with $R \geq 4$. Then the positivity expands, i.e. there exist constants T^* and μ^*, depending only on n, p, $\mathcal{A}_0$, $\mathcal{A}_1$ (=structural constants), ν_U and C_{cap} such that*

$$\operatorname*{ess\,inf}_{Q} u \geq N\mu^*,$$

where $Q = B(0,2) \times (N^{2-p}T^*/2, N^{2-p}T^*)$. Moreover, constants T^* and μ^* are stable, as $p \to 2$.

Remark. In Theorem 5.2, we only need to assume that u is a supersolution in $B(0,4) \times (0, N^{2-p}T^*)$.

A crucial step in the proof of Theorem 5.2 is that, if u is a weak supersolution, then also a time–scaled function is a supersolution.

Lemma 5.3. *Suppose that u is a weak supersolution to (5.1). Then*

$$v(x,t) = \frac{\exp(\kappa t)}{N} u\big(x, \frac{\exp(\kappa(p-2)t) - 1}{\kappa(p-2)N^{p-2}}\big),$$

$\kappa, N > 0$, is a supersolution - not to the same equation, but to an equation with a similar structure.

Proof. We first define

$$g(t) = \frac{1}{N}(1 + \kappa(p-2)N^{p-2}t)^{1/(p-2)}, \quad g'(t) = \kappa g^{3-p}(t).$$

Note that $g(t) \to \exp(\kappa t)/N$ and $(\kappa(p-2)t) - 1)/\kappa(p-2)N^{p-2} \to t$ as $p \to 2$. We formally calculate with

$$\eta(x,t) = g(t)\varphi(x,t), \quad \tilde{v}(x,t) = g(t)u(x,t).$$

This implies

$$\nabla u = \frac{1}{g}\nabla \tilde{v}, \quad u\eta = \tilde{v}\varphi$$

and

$$
\begin{aligned}
u\frac{\partial \eta}{\partial t} &= \frac{\kappa N^{p-2} u g \varphi}{1+\kappa(p-2)N^{p-2}t} + ug\frac{\partial \varphi}{\partial t} \\
&= \frac{\kappa N^{p-2}}{1+\kappa(p-2)N^{p-2}t}\tilde{v}\varphi + \tilde{v}\frac{\partial \varphi}{\partial t}.
\end{aligned}
$$

Furthermore, if we define

$$
\tilde{\mathcal{A}}(x,t,\tilde{v},\nabla\tilde{v}) = g^{p-1}\mathcal{A}\big(x,t,\tilde{v}/g,\nabla\tilde{v}/g\big)
$$

we obtain

$$
\begin{aligned}
&\tilde{\mathcal{A}}(x,t,\tilde{v},\nabla\tilde{v})\cdot\nabla\tilde{v} \geq \mathcal{A}_0|\nabla\tilde{v}|^p, \\
&|\tilde{\mathcal{A}}(x,t,\tilde{v},\nabla\tilde{v})| \leq \mathcal{A}_1|\nabla\tilde{v}|^{p-1}
\end{aligned}
$$

and

$$
\mathcal{A}(x,t,u,\nabla u)\cdot\nabla\eta = \frac{N^{p-2}}{1+\kappa(p-2)N^{p-2}t}\tilde{\mathcal{A}}(x,t,\tilde{v},\nabla\tilde{v})\cdot\nabla\varphi.
$$

We change the time variable

$$
\tau = \tau(t) = \frac{1}{\kappa(p-2)}\ln(1+\kappa(p-2)N^{p-2}t)
$$

so that

$$
\frac{d\tau}{dt} = \frac{N^{p-2}}{1+\kappa(p-2)t}.
$$

Rewritten weak formulation for v and

$$
\psi(x,t) = \varphi\big(x, \frac{\exp(\kappa(p-2)\tau)-1}{\kappa(p-2)N^{p-2}}\big)
$$

is

$$
\begin{aligned}
0 \leq &\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^n}\tilde{\mathcal{A}}\big(x,\frac{\exp(\kappa(p-2)\tau)-1}{\kappa(p-2)N^{p-2}},v,\nabla v\big)\cdot\nabla\psi\,dxd\tau \\
&-\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^n} v\frac{\partial\psi}{\partial\tau}\,dxd\tau - \kappa\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^n} v\psi\,dxd\tau. \\
&+\int_{\mathbb{R}^n} v(x,\tau_2)\psi(x,\tau_2)\,dx - \int_{\mathbb{R}^n} v(x,\tau_1)\psi(x,\tau_1)\,dx.
\end{aligned}
$$

Thus, v is a supersolution. □

Furthermore we need the following energy estimate. We have proved this in Lemma 4.1. Notice that, since u is a supersolution, then $k-u$ is a subsolution and, hence, also $\max(k-u,0)$ is a subsolution. The same result can also be found from [DiB93].

Lemma 5.4. *Let u be a supersolution in $B(x_0,R)\times(t_1,t_2)$ and $k\in\mathbb{R}$. Then there exists a constant C depending only on structural constants* *such that*

$$\int_{t_1}^{t_2}\int_{B(x_0,R)} |\nabla((u-k)_-\varphi)|^p\,dx\,dt + \operatorname*{ess\,sup}_{t_1<t<t_2}\int_{B(x_0,R)} (u-k)_-^2\varphi^p\,dx \\ \le C\int_{t_1}^{t_2}\int_{B(x_0,R)} (u-k)_-^p|\nabla\varphi|^p + (u-k)_-^2\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,dx\,dt,$$

where $(u-k)_- = -\min(0,u-k)$ *and* φ *is a nonnegative smooth test function with a property that* $\varphi(\cdot,t_1)=0$ *and* $\varphi(\cdot,t)\in C_0^\infty(B(x_0,R))$ *for every* $t_1\le t\le t_2$.

We denote

$$\{v>k\} = \big\{x\in B(x_0,R)\ :\ v(x)>k\big\}.$$

Definitions of $\{v<l\}$ and $\{l<v<k\}$ are similar.

The following two Sobolev estimates are proved in [DiB93].

Lemma 5.5. *Let* $v\in W^{1,p}(B(x_0,R))$. *Then there exists a constant* $C=C(n,p)$ *such that*

$$(k-l)\big|\{v<l\}\big| \ \le \frac{CR^{n+1}}{\big|\{v>k\}\big|}\int_{\{l<v<k\}} |\nabla v|\,dx$$

for every $k>l$.

Lemma 5.6. *Let* Ω *be a domain and* $v\in L^p(t_1,t_2;W_0^{1,p}(\Omega))$. *Then there exists a constant* $C=C(n,p)$ *such that*

$$\int_{t_1}^{t_2}\int_\Omega |v|^p\,dx\,dt \le C\big|\{(x,t)\in\Omega\times(t_1,t_2): |v|>0\}\big|^{p/(n+p)} \\ \times\Big(\int_{t_1}^{t_2}\int_\Omega |\nabla v|^p\,dx\,dt + \operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega |v|^p\,dx\Big).$$

Throughout this section, we denote by C and μ constants that depend only on structural constants or, in other words, n, p, $\mathcal{A}_0$ and $\mathcal{A}_1$, and the constants ν_U and C_{cap} of the initial condition.

The first lemma is a straightforward consequence of a choice of a proper test function in the weak formulation.

