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GLOBAL HIGHER INTEGRABILITY FOR NONLINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS IN NONSMOOTH DOMAINS

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Abstract: This thesis studies the global regularity theory for degenerate nonlinear parabolic partial differential equations. Our objective is to show that weak solutions belong to a higher Sobolev space than assumed a priori if the complement of the domain satisfies a capacity density condition and if the boundary values are sufficiently smooth. Moreover, we derive integrability estimates near the lateral and initial boundaries. The results of the thesis extend to parabolic systems as well. The higher integrability estimates provide a useful tool in several applications.

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Keywords: boundary value problem, Caccioppoli inequality, capacity density, Gehring lemma, Giaquinta-Modica lemma, initial value problem, integrability of the gradient, nonlinear parabolic system, parabolic p-Laplace equation, reverse Hölder inequality

Mikko Parviainen: *Epälineaaristen parabolisten osittaisdifferentiaaliyhtälöiden korkeampi integroituvuus epäsäännöllisissä alueissa*; Teknillinen korkeakoulu, Matematiikan laitos, tutkimusraportti A529 (2007); monografia.

Tiivistelmä: Väitöskirjassa tutkitaan epälineaaristen parabolisten osittaisdifferentiaaliyhtälöiden ratkaisujen globaalia säännöllisyyttä. Työssä osoitetaan, että yhtälöiden heikot ratkaisut kuuluvat parempaan Sobolevin avaruuteen kuin määritelmässä oletetaan, jos alueen komplementti toteuttaa kapasiteettitiheysehdon ja reuna-arvot ovat tarpeeksi säännöllisiä. Lisäksi ratkaisujen gradienteille johdetaan integroituvuusestimaatteja sekä lähellä alkuhetkeä että lähellä alueen reunaa. Tämäntyyppiset estimaatit ovat osoittautuneet tärkeiksi monissa sovelluksissa. Väitöskirjan tulokset yleistyvät myös parabolisille systeemeille.

Asiasanat: alkuarvotehtävä, Caccioppolin epäyhtälö, epälineaarinen parabolinen systeemi, Gehringin lemma, Giaquintan-Modican lemma, gradientin integroituvuus, kapasiteettitiheys, käänteinen Hölderin epäyhtälö, parabolinen p-Laplacen yhtälö, reuna-arvotehtävä

Preface

This dissertation has been prepared at the *Institute of Mathematics, Helsinki University of Technology*, during the period 2004–2007. For financial support, I am indebted to *the Magnus Ehrnrooth Foundation and the Finnish Academy of Science and Letters, the Vilho, Yrjö and Kalle Väisälä Foundation.*

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Espoo, September, 2007 Mikko Parviainen

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GLOBAL HIGHER INTEGRABILITY FOR NONLINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS IN NONSMOOTH DOMAINS

Mikko Parviainen

1. Introduction

Higher integrability questions have been extensively studied over the last few decades. In this work, we investigate the parabolic equations of the type

$$\frac{\partial u}{\partial t} = \operatorname{div} A(x, t, \nabla u),$$

where $A(x, t, \nabla u)$ satisfies the well-known Carathéodory-type conditions and p-growth conditions. In particular, the results apply to the parabolic p-Laplace equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right),$$

with $2 \leq p < \infty$.

Weak solutions of the above equations locally belong to a slightly higher Sobolev space than assumed a priori, as Kinnunen and Lewis proved in [KL00]. We intend to show that this also holds globally, that is, up to the boundary. To this end, we prove that the gradient of a weak solution satisfies a global reverse Hölder inequality. In contrast to the local case, the regularity of the boundary, as well as the boundary and initial values, play a role in the proofs. We assume that the complement of the domain satisfies a capacity density condition, which is essentially sharp for our main results. In addition, the boundary values are assumed to belong to an appropriate higher Sobolev space. Note, however, that the results of this work are already nontrivial for regular domains and smooth boundary values.

The proofs are based on Caccioppoli and Sobolev-Poincaré-type inequalities, as well as on the self-improving property of a reverse Hölder inequality. Due to nonquadratic growth conditions, the proofs apply intrinsic scaling and covering arguments. One of the advantages of this method lies in the fact that it can be employed to a wide variety of problems. Indeed, the proofs extend to parabolic systems of the form

$$\frac{\partial u_i}{\partial t} = \operatorname{div} A_i(x, t, \nabla u), \quad i = 1, 2, \ldots, n,$$

although we consider the scalar case for simplicity.