Lemma 5.7. *Let* u *be a weak supersolution of (5.1) with* $R\ge 4$. *Then there exist constants* $\kappa=\kappa(p,\mathcal{A}_0,\mathcal{A}_1,\nu_U,C_{\mathrm{cap}})$ *and* $\nu=\nu(n,\nu_U)$ *such that for*

$$g(t) = \frac{1}{N}(1+\kappa(p-2)N^{p-2}t)^{1/(p-2)}$$

we have

$$|\{x\in B(0,3): u(x,t)g(t)>1\}| \ge \nu\big|B(0,3)\big|$$

for almost every $t>0$.

We have an immediate corollary of Lemma 5.7 for a supersolution v defined in Lemma 5.3.

Corollary 5.8. *Let u be a weak supersolution of (5.1) with $R \geq 4$ and*

$$v(x,t) = \frac{\exp(\kappa t)}{N} u\big(x, \frac{\exp(\kappa(p-2)t) - 1}{\kappa(p-2)N^{p-2}}\big),$$

where κ is as in Lemma 5.7. Let also ν be as in Lemma 5.7. Then v is a supersolution and

$$\big|\{x \in B(0,3) : v(x,t) > 1\}\big| \geq \nu \big|B(0,3)\big|$$

for almost every $t > 0$.

Proof of Lemma 5.7. First, we define the supersolution $v = u + \varepsilon$, $\varepsilon > 0$. We choose the test function $\eta = v^{1-p} w^p$, where w is the function related to the initial data defined at the beginning of this section. The function η is an admissible test function by arguments used to prove Lemma 4.1 and by approximation.

We now study the initial values of function v. Since

$$\int_{B(0,2)} (v_h(x,0) - v_0(x))^2 \, dx \to 0$$

as $h \to 0$ we have that $v_h(x,0) \to v_0(x)$ for almost every $x \in B(0,2)$ as $h \to 0$. Thus, the dominated convergence theorem gives

$$\int_{B(0,2)} v_h^{2-p}(x,0) w^p \, dx \to \int_{B(0,2)} v_0^{2-p}(x) w^p \, dx \leq \int_{B(0,2)} N^{2-p} w^p \, dx$$

as $h \to 0$ since $v_0 \geq N + \varepsilon$ almost everywhere in the support of w. This holds for every $\varepsilon > 0$. Therefore, for any $\delta > 0$, we find a small time $t_\delta > 0$ such that

$$\int_{B(0,2)} u^{2-p}(x, t_\delta) w^p \, dx \leq \int_{B(0,2)} N^{2-p} w^p \, dx + \delta.$$

We choose such a time t_δ. We then proceed formally and obtain

$$\begin{aligned}\int_{t_\delta}^T \int_{B(0,2)} u \frac{\partial \eta}{\partial t} \, dx \, dt &= \frac{p-1}{p-2} \int_{t_\delta}^T \int_{B(0,2)} \frac{\partial u^{2-p}}{\partial t} w^p \, dx \, dt \\ &= \frac{p-1}{p-2} \Big(\int_{B(0,2)} u^{2-p}(x,T) w^p \, dx - \int_{B(0,2)} u^{2-p}(x,t_\delta) w^p \, dx \Big)\end{aligned}$$

so that

$$\begin{aligned}&- \int_{t_\delta}^T \int_{B(0,2)} u \frac{\partial \eta}{\partial t} \, dx \, dt \\ &+ \int_{B(0,2)} u^{2-p}(x,T) w^p \, dx - \int_{B(0,2)} u^{2-p}(x,t_\delta) w^p \, dx \\ &= - \frac{1}{p-2} \Big(\int_{B(0,2)} u^{2-p}(x,T) w^p \, dx - \int_{B(0,2)} u^{2-p}(x,t_\delta) w^p \, dx \Big)\end{aligned}$$

for almost every $T > t_\delta$. Furthermore, we use the structural conditions (3.1) and Young's inequality and obtain

$$
\begin{aligned}
\mathcal{A}(x,&t, u, \nabla u) \cdot \nabla \eta \\
&\le (1-p)\mathcal{A}_0 |\nabla u|^p u^{-p} w^p + p\mathcal{A}_1 |\nabla u|^{p-1} u^{1-p} |\nabla w| w^{p-1} \\
&= (1-p)\mathcal{A}_0 |\nabla u|^p u^{-p} w^p + p(\mathcal{A}_0^{1/p} |\nabla u| u^{-1} w)^{p-1} (\mathcal{A}_1 \mathcal{A}_0^{-(p-1)/p} |\nabla w|) \\
&\le \mathcal{A}_1 \Big(\frac{\mathcal{A}_1}{\mathcal{A}_0}\Big)^{p-1} |\nabla w|^p.
\end{aligned}
$$

We now send $\delta \to 0$ and conclude

$$
\limsup_{t_\delta \to 0} \int_{B(0,2)} \frac{u^{2-p}(x, t_\delta) - N^{2-p}}{p-2} w^p \, dx \le 0.
$$

Hence,

$$
\begin{aligned}
&\int_{B(0,2)} \frac{u^{2-p}(x,T) - N^{2-p}}{p-2} w^p \, dx \\
&\qquad \le \mathcal{A}_1 \Big(\frac{\mathcal{A}_1}{\mathcal{A}_0}\Big)^{p-1} \int_0^T \int_{B(0,2)} |\nabla w|^p \, dx \, dt = C_{\text{cap}} \mathcal{A}_1 \Big(\frac{\mathcal{A}_1}{\mathcal{A}_0}\Big)^{p-1} T.
\end{aligned}
$$

for almost every $T > 0$. It follows that, for every $\varepsilon > 0$, we have

$$
\begin{aligned}
\big|\{x \in U : u(x,T) \le \varepsilon\}\big| &\frac{\varepsilon^{2-p} - N^{2-p}}{p-2} \\
&\le \int_U \frac{u^{2-p}(x,T) - N^{2-p}}{p-2} \, dx \le C_{\text{cap}} C T.
\end{aligned}
$$

We then choose

$$
\varepsilon = N\big(1 + 2(p-2) N^{p-2} \big|U\big|^{-1} C_{\text{cap}} C T\big)^{-1/(p-2)}
$$

and obtain

$$
\big|\{x \in U : u(x,T) \le \varepsilon\}\big| \le \frac{1}{2} \big|U\big|.
$$

Consequently, since $U \subset B(0,3)$, we get

$$
\big|\{x \in B(0,3) : u(x,T) > \varepsilon\}\big| \ge \frac{1}{2} \big|U\big| \ge \frac{\nu_U}{2} \big|B(0,1)\big|
$$

and the result follows with $\kappa = C\big|U\big|^{-1} C_{\text{cap}}$ and $\nu = \nu_U / 2^{2n+1}$. □

By the corollary above, we know that the time scaled supersolutions have values bigger than one for almost every $t > 0$, at least in a set of uniformly positive measure. Next, lemma tells that, for later times, we can find positivity in a larger set. This type of argument has been used in the 'Second Alternative' in the proof of the Hölder continuity of the solution, see [DiB93].

Lemma 5.9. *Let v be a weak supersolution of (5.1) with $R \ge 4$. Suppose that*

$$
\big|\{x \in B(0,3) : v(x,t) > 1\}\big| \ge \nu \big|B(0,3)\big|.
$$

for some $\nu > 0$. *Then, for every* $0 < \nu^* < 1$, *there exists a constant* $M = M(n, p, \mathcal{A}_0, \mathcal{A}_1, \nu, \nu^*)$ *such that*

$$\begin{aligned}\big|\big\{(x,t) \in & B(0,3) \times (T,4T)\ : v(x,t) \le 2^{-M}\big\}\big|\\ \le & \nu^*\big|B(0,3) \times (T,4T)\big|,\end{aligned}$$

where $T = 2^{M(p-2)}$.

Proof. We define $k_j = 2^{-j}$, $j = 0, 1, 2, \dots, M$, and $T = k_M^{2-p}$, where M is going to be fixed. Let θ be a test function vanishing on the parabolic boundary of $B(0,4) \times (0,4T)$, $\theta = 1$ in $B(0,3) \times (T,4T)$ and

$$|\nabla\theta| \le C, \quad \Big|\frac{\partial\theta}{\partial t}\Big| \le \frac{C}{T}.$$

Now

$$\begin{aligned}\int_0^{4T}\int_{B(0,4)} & \theta^{p-1}\Big|\frac{\partial\theta}{\partial t}\Big|(v-k_j)_-^2\,dx\,dt\\ & \le \frac{Ck_j^2}{T}\big|B(0,3)\times(T,4T)\big| \le Ck_j^p\big|B(0,3)\times(T,4T)\big|\end{aligned}$$

and

$$\int_0^{4T}\int_{B(0,3)} |\nabla\theta|^p(v-k_j)_-^p\,dx\,dt \le Ck_j^p\big|B(0,3)\times(T,4T)\big|.$$

It follows from lemma 5.4 that

$$\int_T^{4T}\int_{B(0,3)} |\nabla(v-k_j)_-|^p\,dx\,dt \le Ck_j^p\big|B(0,3)\times(T,4T)\big|.$$

Lemma 5.5 together with assumptions yields

$$\begin{aligned}k_{j+1}&\big|\big\{x \in B(0,3)\ :\ v(x,t) \le k_{j+1}\big\}\big|\\ &\le \frac{C\int_{B(0,3)}|\nabla v|\chi_{\{k_{j+1}<v<k_j\}}\,dx}{\big|\big\{x \in B(0,3)\ :\ v(x,t) \ge k_j\big\}\big|} \le C\int_{B(0,3)}|\nabla v|\chi_{\{k_{j+1}<v<k_j\}}\,dx\end{aligned}$$

for almost every $t > 0$. We integrate this in time from T to $4T$. We first notice that

$$\begin{aligned}&\big|\big\{(x,t) \in B(0,3)\times(T,4T) : v(x,t) \le k_M\big\}\big|\\ \le&\big|\big\{(x,t) \in B(0,3)\times(T,4T) : v(x,t) \le k_{j+1}\big\}\big|.\end{aligned}$$

The use of Hölder's inequality then gives

$$\begin{aligned}\big|\big\{(x,t) \in & B(0,3)\times(T,4T)\ :\ v(x,t) \le k_M\big\}\big|\\ \le&\frac{C}{k_{j+1}}\Big(\int_T^{4T}\int_{B(0,3)}|\nabla(v-k_j)_-|^p\,dx\,dt\Big)^{1/p}\\ &\times\Big(\int_T^{4T}\int_{B(0,3)}\chi_{\{k_{j+1}<v<k_j\}}\,dx\,dt\Big)^{(p-1)/p}\\ \le& C\Big(\big|B(0,3)\times(T,4T)\big|\Big)^{1/p}\Big(\int_T^{4T}\int_{B(0,3)}\chi_{\{k_{j+1}<v<k_j\}}\,dx\,dt\Big)^{(p-1)/p}.\end{aligned}$$

We take power $p/(p-1)$ on both sides and sum up from $j=0$ to $M-1$. Note that sets $\{k_{j+1} < v < k_j\}$ are disjoint for different j's. This implies

$$\begin{aligned}\big|\big\{(x,t) \in B(0,3)\times(T,4T) : v(x,t) \le k_M\big\}\big| \\ \le \frac{C}{M^{(p-1)/p}}\big|B(0,3)\times(T,4T)\big|.\end{aligned}$$

Therefore, if we take M so large that

$$\Big(\frac{C}{M}\Big)^{(p-1)/p} \le \nu^*,$$

the result follows. □

We are ready to proceed to the proof of Theorem 5.2. It follows the proof of Lemma 4.1, p. 49, in [DiB93].

Proof of Theorem 5.2. We have shown in Lemma 5.3 that the function

$$v(x,t) = \frac{\exp(\kappa t)}{N}\, u\big(x, \frac{\exp(\kappa(p-2)t)-1}{\kappa(p-2)N^{p-2}}\big)$$

is a supersolution. Moreover, by Corollary 5.8, it satisfies the assumptions of Lemma 5.9. Thus, for every ν^* we find M such that

$$\big|\big\{(x,t) \in B(0,3)\times(T,4T)\ : v(x,t) \le 2^{-M}\big\}\big| \le \nu^*\big|B(0,3)\times(T,4T)\big|,$$

where $T = 2^{M(p-2)}$. We define

$$k_j = 2^{-M-1}(1+2^{-j}), \quad r_j = 2+2^{-j}, \quad T_j = 2T\big(1-2^{-j-1}\big)$$

for $j = 0,1,2,\dots$. We also denote

$$Q_j = B_j \times \Gamma_j = B(r_j,0)\times(T_j,4T).$$

We then have

$$k_j - k_{j+1} = 2^{-M-j-2} \quad \text{and} \quad 2^{-M-1} \le k_j \le 2^{-M}.$$

Furthermore, let θ_j be a test function such that it vanishes on the parabolic boundary of Q_j and $\theta_j = 1$ in Q_{j+1}. We may choose it so that

$$|\nabla\theta_j| \le C2^j, \quad \left|\frac{\partial\theta_j}{\partial t}\right| \le \frac{C2^j}{T} \le C2^j k_j^{p-2}.$$

Also the estimate

$$(v-k_j)_-^2 \ge \frac{(v-k_j)_-^p}{k_j^{p-2}} \ge \frac{T}{2}(v-k_j)_-^p$$

will be used. Collecting the results so far, we have from Lemma 4.1 that

(5.10)

$$
\begin{aligned}
\frac{1}{T}\int_{Q_j}|\nabla((v-k_j)_-\theta_j)|^p\,dx\,dt + \operatorname*{ess\,sup}_{\Gamma_j}\int_{B_j}(v-k_j)_-^p\theta_j^p\,dx \\
\le \frac{C}{T}\int_{Q_j}(v-k_j)_-^p|\nabla\theta_j|^p\,dx\,dt + \int_{Q_j}(v-k_j)_-^2\varphi^{p-1}\left|\frac{\partial\theta_j}{\partial t}\right|\,dx\,dt \\
\le \frac{C2^{pj}}{T}\int_{Q_j}(v-k_j)_-^p + (v-k_j)_-^2k_j^{p-2}\,dx\,dt.
\end{aligned}
$$

A change of variables $z = t/T$ now gives

(5.11)

$$
\begin{aligned}
\int_{T_j/T}^4\int_{B_j}|\nabla((w-k_j)_-\psi_j)|^p\,dx\,dz + \operatorname*{ess\,sup}_{T_j/T<t<4}\int_{B_j}(w-k_j)_-^p\psi_j^p\,dx \\
\le C2^{pj}\int_{T_j/T}^4\int_{B_j}(w-k_j)_-^p + (w-k_j)_-^2k_j^{p-2}\,dx\,dz,
\end{aligned}
$$

where $w(x,t) = v(x,Tt)$ and $\psi_j(x,t) = \theta_j(x,Tt)$. Now let

$$
A_j = \int_{T_j/T}^4\int_{B_j}\chi_{\{w<k_j\}}\,dx\,dz = \frac{1}{T}\int_{T_j}^{4T}\int_{B_j}\chi_{\{v<k_j\}}\,dx\,dt.
$$