Motivation for studying the higher integrability comes from applications to partial regularity (see, for example, [GM79]) and stability questions, to mention a few. On the other hand, the regularity properties of solutions are often interesting in their own right.

The first higher integrability results apparently date back to a 1957 paper of Bojarski, [Boj57]. Later, Elcrat and Meyers proved the local higher integrability for nonlinear elliptic systems in [EM75] (see also [Gia83]). In [GS82], Giaquinta and Struwe studied similar questions for systems of parabolic equations with quadratic growth conditions. In addition, Arkhipova has considered the global integrability questions for parabolic systems, for example, in [Ark92] and [Ark95]. For recent higher integrability results, see [AM07].

In [Gra82], Granlund showed that an elliptic minimizer has the higher integrability property if the complement of the domain satisfies a measure density condition. Later, Kilpeläinen and Koskela generalized the elliptic results to the uniform capacity density condition in [KK94]. For a good survey of boundary regularity, see Section 8 of [Mik96]. Recently, it was shown in [Par] that parabolic quasiminimizers with quadratic growth conditions have a global higher integrability property.

This work is organized as follows. Section 2 introduces the problem and notation, while the following sections consider the higher integrability near the lateral and initial boundaries separately: Sections 3 and 4 concentrate on the lateral boundary case while Sections 5 and 6 are devoted to estimates near the initial boundary. Theorem 4.7 provides the main result.

2. Preliminaries

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$ and let $p \geq 2$. We study the equation

$$\frac{\partial u}{\partial t} = \operatorname{div} A(x,t,\nabla u), \qquad (x,t) \in \Omega \times (0,T), \tag{2.1}$$

where $u : \Omega \times (0,T) \to \mathbb{R}$, $A : \Omega \times (0,T) \times \mathbb{R}^n \to \mathbb{R}^n$, and A satisfies the following conditions.

(1) $x \mapsto A(x,t,\xi)$ and $t \mapsto A(x,t,\xi)$ are measurable for every ξ,
(2) $\xi \mapsto A(x,t,\xi)$ is continuous for almost every (x,t),
(3) there exist constants $0 < \alpha \leq \beta < \infty$ such that for every ξ and for almost every (x,t), we have $A(x,t,\xi)\cdot\xi \geq \alpha\,|\xi|^p$ and $|A(x,t,\xi)| \leq \beta\,|\xi|^{p-1}$.

As usual, $W^{1,p}(\Omega)$ denotes the Sobolev space of functions in $L^p(\Omega)$ whose first distributional partial derivatives belong to $L^p(\Omega)$ with the norm

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)} .$$

The Sobolev space $W_0^{1,p}(\Omega)$ is a completion of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.

The parabolic space $L^p(0,T;W^{1,p}(\Omega))$ is a collection of measurable functions $u(x,t)$ such that for almost every $t \in (0,T)$, the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(\Omega)$, and the norm

$$||u||_{L^p(0,T;W^{1,p}(\Omega))} = \left(\int_0^T ||u||^p_{W^{1,p}(\Omega)} \, \mathrm{d}t \right)^{1/p}$$

is finite. Analogously, the space $L^p(0,T;W_0^{1,p}(\Omega))$ is a collection of measurable functions $u(x,t)$ such that for almost every $t \in (0,T)$, the function $x \mapsto u(x,t)$ belongs to $W_0^{1,p}(\Omega)$ and

$$||u||_{L^p(0,T;W^{1,p}(\Omega))} < \infty.$$

The parabolic Sobolev space $W^{1,2}(0,T;L^2(\Omega))$ is defined as

$$\begin{aligned} &W^{1,2}(0,T;L^2(\Omega)) \\ &\quad = \{\varphi \in L^2(0,T;L^2(\Omega)) \, : \, \frac{\partial \varphi}{\partial t} \in L^2(0,T;L^2(\Omega))\} \end{aligned}$$

with the norm

$$||\varphi||_{W^{1,2}(0,T;L^2(\Omega))} = ||\varphi||_{L^2(0,T;L^2(\Omega))} + \left|\left| \frac{\partial \varphi}{\partial t} \right|\right|_{L^2(0,T;L^2(\Omega))} .$$