Note that from Lemma 5.9 we have an estimate $A_1 \le \nu^*\big|B(0,3)\big|$ for the first level set. From Lemma 5.6 and inequality (5.11) we get

$$
\begin{aligned}
&\int_{T_j/T}^4\int_{B_j}(w-k_j)_-^p\psi_j^p\,dx\,dz \\
&\le CA_j^{p/(n+p)}\int_{T_j/T}^4\int_{B_j}|\nabla((w-k_j)_-\psi_j)|^p\,dx\,dz \\
&\quad + CA_j^{p/(n+p)}\operatorname*{ess\,sup}_{T_j/T<t<4}\int_{B_j}(w-k_j)_-^p\psi_j^p\,dx \\
&\le C2^{pj}A_j^{1+p/(n+p)}k_j^p.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\int_{T_j/T}^4\int_{B_j}(w-k_j)_-^p\psi_j^p\,dx\,dz &\ge \int_{T_{j+1}/T}^4\int_{B_{j+1}}(w-k_j)_-^p\,dx\,dz \\
&\ge (k_j-k_{j+1})^pA_{j+1} \ge \frac{1}{C2^{pj}}A_{j+1}.
\end{aligned}
$$

This yields an iteration inequality

$$
A_{j+1} \le C4^{pj}A_j^{1+p/(n+p)}.
$$

By a standard argument (see Lemma 4.1, p. 12, in [DiB93]), $A_j \to 0$ if

$$
A_1 \le C^{-(n+p)/p}4^{-(n+p)^2/p}.
$$

By taking

$$
\nu^* = C^{-(n+p)/p}4^{-(n+p)^2/p}\big|B(0,3)\big|^{-1}.
$$

we indeed have that

$$v(x,t) \geq 2^{-M-1}$$

for almost every $(x,t) \in B(0,4) \times (2T, 4T)$, $T = 2^{M(p-2)}$. Note that ν^* may be chosen so that it depends only on the structural constants. That is why also M depends only on the structural constants. The result is now proved with

$$T^* = \frac{\exp(\kappa(p-2)2^{M(p-2)+2}) - 1}{\kappa(p-2)}$$

and

$$\mu^* = 2^{-M-1} \exp(-\kappa 2^{M(p-2)+2}).$$

□

6. Proofs of Theorems 2.2 and 2.5

We now proceed to the proofs of main theorems. In what follows, we extensively use the results of previous sections.

In the proof of Theorem 5.2, we needed a special test function related to the initial data. There is no *a priori* information as to whether such a test function exists. As the next lemma shows, we indeed find such a test function. It depends on the supersolution itself.

Lemma 6.1. *Let u be a nonnegative supersolution in $B(x_0, 2R_0) \times (t_0, t_0 + 2T_0)$. Then for every $k > 0$ there is a Sobolev function*

$$w \in W_0^{1,p}(B(x_0, 2R_0))$$

and a time $t_0 + T_0 < t^* < t_0 + 2T_0$ such that

$$w = 1 \quad \text{almost everywhere in} \quad \{x \in B(x_0, R_0) : u(x, t^*) \geq 2k\}$$

and

$$w = 0 \quad \text{almost everywhere in} \quad \{x \in B(x_0, 2R_0) : u(x, t^*) \leq k\}.$$

Moreover, there is a constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ such that

$$\int_{B(x_0,2R_0)} |\nabla w|^p \, dx \leq C R_0^n \Big(\frac{1}{R_0^p} + \frac{1}{T_0 k^{p-2}} \Big).$$

Proof. We fix $k > 0$. Since u is a supersolution, $\min(u, 2k)$ is also a supersolution. This implies that $v = 2k - \min(u, 2k)$ is a nonnegative subsolution. Let $\psi \in C_0^\infty(B(x_0, 2R_0))$ now be such that

$$0 \leq \psi \leq 1, \quad \psi = 1 \quad \text{in} \quad B(x_0, R_0), \quad \text{and} \quad |\nabla \psi| \leq \frac{C}{R_0}$$

and let further $\zeta \in C^\infty(t_0, t_0 + 2T_0)$ be such that

$$0 \leq \zeta \leq 1, \quad \zeta(t) = 1 \quad \text{as} \quad t_0 + T_0 \leq t \leq t_0 + 2T_0, \quad \zeta(t_0) = 0$$

and $|\partial\zeta/\partial t| \le C/T_0$. By Caccioppoli estimate, Lemma 4.1, there is a constant $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ such that

$$\int_{t_0+T_0}^{t_0+2T_0}\int_{B(x_0,2R_0)} |\nabla(\psi v)|^p\,dx\,dt \le C\big(k^pT_0R_0^{n-p} + k^2R_0^n\big).$$

This is seen with the aid of the test function $\varphi = \psi\zeta$. Furthermore,

$$\eta = \frac{1}{k}(k-v)_+$$

is a function such that $\eta = 0$ almost everywhere in $\{u \le k\}$ and $\eta = 1$ almost everywhere in $\{u \ge 2k\}$. Moreover,

$$\begin{aligned}\int_{t_0+T_0}^{t_0+2T_0}\int_{B(x_0,2R_0)} |\nabla(\psi\eta)|^p\,dx\,dt &\le \frac{1}{k^p}\int_{t_0+T_0}^{t_0+2T_0}\int_{B(x_0,2R_0)} |\nabla(\psi v)|^p\,dx\,dt\\ &\le CR_0^n\big(\frac{T_0}{R_0^p} + k^{2-p}\big).\end{aligned}$$

Therefore, there exists a time $t_0+T_0 < t^* < t_0+2T_0$ such that

$$\int_{B(x_0,2R_0)} |\nabla(\psi\eta(t^*,x))|^p\,dx \le CR_0^n\big(\frac{1}{R_0^p} + \frac{1}{T_0k^{p-2}}\big).$$

Thus, we may choose $w = \eta(\cdot,t^*)\psi$ and it satisfies all the requirements we asserted. $\square$

Much of the work done in Section 4 aimed to prove the following result. The technique is fairly simple. We have a solution to the Dirichlet problem with zero boundary values and initial mass of one. The goal of the lemma is twofold. On the one hand, we want to show that there exists a time depending only on data such that the solution's mass has not grown too much. On the other hand, we want to show that some quantity of the initial mass is preserved for such a time.