Finally, the space $C([0,T];L^2(\Omega))$ comprises all continuous functions $u : [0,T] \to L^2(\Omega)$ (that is, u is continuous with respect to t in the norm $|| \cdot ||_{L^2(\Omega)}$) such that

$$\max_{t \in [0,T]} ||u(\cdot,t)||_{L^2(\Omega)} < \infty.$$

In the Bochner integration theory, the space $L^p(0,T;W^{1,p}(\Omega))$ is defined as a collection of strongly measurable functions $u : (0,T) \to W^{1,p}(\Omega)$ for which

$$\left(\int_0^T ||u||^p_{W^{1,p}(\Omega)} \, \mathrm{d}t \right)^{1/p} < \infty.$$

We could take this definition as a starting point as well. Indeed, $u(x,t)$ is not, in general, product measurable, but there always exists a measurable representative. Consequently, Fubini's theorem is available in this setting also. The reader is referred to Chapter 4 of [Soh01] and Chapter 23 of [Kut98] for further information.

A function u belonging to the space $L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution to (2.1) if

$$-\int_0^T\int_\Omega u\frac{\partial\phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega A(x,t,\nabla u)\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t=0, \tag{2.2}$$

for every $\phi\in C_0^\infty(\Omega\times(0,T))$.

There is a well-recognized difficulty in proving Caccioppoli-type estimates for weak solutions: One often needs a test function depending on u itself, but u may not be admissible. For example, the time derivative of the test function contains $\frac{\partial u}{\partial t}$, which does not necessarily exist as a function. There are several ways to treat this difficulty: We may, for example, use the Steklov averages, as on page 25 in [DiB93], or we may use the standard mollifications. We adopt the latter approach and set

$$\tilde\phi(x,t)=\int_{\mathbb{R}}\phi(x,t-s)\zeta_\varepsilon(s)\,\mathrm{d}s,$$

where $\phi\in C_0^\infty(\Omega\times(0,T))$ and $\zeta_\varepsilon(s)$ is a standard mollifier, whose support is contained in $(-\varepsilon,\varepsilon)$ with $\varepsilon<\operatorname{dist}(\operatorname{spt}(\phi),\Omega\times\{0,T\})$. We insert $\tilde\phi$ into (2.2), change variables, and apply Fubini's theorem to obtain

$$-\int_0^T\int_\Omega u_\varepsilon\frac{\partial\phi}{\partial t}\,\mathrm{d}z+\int_0^T\int_\Omega A(x,t,\nabla u)_\varepsilon\cdot\nabla\phi\,\mathrm{d}z=0. \tag{2.3}$$

Here u_ε and $A(x,t,\nabla u)_\varepsilon$ denote the standard mollifications in the time direction.

We finish this section with the notation used throughout the work. Let

$$D=\Omega\times(0,T)$$

be a space-time cylinder. We denote the points of the cylinder by $z=(x,t)$ and employ a shorthand notation $\mathrm{d}z=\mathrm{d}x\,\mathrm{d}t$. Let $z_0=(x_0,t_0)\in D$ and $\theta,\rho>0$. Then we denote

$$B_\rho(x_0)=\{\,x\in\mathbb{R}^n\,:\,|x-x_0|<\rho\,\},$$
$$\overline{B}_\rho(x_0)=\{\,x\in\mathbb{R}^n\,:\,|x-x_0|\le\rho\,\},$$

and

$$\Lambda_{\theta\rho^2}(t_0)=(t_0-\frac{1}{2}\theta\rho^2,\,t_0+\frac{1}{2}\theta\rho^2).$$

Further, a space-time cylinder in $\mathbb{R}^{n+1}$ is denoted by

$$Q_{\rho,\theta\rho^2}(z_0)=Q_{\rho,\theta\rho^2}(x_0,t_0)=B_\rho(x_0)\times\Lambda_{\theta\rho^2}(t_0).$$

When no confusion arises, we shall omit the reference points and simply write B_ρ, $\Lambda_{\theta\rho^2}$ and $Q_{\rho,\theta\rho^2}$. The integral average of u is denoted by

$$u_\rho(t)=\fint_{B_\rho}u(x,t)\,\mathrm{d}x=\frac{1}{|B_\rho|}\int_{B_\rho}u(x,t)\,\mathrm{d}x,$$

where $|B_\rho|$ denotes the Lebesgue measure of B_ρ. Finally, ϕ' sometimes denotes the time derivative of ϕ instead of $\frac{\partial\phi}{\partial t}$.