Lemma 6.2. *Let*

$$u \in L^p(0,1;W_0^{1,p}(B(0,6)))$$

be a nonnegative solution in $B(0,6)\times(0,1)$*. Suppose further that the initial data satisfies*

$$u_0 = 0 \quad in \quad B(0,6)\backslash B(0,1)$$

and

$$\int_{B(0,1)} u_0\,dx = 1.$$

Then there are constants $\tau_0 = \tau_0(n,p,\mathcal{A}_0,\mathcal{A}_1)$ *and* $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ *such that*

$$\int_{B(0,3)} u(x,t)\,dx \ge \frac{1}{2}$$

and

$$\int_{B(0,6)} u(x,t)\,dx \le C$$

for every $0 \le t < \tau_0$*.*

Proof. We want to apply Lemma 4.24 in such a way that the right hand side does not depend on u. We cancel the dependence by taking T to be the smallest root of the equation

$$T = \Big(\sup_{0<t<T} \int_{B(0,6)} u\, dx \Big)^{-\beta},$$

where $\beta \geq 0$ is to be chosen. We denote

$$N = N(T) = \sup_{0<t<T} \int_{B(0,6)} u\, dx$$

which is larger or equal to 1 by the initial condition. We infer that N is a continuous function. Since u is a solution we see from the equation that $\partial u/\partial t$ belongs to the dual of $L^p(0,1; W_0^{1,p}(B(0,6)))$. It is then a standard result that u belongs to $C(0,1; L^2(B(0,6))$, see, for example, [Sho97], and thus also N is continuous. Hence, there exists the smallest root of $T = N(T)^{-\beta}$ by Rolle's theorem. We choose

$$\beta = \max\Big\{ p-1, p-2+\frac{p}{3n}+\frac{3(n(p-2)+p)}{2} \Big\}.$$

We then obtain from Lemma 4.22 that

$$\operatorname*{ess\,sup}_{(B(0,6)\setminus B(0,2))\times(0,T)} u \leq C$$

and from Lemma 4.24 and the remark thereafter that

$$\int_0^T \int_{B(0,3)} |\nabla u|^{p-1}\, dx\, dt \leq C.$$

Furthermore, from the weak formulation with the cut-off function $\varphi \in C_0^\infty(B(0,3))$, $\varphi = 1$ in $B(0,2)$, $|\nabla \varphi| \leq C$, we deduce

$$\int_{B(0,3)} u(x,t)\varphi\, dx \leq \int_{B(0,3)} u(x,0)\varphi\, dx + \mathcal{A}_1 \|\nabla\varphi\|_\infty \int_0^T \int_{B(0,3)} |\nabla u|^{p-1}\, dx\, dt \leq C$$

for every $0 < t < T$. Consequently, we have

$$\int_{B(0,6)} u(x,t)\, dx = \int_{B(0,2)} u(x,t)\, dx + \int_{B(0,6)\setminus B(0,2)} u(x,t)\, dx \leq C$$

for every $0 < t < T$. With β fixed, we get

$$T = N^{-\beta} \geq C^{-\beta}.$$

Hence, we obtain a lower bound for T, which depends only on n, p, $\mathcal{A}_0$ and $\mathcal{A}_1$. Let $0 < \tau_0 \leq T$ be chosen so that

$$\int_0^{\tau_0} \int_{B(0,3)} |\nabla u|^{p-1}\, dx\, dt \leq \frac{1}{2}\big(\mathcal{A}_1 \|\nabla\varphi\|_\infty\big)^{-1}.$$

Such a constant τ_0 can be found by Lemma 4.24. Moreover, τ_0 may be chosen to depend only on n, p, $\mathcal{A}_0$ and $\mathcal{A}_1$. Therefore, again from the weak formulation, we obtain

$$\begin{aligned}1 = \int_{B(0,1)} u_0\,dx \le &\int_{B(0,3)} u(x,t)\varphi\,dx \\ &+ \mathcal{A}_1\|\nabla\varphi\|_\infty \int_0^{\tau_0}\int_{B(0,3)} |\nabla u|^{p-1}\,dx\,ds \\ \le &\int_{B(0,3)} u(x,t)\,dx + \frac{1}{2}\end{aligned}$$

for every $0 < t < \tau_0$. The result now follows. □

We also need the following simple tool.

Lemma 6.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that f is a measurable function in Ω satisfying*

$$\fint_\Omega f\,dx \ge 2N$$

and

$$\Big(\fint_\Omega f^q\,dx\Big)^{1/q} \le C_0 N$$

for some $1 < q \le +\infty$. Then

$$\big|\{x \in \Omega : f(x) > N\}\big| \ge C_0^{-q/(q-1)}\big|\Omega\big|.$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}2N \le &\fint_\Omega f\,dx = \frac{1}{|\Omega|}\Big(\int_{\{f>N\}} f\,dx + \int_{\{f\le N\}} f\,dx\Big) \\ \le &C_0 N\Big(\frac{1}{|\Omega|}\big|\{f>N\}\big|\Big)^{1-1/q} + N,\end{aligned}$$

which implies the result. □

Finally, we are ready to prove the local weak Harnack estimate. This is the only proof of this work that uses the comparison principle and the existence result.

Proof of Theorem 2.5. By the existence result in [Hun01], we have the solution

$$v \in L^p(t_1, t_0 + T_0; W_0^{1,p}(B(x_0, 6R_0)))$$

to the problem

$$\begin{cases}\operatorname{div}\big(\mathcal{A}(x,t,v,\nabla v\big) = \dfrac{\partial v}{\partial t} & \text{in} \quad B(x_0, 6R_0) \times (t_1, t_0 + T_0) \\ v(\cdot, t_1) = v_0 & \end{cases},$$

where

$$v_0 = \chi_{B(x_0,R_0)} u(\cdot, t_1),$$

for almost every $t_0 < t_1 < t_0 + T_0$. By the comparison principle, Theorem 3.5,

$$v \le u \quad \text{in} \quad B(x_0, 6R_0) \times (t_1, t_0 + T_0).$$

We now denote

$$N = \fint_{B(x_0,R_0)} u(x,t_1)\,dx$$

and assume that

$$N \geq \Big(\frac{C_1 R_0^p}{T_0+t_0-t_1}\Big)^{1/(p-2)}.$$

We scale the solution v as

$$w(x,t) = \frac{1}{\big|B(0,1)\big|N} v(x_0+R_0x, t_1 + (\big|B(0,1)\big|N)^{2-p}R_0^p t)$$

so that w is a solution in $B(0,6)\times(0,1)$ provided that

$$N \geq \frac{1}{\big|B(0,1)\big|}\Big(\frac{R_0^p}{T_0+t_0-t_1}\Big)^{1/(p-2)}.$$

This is certainly true if $C_1 \geq 1$. Moreover, the support of the initial data w_0 belongs to $B(0,1)$ and

$$\int_{B(0,1)} w_0\,dx = 1.$$

By Corollary 4.19 and Lemma 6.2 we obtain

$$\operatorname*{ess\,sup}_{x\in B(0,6)} w(x,\tau) \leq C\tau^{-n/\lambda}$$

for every $0<\tau<\tau_0$, where $\tau_0 = \tau_0(n,p,\mathcal{A}_0,\mathcal{A}_1)$ is as in Lemma 6.2. For such τ, we also have

$$\fint_{B(0,2)} w(x,\tau) \geq \frac{1}{2\big|B(0,2)\big|}.$$

It readily follows from Lemma 6.3 that

$$\Big|\{x\in B(0,2): w(x,t) > \frac{1}{4\big|B(0,2)\big|}\}\Big| \geq \frac{\tau_0^{n/\lambda}}{C}\big|B(0,2)\big|$$

for every $\tau_0/2 < t < \tau_0$. We then define yet another supersolution

$$v(x,t) = \frac{1}{N}u(x_0+2R_0x, t_1+N^{2-p}(2R_0)^p t)$$

in the domain $B(0,4)\times(0, N^{p-2}T_0/(2R_0)^p)$. By the comparison principle, we have

$$\Big|\{x\in B(0,1): v(x,t) > 1/C\}\Big| \geq \frac{1}{C}\big|B(0,1)\big|$$

for almost every $\tau_1/2 < t < \tau_1$, $\tau_1 = 2^{-p}\big|B(0,1)\big|^{2-p}\tau_0$, and for some $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$. We now define

$$U(t) = \{x\in B(0,1): v(x,t) \geq 1/C\}$$

and

$$V(t) = \{x\in B(0,2): v(x,t) \leq 1/2C\}.$$

Since v is a supersolution, it follows from Lemma 6.1 that we find a test function $w\in W_0^{1,p}(B(0,2))$ and some time $\tau_1/2 < t^* < \tau_1$ such that

$$w = 1 \quad \text{almost everywhere in} \quad U(t^*),$$

$$w = 0 \quad \text{almost everywhere in} \quad V(t^*)$$

and

$$\int_{B(0,2)} |\nabla w|^p \, dx \le C.$$

Moreover, we have already shown that

$$\big|U(t^*)\big| \ge \frac{1}{C}\big|B(0,1)\big|.$$

Consequently, Theorem 5.2 gives constants $\tilde{T}$ and $\tilde{\mu}$, depending only on n, p, $\mathcal{A}_0$ and $\mathcal{A}_1$ such that

$$\operatorname*{ess\,inf}_{Q_1} v \ge \tilde{\mu},$$

where $Q_1 = B(0,2) \times (t^* + \tilde{T}/2, t^* + \tilde{T})$. We scale back and have for $C_1 = 2^p(\tilde{T} + t^*)$

$$\operatorname*{ess\,inf}_{Q} u \ge N\tilde{\mu},$$

where $Q = B(x_0, 4R_0) \times (t_1 + C_1 N^{2-p}/2, t_1 + C_1 N^{2-p})$. To guarantee that u still is a supersolution up to the time $t_1 + C_1 N^{2-p}$ we need the condition

$$C_1 N^{2-p} R_0^p \le T_0 + t_0 - t_1,$$

that is

$$\fint_{B(x_0,R_0)} u(x,t_1)\, dx \ge \Big(\frac{C_1 R_0^p}{T_0 + t_0 - t_1}\Big)^{1/(p-2)}.$$

We now have finished, since we may choose $C_2 = 1/\tilde{\mu}$. □

We use the local Harnack estimate to prove the other main result, the global Harnack estimate.

Proof of Theorem 2.2. We use the blow-up argument. First, we scale the supersolution as

$$v(x,t) = u(x_0 + (R/M)x, (R/M)^p t),$$

where $M \le 1$. Then v is a supersolution in

$$\mathbb{R}^n \times (0, T_0(M/R)^p).$$

We have

$$\fint_{B(x_0,R)} u(x,t_0)\, dx = \fint_{B(0,M)} v(x,t_1)\, dx \le M^{-n} \fint_{B(0,1)} v(x,t_1)\, dx, \tag{6.4}$$

where $t_1 = (M/R)^p t_0$. Let C_1 and C_2 be as in Theorem 2.5 and suppose that

$$\fint_{B(0,1)} v(x,t_1)\, dx > 2\Big(\frac{C_1}{(T_0 - t_0)(M/R)^p}\Big)^{1/(p-2)}. \tag{6.5}$$

We then apply Theorem 2.5 and obtain

$$\fint_{B(0,1)} v(x,t_1)\, dx \le \Big(\frac{C_1}{(T_0 - t_0)(M/R)^p}\Big)^{1/(p-2)} + C_2 \operatorname*{ess\,inf}_{Q_1} v,$$

where $Q_1 = B(0,4) \times (t_1 + T_1/2, t_1 + T_1)$ and

$$T_1 = C_1\Big(\frac{1}{2}\fint_{B(0,1)} v(x,t_1)\,dx\Big)^{2-p} < (T_0 - t_0)(M/R)^p$$

by the condition (6.5). It follows that

$$\fint_{B(0,1)} v(x,t_1)\,dx \le 2C_2 \operatorname*{ess\,inf}_{Q_1} v. \tag{6.6}$$

Furthermore, we choose

$$M^p = \frac{C_1 R^p}{T}\Big(\frac{1}{2}\fint_{B(0,1)} v(x,t_1)\,dx\Big)^{2-p}$$

so that

$$T = (R/M)^p C_1\Big(\frac{1}{2}\fint_{B(0,1)} v(x,t_1)\,dx\Big)^{2-p} < T_0 - t_0.$$

The choice of M leads by (6.4) to the inequality

$$M^\lambda \le \frac{C_1 R^p}{T}\Big(\frac{1}{2}\fint_{B(x_0,R)} u(x,t_0)\,dx\Big)^{2-p}.$$

Thus, the requirement $M \le 1$ is certainly fulfilled if

$$\fint_{B(x_0,R)} u(x,t_0)\,dx \ge \Big(\frac{2^{p-2}C_1 R^p}{T}\Big)^{1/(p-2)}.$$

For such initial masses, we have by (6.6) that

$$M^n \fint_{B(x_0,R)} u(x,t_0)\,dx \le 2C_2 \operatorname*{ess\,inf}_{Q_2} u,$$

where $Q_2 = B(x_0, 4R/M) \times (t_0 + T/2, t_0 + T)$. Note that

$$\operatorname*{ess\,inf}_{Q_1} v = \operatorname*{ess\,inf}_{Q_2} u$$

by the definition of M. Moreover, we obtain, again by (6.6), that

$$\begin{aligned} M^{-n} =& C\Big(\frac{T}{R^p}\Big)^{n/p}\Big(\fint_{B(0,1)} v(x,t_1)\,dx\Big)^{n(p-2)/p} \\ \le& C\Big(\frac{T}{R^p}\Big)^{n/p} \operatorname*{ess\,inf}_{Q_2} u^{n(p-2)/p}. \end{aligned}$$

We combine estimates and conclude that

$$\fint_{B(x_0,R)} u(x,t_0)\,dx \le C\Big(\frac{T}{R^p}\Big)^{n/p} \operatorname*{ess\,inf}_{Q_2} u^{\lambda/p}.$$

Recall now that $Q \subset Q_2$, where $Q = B(x_0, 4R) \times (t_0 + T/2, t_0 + T)$, since $M \le 1$. This proves the result. □

7. Hölder continuity of solutions

We now turn our focus to the solutions and prove that every solution has an Hölder continuous representative. We first prove the local intrinsic Harnack estimate and then use this to prove the Hölder continuity of the solution. The proof presented here follows [DGV06] and it is similar to the one due to Moser in [Mos64].

We again want to emphasize the fact that we have stronger assumptions than Theorem 1.1 has in [DGV06]. Our proof uses the comparison principle and the existence result.

We start with a lemma that implies the local Harnack estimate.

Lemma 7.1. *Suppose that u is a local nonnegative weak solution to (3.3) in $B(0,4)\times(-C_0,C_0)$, where $C_0 = C_0(n,p,\mathcal{A}_0,\mathcal{A}_1) > 1$ is given in the proof. Moreover, suppose that*

$$\operatorname*{ess\,sup}_{B(0,1/4)\times(-1/4,0)} u \geq 1.$$

Then there exists a constant $C = C(n,p,\mathcal{A}_0,\mathcal{A}_1)$ such that

$$\operatorname*{ess\,inf}_{B(0,1)\times(C_0/2,C_0)} u \geq \frac{1}{C}.$$

Moreover, the constants C and C_0 are stable as $p\to 2$.