3. Estimates near the lateral boundary

In this section, we derive estimates near the lateral boundary $\partial\Omega\times(0,T)$. These estimates are applied in Section 4 in order to prove a reverse Hölder inequality.

A Lebesgue-type initial condition and a Sobolev-type boundary condition turn out to be convenient for our purposes. To be more specific, we say that u is a global solution if $u\in L^p(0,T;W^{1,p}(\Omega))$ satisfies (2.2) as well as the initial and boundary conditions:

$$\begin{gathered} u(\cdot,t)-\varphi(\cdot,t)\in W_0^{1,p}(\Omega)\quad\text{for almost every}\quad t\in(0,T)\\ \text{and}\\ \frac{1}{h}\int_0^h\int_\Omega|u-\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\to 0\quad\text{as}\quad h\to 0, \end{gathered}\tag{3.1}$$

for a given

$$\varphi\in W^{1,2}(0,T;L^2(\Omega))\cap L^p(0,T;W^{1,p}(\Omega)).$$

Observe that already smooth φ leads to a nontrivial theory. We start with a Caccioppoli-type inequality.

Lemma 3.2 (Caccioppoli)**.** *Let u be a global solution with the boundary and initial conditions* (3.1)*. Let $\theta>0$, suppose that $0<\rho<M$ for some $M>0$, and let $Q_{\rho,\theta\rho^2}=Q_{\rho,\theta\rho^2}(x_0,t_0)\subset\mathbb{R}^{n+1}$. Then there exists a constant $c=c(n,p,M,\alpha,\beta)>0$ such that*

$$\begin{aligned} \int_{Q_{\rho,\theta\rho^2}\cap D}|\nabla u|^p\,\mathrm{d}z+\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)}\int_{B_\rho\cap\Omega}|u-\varphi|^2\,\mathrm{d}x\\ \le\frac{c}{\theta\rho^2}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^2\,\mathrm{d}z+\frac{c}{\rho^p}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^p\,\mathrm{d}z\\ +c\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}f^p\,\mathrm{d}z, \end{aligned}$$

where $D=\Omega\times(0,T)$ and $f=\big(|\varphi'|^{p/(p-1)}+|\nabla\varphi|^p\big)^{1/p}$.

Proof: We may assume that $Q_{\rho,\theta\rho^2}\cap D\neq\emptyset$ since otherwise the claim is trivial. Let $t_1\in\Lambda_{\theta\rho^2}\cap(0,T)$. We define $\chi^h_{0,t_1}(t)$ to be a piecewise linear approximation of a characteristic function such that

$$\chi^h_{0,t_1}(t)=1\quad\text{as}\quad h\le t\le t_1-h,$$

$$\chi^h_{0,t_1}(t)=0\quad\text{as}\quad t\le h/10\quad\text{or}\quad t\ge t_1-h/10,$$

and

$$\left|(\chi^h_{0,t_1}(t))'\right| \le \frac{10}{9h}.$$

Further, denote by $\chi^{h,\varepsilon}_{0,t_1}(t)$, u_ε and φ_ε the standard mollifications in the time direction for $\varepsilon < h/20$. We choose a test function

$$\phi_\varepsilon(x,t) = \eta^p(x,t)(u_\varepsilon(x,t) - \varphi_\varepsilon(x,t))\chi^{h,\varepsilon}_{0,t_1}(t),$$

where $\eta \in C_0^\infty(\mathbb{R}^{n+1})$ is a cut-off function such that $\operatorname{spt}\eta \subset Q_{4\rho,\theta(4\rho)^2}$, $\eta(x,t) = 1$ in $Q_{\rho,\theta\rho^2}$, $0 \le \eta \le 1$, and

$$\rho\,|\nabla\eta| + \theta\rho^2\left|\frac{\partial\eta}{\partial t}\right| \le c. \tag{3.3}$$

The mollification in the time direction does not affect the lateral boundary values, and thus $\phi_\varepsilon(\cdot,t) \in W_0^{1,p}(\Omega)$ for almost every $t \in (0,T)$.