Proof. Basic ingredients of the proof are Theorem 2.5 and Theorem 4.10. For the sake of clarity, we enumerate our constants. They will all, however, depend only on n, p, $\mathcal{A}_0$ and $\mathcal{A}_1$. We first use Theorem 4.10 for *subsolutions* and get

$$1 \leq C_1\left(\left(\frac{R_1^p}{T_1}\right)^{1/(p-2)} + \frac{T_1}{R_1^p}\left(\operatorname*{ess\,sup}_{-T_1<t<0} \fint_{B(0,R_1)} u\,dx\right)^{p-1}\right)$$

with

$$R_1 = \frac{1}{2}, \quad \frac{R_1^p}{T_1} = (2C_1)^{2-p}, \quad 1 < T_1 < C_0.$$

We obtain for some $-T_1 < t_1 < 0$ and constant C_2 that

$$\fint_{B(0,1/2)} u(x,t_1)\,dx \geq \frac{2}{C_2}.$$

We then apply Theorem 2.5 for *supersolutions* with $R_0 = 1/2$ and $T_0 = C_0 - t_1$. We may choose C_0 so that the condition

$$\frac{1}{C_2} \geq \left(\frac{C_3 R_0^p}{T_0}\right)^{1/(p-2)} = \left(\frac{C_3}{2^p(C_0-t_1)}\right)^{1/(p-2)}$$

is satisfied and consequently

$$\frac{1}{C_2} \leq C_4 \operatorname*{ess\,inf}_{B(0,2)\times(T_2/2,T_2)} v,$$

where

$$T_2 = C_3\Big(\fint_{B(0,1/2)} u(x,t_1)\,dx\Big)^{2-p} \le \frac{C_0}{2}.$$

We still need to carry the obtained positivity up to the time C_0. We study a function

$$v(x,t) = u(x, T_2 + t),$$

which is a supersolution in $B(0,4) \times (0, C_0 - T_2)$. We now follow the proof of Theorem 5.2 and comment on the details briefly. First, repeating the calculations in the proof of Lemma 5.7, we may choose

$$\varepsilon = \frac{1}{C_5}\big(1 + \frac{C_5(p-2)}{\nu^*}(C_0 - T_2)\big)^{-1/(p-2)},$$

$C_5 = C_5(n, p, \mathcal{A}_0, \mathcal{A}_1)$ and $\nu^* > 0$, so that

$$\big|\{x \in B(0,3/2) : v(x,t) \le \varepsilon\}\big| \le \nu^*\big|B(0,3/2)\big|$$

for almost every $0 < t < C_0 - T_2$. We then choose $r_j = 1 + 2^{-1-j}$ and

$$Q_j = B_j \times \Gamma = B(0, r_j) \times (0, C_0 - T_2).$$

We also choose the test functions $\theta_j \in C_0^\infty(B(0,r_j))$ such that $\theta_j = 1$ in $B(0, r_{j+1})$ and $|\nabla\theta_j| \le C2^j$. The truncation levels k_j are chosen as

$$k_j = \frac{\varepsilon}{2}(1 + 2^{-j}).$$

Then $(v(x,0) - k_j)_- = 0$ for almost every $x \in B(0,3/2)$ and (5.10) becomes

$$\begin{aligned}\int_{Q_j} |\nabla((v-k_j)_-\theta_j)|^p\,dx\,dt + \varepsilon^{2-p}\operatorname*{ess\,sup}_{\Gamma}\int_{B_j}(v-k_j)_-^p\theta_j^p\,dx \\ \le C\int_{Q_j}(v-k_j)_-^p|\nabla\theta_j|^p\,dx\,dt.\end{aligned}$$

We change the time variable $z = \varepsilon^{p-2}t$. Using the notation of the proof of Theorem 5.2, we conclude that

$$A_1 \le \varepsilon\nu^*(C_0 - T_2)\big|B(0,2)\big|.$$

We may now choose a suitably small ν^* so that it depends only on n, p, $\mathcal{A}_0$ and $\mathcal{A}_1$ and repeat the argument in the proof of Theorem 5.2 to show the assertion of the lemma. □

We now define the values of u pointwise via

$$u(x,t) = \lim_{R\to 0}\operatorname*{ess\,sup}_{B(x,R)\times(t,t-R^p)} u.$$

We then obtain the local Harnack estimate.

Theorem 7.2. *Suppose that u is a nonnegative local weak solution in $Q_1 = B(x_1, R_1) \times (t_1, t_1 + T_1)$ to*

$$\operatorname{div}\big(\mathcal{A}(x,t,u,\nabla u)\big) = \frac{\partial u}{\partial t}.$$

Then there exist constants $C_i = C_i(n, p, \mathcal{A}_0, \mathcal{A}_1)$, $i = 1, 2$, such that, if

$$Q_0 = B(x_0, 4R) \times (t_0 - C_1 u(x_0,t_0)^{2-p}R^p, t_0 + C_1 u(x_0,t_0)^{2-p}R^p)$$

belongs to Q_1 for $R > 0$, then

$$u(x_0, t_0) \le C_2 \operatorname*{ess\,inf}_Q u$$

where

$$Q = B(x_0, R) \times (t_0 + C_1 u(x_0, t_0)^{2-p} R^p/2, t_0 + C_1 u(x_0, t_0)^{2-p} R^p).$$

The constants C and C_0 are stable as $p \to 2$.

Proof. We assume that $u(x_0, t_0) > 0$. Let C_0 be as in Lemma 7.1. Then the scaled solution

$$v(x,t) = \frac{1}{u(x_0, t_0)} u(x_0 + x/R, t_0 + u(x_0, t_0)^{2-p} t/R^p)$$

satisfy the assumptions of Lemma 7.1 and by scaling back we get the result. □

We will now use previous Harnack estimates to show that solutions are Hölder continuous. We define

$$\operatorname*{ess\,osc}_Q u = \operatorname*{ess\,sup}_Q u - \operatorname*{ess\,inf}_Q u.$$

Lemma 7.3. *Let u be a local weak solution in $B(x_0, R_0) \times (t_0 - T_0, t_0)$ and let*

$$\max\Big(\operatorname*{ess\,osc}_{B(x_0,R_0)\times(t_0-T_0,t_0)} u, \big(\frac{C_0 R_0^p}{T_0}\big)^{1/(p-2)}\Big) \le \omega < \infty,$$

where $C_0 \ge 32$ is as in Lemma 7.1. Then there exists a constant $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ and $\alpha = \alpha(n, p, \mathcal{A}_0, \mathcal{A}_1) < 1$ such that

$$\operatorname*{ess\,osc}_Q u \le C\omega\Big(\frac{\rho}{R_0}\Big)^{\alpha},$$

where

$$Q = B(x_0, \rho) \times (t_0 - \omega^{2-p}\rho^p, t_0),$$

$\rho \le R_0/4$.

Proof. Let $\omega_k = \delta^k \omega$, $k = 0, 1, 2, \dots$, where $1/2 < \delta < 1$ is to be chosen. We define the intrinsic geometry as follows. Let

$$R_k = 4^{-k-1} R_0, \quad S_k = \frac{C_0}{2}\big(\frac{\omega_k}{4}\big)^{2-p} R_k^p, \quad s_k = t_0 - S_k$$

and

$$Q_k = B(x_0, R_k) \times (s_k, t_0), \quad U_k = B(x_0, R_k) \times (s_k - 3S_k, t_0).$$

We first infer that U_{k+1} is a subset of Q_k. This is true if the condition

$$s_k \le s_{k+1} - 3S_{k+1}$$

holds. For $k = 0$ it follows from the assumptions. For $k \ge 1$ we rewrite the condition equivalently as

$$\big(\frac{\omega_k}{4}\big)^{2-p} R_k^p \ge 4\big(\frac{\omega_{k+1}}{4}\big)^{2-p} R_{k+1}^p$$

or

$$\frac{R_{k+1}}{R_k} \le \Big(\frac{\delta^{p-2}}{4}\Big)^{1/p}.$$

This implies the inclusion because $R_k/R_{k+1} = 4$ and $\delta > 1/2$.