To begin with, we insert the test function into (2.3) and manipulate the first term to have

$$-\int_D u_\varepsilon\phi_\varepsilon'\,\mathrm{d}z = -\int_D (u_\varepsilon - \varphi_\varepsilon)\phi_\varepsilon'\,\mathrm{d}z - \int_D \varphi_\varepsilon\phi_\varepsilon'\,\mathrm{d}z. \tag{3.4}$$

By integrating the first term on the right hand side of (3.4) by parts, we obtain

$$\begin{aligned}
-\int_D &(u_\varepsilon - \varphi_\varepsilon)\phi_\varepsilon'\,\mathrm{d}z \\
&= -\int_D \left((u_\varepsilon - \varphi_\varepsilon)^2(\eta^p\chi^{h,\varepsilon}_{0,t_1})' + \frac{1}{2}\left[(u_\varepsilon - \varphi_\varepsilon)^2\right]'\eta^p\chi^{h,\varepsilon}_{0,t_1}\right)\mathrm{d}z \\
&= -\frac{1}{2}\int_D (u_\varepsilon - \varphi_\varepsilon)^2(\eta^p\chi^{h,\varepsilon}_{0,t_1})'\,\mathrm{d}z.
\end{aligned}$$

As a next step, we take limits, apply the initial condition, and use the well-known convergence properties of mollified functions. We deduce for almost every $t_1 \in \Lambda_{\theta\rho^2} \cap (0,T)$ that

$$\begin{aligned}
-\int_D (u_\varepsilon - \varphi_\varepsilon)\phi_\varepsilon'\,\mathrm{d}z \to &-\frac{1}{2}\int_{\Omega\times(0,t_1)} |u-\varphi|^2 p\eta^{p-1}\eta'\,\mathrm{d}z \\
&+ \frac{1}{2}\int_\Omega |u(x,t_1) - \varphi(x,t_1)|^2\eta^p(x,t_1)\,\mathrm{d}x,
\end{aligned}$$

as first $\varepsilon \to 0$ and then $h \to 0$. Because we take the limits in this order, the mollifications are well defined. Observe also that the initial boundary term disappears at $t = 0$ because of the initial condition.

Then we combine the previous estimates, integrate the last term of (3.4) by parts, and obtain

$$
\begin{aligned}
-\int_D u_\varepsilon \phi_\varepsilon' \,\mathrm{d}z \to -\frac{1}{2}\int_{\Omega\times(0,t_1)} &|u-\varphi|^2 p\eta^{p-1}\eta' \,\mathrm{d}z \\
&+\frac{1}{2}\int_\Omega (u(x,t_1)-\varphi(x,t_1))^2\eta^p(x,t_1)\,\mathrm{d}x \\
&+\int_{\Omega\times(0,t_1)} \varphi'\eta^p(u-\varphi)\,\mathrm{d}z,
\end{aligned}
$$

as first $\varepsilon\to 0$ and then $h\to 0$.

Inserting the test function into the second term of (2.3) implies

$$
\begin{aligned}
&\int_D A(x,t,\nabla u)_\varepsilon\cdot\nabla\left(\eta^p(u_\varepsilon-\varphi_\varepsilon)\chi_{0,t_1}^{h,\varepsilon}\right)\,\mathrm{d}z \\
&\quad\to \int_{\Omega\times(0,t_1)} A(x,t,\nabla u)\cdot\left[p\eta^{p-1}\nabla\eta(u-\varphi)+\eta^p(\nabla u-\nabla\varphi)\right]\,\mathrm{d}z,
\end{aligned}
$$

as first $\varepsilon\to 0$ and then $h\to 0$.

Collecting the facts, we arrive at

$$
\begin{aligned}
\int_{\Omega\times(0,t_1)} &\eta^p A(x,t,\nabla u)\cdot\nabla u\,\mathrm{d}z + \frac{1}{2}\int_\Omega |u(x,t_1)-\varphi(x,t_1)|^2\eta^p(x,t_1)\,\mathrm{d}x \\
&\le \frac{1}{2}\int_{\Omega\times(0,t_1)} |u-\varphi|^2 p\eta^{p-1}|\eta'|\,\mathrm{d}z + \int_{\Omega\times(0,t_1)} |\varphi'|\,\eta^p|u-\varphi|\,\mathrm{d}z \\
&\quad+\int_{\Omega\times(0,t_1)} |A(x,t,\nabla u)|\,p\eta^{p-1}\,|\nabla\eta|\,|u-\varphi|\,\mathrm{d}z \\
&\quad+\int_{\Omega\times(0,t_1)} |A(x,t,\nabla u)|\,\eta^p\,|\nabla\varphi|\,\mathrm{d}z.
\end{aligned}
\tag{3.5}
$$

In view of our hypotheses on A, the first term on the left hand side satisfies the inequality

$$
\alpha\int_{\Omega\times(0,t_1)} \eta^p\,|\nabla u|^p\,\mathrm{d}z \le \int_{\Omega\times(0,t_1)} \eta^p A(x,t,\nabla u)\cdot\nabla u\,\mathrm{d}z.
$$

Since $\rho<M$, there exists a constant $c>0$ such that $1\le c/\rho^p$, where c, of course, depends on M. Consequently, Young's inequality implies

$$
\begin{aligned}
&\int_{\Omega\times(0,t_1)} |\varphi'|\,\eta^p|u-\varphi|\,\mathrm{d}z \\
&\qquad\le \varepsilon\int_{\Omega\times(0,t_1)} |\varphi'|^{p/(p-1)}\,\eta^p\,\mathrm{d}z + \frac{c}{\rho^p}\int_{\Omega\times(0,t_1)} |u-\varphi|^p\eta^p\,\mathrm{d}z,
\end{aligned}
$$

where the constant depends on M and $\varepsilon > 0$. Next we estimate the third term on the right hand side of (3.5). Young's inequality and the structural assumptions on A lead to

$$\begin{aligned}\int_{\Omega\times(0,t_1)} &|A(x,t,\nabla u)|\, p\eta^{p-1}\, |\nabla\eta|\, |u-\varphi| \,\mathrm{d}z \\ &\le \varepsilon \int_{\Omega\times(0,t_1)} |\nabla u|^p\, \eta^p \,\mathrm{d}z + c\int_{\Omega\times(0,t_1)} |\nabla\eta|^p\, |u-\varphi|^p \,\mathrm{d}z.\end{aligned}$$

A similar reasoning allows us to estimate the fourth term on the right hand side of (3.5) as

$$\begin{aligned}\int_{\Omega\times(0,t_1)} &|A(x,t,\nabla u)|\, \eta^p\, |\nabla\varphi| \,\mathrm{d}z \\ &\le \varepsilon \int_{\Omega\times(0,t_1)} |\nabla u|^p\, \eta^p \,\mathrm{d}z + c\int_{\Omega\times(0,t_1)} \eta^p\, |\nabla\varphi|^p \,\mathrm{d}z.\end{aligned}$$

Let us then estimate the second term on the left hand side of (3.5). We can choose $t_1 \in \Lambda_{\theta\rho^2} \cap (0,T)$ such that

$$\begin{aligned}\frac{1}{2} \operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} &\int_{B_\rho\cap\Omega} |u-\varphi|^2\eta^p \,\mathrm{d}x \\ &\le \int_{\Omega} |u(x,t_1)-\varphi(x,t_1)|^2 \eta^p(x,t_1)\,\mathrm{d}x.\end{aligned}$$

Finally, we combine the above estimates with (3.5) and choose $\varepsilon > 0$ small enough to absorb

$$\varepsilon \int_{\Omega\times(0,t_1)} \eta^p\, |\nabla u|^p \,\mathrm{d}z$$

into the left hand side. Since η satisfies condition (3.3), we obtain the claim. □

The regularity of the boundary plays a role in the global higher integrability. In this work, we assume that the complement of the domain satisfies a uniform capacity density condition.