We then define two scaled solutions

$$v(x,t) = \frac{4}{\omega_k}\big(\operatorname*{ess\,sup}_{Q_k} u - u(x_0 + R_{k+1}x, s_{k+1} - S_{k+1} + \frac{2S_{k+1}}{C_0}t)\big)$$

and

$$w(x,t) = \frac{4}{\omega_k}\big(u(x_0 + R_{k+1}x, s_{k+1} - S_{k+1} + \frac{2S_{k+1}}{C_0}t) - \operatorname*{ess\,inf}_{Q_k} u\big).$$

Note that they are nonnegative solutions in $B(0,4) \times (-C_0, C_0)$. We study different cases. Let first

$$\operatorname*{ess\,sup}_{B(0,1/4)\times(-1/4,0)} v \ge 1. \tag{7.4}$$

It then follows from Lemma 7.1 that

$$\operatorname*{ess\,inf}_{B(0,1)\times(C_0/2,C_0)} v \ge \frac{1}{C},$$

$C \ge 1$. In the original coordinates this implies

$$\frac{4}{\omega_k}\big(\operatorname*{ess\,sup}_{Q_k} u - \operatorname*{ess\,sup}_{Q_{k+1}} u\big) \ge \frac{1}{C}.$$

Consequently, we have

$$\operatorname*{ess\,osc}_{Q_{k+1}} u \le \operatorname*{ess\,sup}_{Q_k} u - \frac{\omega_k}{4C} - \operatorname*{ess\,inf}_{Q_{k+1}} u \le \omega_k\big(1 - \frac{1}{4C}\big). \tag{7.5}$$

Suppose then that

$$\operatorname*{ess\,osc}_{Q_k} u \ge \frac{1}{2}\omega_k. \tag{7.6}$$

The definition of v and w now gives

$$v(x,t) + w(x,t) = \frac{4}{\omega_k}\operatorname*{ess\,osc}_{Q_k} u \ge 2.$$

Therefore, if (7.4) is not the case, then

$$\operatorname*{ess\,sup}_{B(0,1/4)\times(-1/4,0)} w \ge \operatorname*{ess\,inf}_{B(0,1/4)\times(-1/4,0)} w \ge 1$$

and it follows that

$$\operatorname*{ess\,inf}_{B(0,1)\times(C_0/2,C_0)} w \ge \frac{1}{C}.$$

This means

$$\operatorname*{ess\,inf}_{Q_{k+1}} u - \operatorname*{ess\,inf}_{Q_k} u \ge \frac{\omega_k}{4C}$$

and again we obtain (7.5). We now choose

$$\delta = 1 - \frac{1}{4C} > \frac{1}{2}.$$

If (7.6) is not the case, then trivially

$$\operatorname*{ess\,osc}_{Q_{k+1}} u \le \operatorname*{ess\,osc}_{Q_k} u \le \frac{1}{2}\omega_k \le \omega_{k+1}.$$

Hence, we obtain in all cases

$$\operatorname*{ess\,osc}_{Q_k} u \le \delta^k \omega.$$

We then choose k such that

$$4^{-k-2}R_0 < \rho \le 4^{-k-1}R_0. \tag{7.7}$$

Remark that

$$S_{k+1} = \frac{C_0}{2}4^{p-2}4^{-p}\omega^{2-p}\delta^{(k+1)(2-p)}4^{-(k+1)p}R_0^p \ge \frac{C_0}{32}\omega^{2-p}\rho^p \ge \omega^{2-p}\rho^p$$

provided that $C_0 \ge 32$. It is clear from the proof of Lemma 7.1 that we may indeed choose such C_0. We now define $\alpha = -\log\delta/\log 4$ and the result follows since we have

$$\operatorname*{ess\,osc}_{Q} u \le \operatorname*{ess\,osc}_{Q_{k+1}} u \le \delta^{k+1}\omega = \omega 4^{-(k+1)\alpha} \le C\omega\Big(\frac{\rho}{R_0}\Big)^{\alpha}$$

because of (7.7). □

The Hölder continuity of solutions is a consequence of Lemma 7.3. Let $\Omega_T = \Omega \times (\tau_1, \tau_2)$ be an open set in $\mathbb{R}^{n+1}$. Its parabolic boundary is defined as

$$\partial_p\Omega_T = \Omega \times \{\tau_1\} \cup \partial\Omega \times (\tau_1, \tau_2).$$

Let $K \Subset \Omega_T$. We define the general intrinsic parabolic distance from K to Ω_T as

$$\Gamma_p - \operatorname{dist}(M) = \inf_{(x,t)\in\partial_p\Omega_T,(y,s)\in K} \min(|x-y|, M^{(p-2)/p}|t-s|^{1/p}),$$

where $M > 0$.

Theorem 7.8. *Let u be a local weak solution in $\Omega \times (\tau_1, \tau_2)$. Suppose that $\omega = \operatorname{ess\,osc}_{\Omega_T} u < \infty$. Then u is, after a proper redefinition on a set of measure zero, locally Hölder continuous. Moreover, let $K \Subset \Omega_T$. Then there exist constants $C = C(n, p, \mathcal{A}_0, \mathcal{A}_1)$ and $\alpha = \alpha(n, p, \mathcal{A}_0, \mathcal{A}_1)$ such that*

$$\big|u(x_0, t_0) - u(x_0, t_1)\big| \le C\omega\Big(\frac{|x_0 - y_0| + \omega^{(p-2)/p}|t_0 - t_1|^{1/p}}{\Gamma_p - \operatorname{dist}(\omega)}\Big)^{\alpha}$$

for every $(x_0, t_0), (x_1, t_1) \in K$.

Proof. We show the estimate for arbitrary $K \Subset \Omega_T$, which shows the local Hölder continuity. Let (x_0, t_0) and (x_1, t_1) be two given points in K. Without losing the generality, we may assume that $t_0 > t_1$. We define

$$T_0 = \inf_{(x,t)\in\partial_p\Omega_T,(y,s)\in K} |t-s|$$

and

$$R_0 = \frac{1}{C_0^{1/p}}\Gamma_p - \operatorname{dist}(\omega) \le \frac{1}{C_0^{1/p}}\omega^{(p-2)/p}T_0^{1/p},$$

where C_0 as in Lemma 7.1. The definitions imply that

$$B(x_0, R_0) \times (t_0 - T_0, t_0) \subset \Omega_T$$

and

$$\omega \geq \max\Big(\operatorname*{ess\,osc}_{B(x_0,R_0)\times(t_0-T_0,t_0)} u, \big(\frac{C_0 R_0^p}{T_0}\big)^{1/(p-2)}\Big).$$

Suppose first that

$$\max(|x_0 - y_0|, \omega^{(p-2)/p}(t_0 - t_1)^{1/p}) = \rho \leq R_0/4. \tag{7.9}$$

It then follows from Lemma 7.3 that

$$\big|u(x_0, t_0) - u(x_1, t_1)\big| \leq C\omega\Big(\frac{\rho}{R_0}\Big)^\alpha$$

which is the result of the theorem. If (7.9) does not hold, then the result follows trivially since

$$1 \leq C\Big(\frac{|x_0 - y_0| + \omega^{(p-2)/p}(t_0 - t_1)^{1/p}}{\Gamma_p - \operatorname{dist}(\omega)}\Big)^\alpha.$$

□

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