Let $1 < p < \infty$. The *variational p-capacity* of a compact set $C \subset \Omega$ is defined to be

$$\operatorname{cap}_p(C,\Omega) = \inf_g \int_\Omega |\nabla g|^p \,\mathrm{d}x,$$

where the infimum is taken over all the functions $g \in C_0^\infty(\Omega)$ such that $g = 1$ in C. To define the variational p-capacity of an open set $U \subset \Omega$, we take the supremum over the capacities of the compact sets belonging to U. The variational p-capacity of an arbitrary set $E \subset \Omega$ is defined by taking the infimum over the capacities of the open sets containing E. For further details, see Chapter 2 of [HKM93], Chapter 2 of [MZ97], or Chapter 4 of [EG92].

A set $E \subset \mathbb{R}^n$ is said to be of p-capacity zero if

$$\operatorname{cap}_p(E \cap U, U) = 0$$

for all open $U \subset \mathbb{R}^n$. For the capacity of a ball, we obtain the following simple formula

$$\operatorname{cap}_p(\overline{B}_\rho, B_{2\rho}) = c\rho^{n-p}, \tag{3.6}$$

where $c > 0$ depends only on n and p.

Let us now introduce the capacity density condition which we later impose on the complement of the domain. For the higher integrability results, this condition is essentially sharp as pointed out in Remark 3.3 of [KK94] in the elliptic case.

Definition 3.7. A set $E \subset \mathbb{R}^n$ is *uniformly p-thick* if there exist constants $\mu, \rho_0 > 0$ such that

$$\operatorname{cap}_p(E \cap \overline{B}_\rho(x), B_{2\rho}(x)) \geq \mu \operatorname{cap}_p(\overline{B}_\rho(x), B_{2\rho}(x)),$$

for all $x \in E$ and for all $0 < \rho < \rho_0$.

If we replace the capacity with the Lebesgue measure in the definition above, then we obtain a measure density condition. A set E, satisfying the measure density condition, is uniformly p-thick for all $p > 1$. If $p > n$, then every nonempty set is uniformly p-thick. The following lemma extends the capacity estimate in Definition 3.7.

Lemma 3.8. *Let Ω be a bounded open set, and suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick. Choose $y \in \Omega$ such that $B_{\frac{4}{3}\rho}(y) \setminus \Omega \neq \emptyset$. Then there exists a constant $\tilde{\mu} = \tilde{\mu}(\mu, \rho_0, n, p) > 0$ such that*

$$\operatorname{cap}_p(\overline{B}_{2\rho}(y) \setminus \Omega, B_{4\rho}(y)) \geq \tilde{\mu} \operatorname{cap}_p(\overline{B}_{2\rho}(y), B_{4\rho}(y)).$$

Proof: Since $B_{\frac{4}{3}\rho}(y) \setminus \Omega \neq \emptyset$, we may choose $x \in \mathbb{R}^n \setminus \Omega$ such that $\operatorname{dist}(x, y) < \frac{4}{3}\rho$. Then

$$B_{4\rho}(y) \subset B_{(\frac{4}{3}+4)\rho}(x) \quad \text{and} \quad B_{\frac{2}{3}\rho}(x) \subset B_{2\rho}(y),$$

and hence due to the properties of the capacity, we obtain

$$\begin{aligned} \operatorname{cap}_p(\overline{B}_{2\rho}(y) \setminus \Omega, B_{4\rho}(y)) &\geq \operatorname{cap}_p(\overline{B}_{2\rho}(y) \setminus \Omega, B_{(\frac{4}{3}+4)\rho}(x)) \\ &\geq \operatorname{cap}_p(\overline{B}_{\frac{2}{3}\rho}(x) \setminus \Omega, B_{(\frac{4}{3}+4)\rho}(x)). \end{aligned} \tag{3.9}$$

Lemma 2.16 of [HKM93] provides the estimate

$$\operatorname{cap}_p(\overline{B}_{\frac{2}{3}\rho}(x) \setminus \Omega, B_{(\frac{4}{3}+4)\rho}(x)) \geq c \operatorname{cap}_p(\overline{B}_{\frac{2}{3}\rho}(x) \setminus \Omega, B_{\frac{4}{3}\rho}(x)),$$

and hence the uniform p-thickness condition implies

$$\operatorname{cap}_p(\overline{B}_{\frac{2}{3}\rho}(x) \setminus \Omega, B_{(\frac{4}{3}+4)\rho}(x)) \geq c\mu \operatorname{cap}_p(\overline{B}_{\frac{2}{3}\rho}(x), B_{\frac{4}{3}\rho}(x)). \tag{3.10}$$

According to (3.6), there exists a constant $c > 0$ such that

$$\operatorname{cap}_p(\overline{B}_{\frac{2}{3}\rho}(x), B_{\frac{4}{3}\rho}(x)) \geq c \operatorname{cap}_p(\overline{B}_{2\rho}(y), B_{4\rho}(y)). \tag{3.11}$$

A combination of (3.9), (3.10), and (3.11) implies the result. □

A uniformly p-thick domain has a deep self-improving property. This result was shown by Lewis in [Lew88]. See also [Anc86] and [Mik96].

Theorem 3.12. *Let $1 < p \leq n$. If a set E is uniformly p-thick, then there exists a constant $q = q(n, p, \mu)$ such that $1 < q < p$ for which E is uniformly q-thick.*

A uniformly q-thick set is also uniformly p-thick for all $p \geq q$. This is a simple consequence of Hölder's and Young's inequalities. We prove the claim for a compact set.

Lemma 3.13. *If a compact set E is uniformly q-thick, then E is uniformly p-thick for all $p \geq q$.*

Proof: Choose $x \in E$ and ρ such that $0 < \rho < \rho_0$, where ρ_0 is the constant in Definition 3.7. Denote $B_\rho = B_\rho(x)$. By (3.6), we have

$$\operatorname{cap}_p(\overline{B}_\rho, B_{2\rho}) = c\rho^{n-p} = c\rho^{q-p} \operatorname{cap}_q(\overline{B}_\rho, B_{2\rho}),$$

where the constant in the last expression depends on n, p and q.

We choose $g \in C_0^\infty(B_{2\rho})$ such that $g = 1$ in $E \cap \overline{B}_\rho$. Consequently, g is admissible in calculating the q-capacity for $E \cap \overline{B}_\rho$, and thus Hölder's inequality implies

$$\begin{aligned} &\operatorname{cap}_q(E \cap \overline{B}_\rho, B_{2\rho}) \\ &\qquad \leq \int_{B_{2\rho}} |\nabla g|^q \,\mathrm{d}x \leq c\rho^{n(1-q/p)} \left(\int_{B_{2\rho}} |\nabla g|^p \,\mathrm{d}x \right)^{q/p}. \end{aligned}$$

By the uniform q-thickness of E and the above estimates, we get

$$\begin{aligned} \operatorname{cap}_p(\overline{B}_\rho, B_{2\rho}) &= c\rho^{q-p} \operatorname{cap}_q(\overline{B}_\rho, B_{2\rho}) \\ &\leq \mu^{-1} c\rho^{q-p} \operatorname{cap}_q(E \cap \overline{B}_\rho, B_{2\rho}) \\ &\leq c\rho^{(q-p)(1-n/p)} \left(\int_{B_{2\rho}} |\nabla g|^p \,\mathrm{d}x \right)^{q/p}. \end{aligned}$$

Then we apply Young's inequality and have

$$\operatorname{cap}_p(\overline{B}_\rho, B_{2\rho}) \leq \varepsilon\rho^{n-p} + c \int_{B_{2\rho}} |\nabla g|^p \,\mathrm{d}x.$$

The first term on the right can be absorbed into the left side by choosing $\varepsilon > 0$ small enough. The result follows by taking the infimum with respect to g. □

Next we establish a well-known version of the Sobolev-type inequality (see [Hed81], Chapter 10 of [Maz85] and also Lemma 3.1 of [KK94]). Later, we combine this estimate with the boundary regularity condition and obtain a boundary version of Sobolev's inequality. We repeat the proof for the convenience of the reader.

The proof uses quasicontinuous representatives of the Sobolev functions. We call $u \in W^{1,p}(\Omega)$ *p-quasicontinuous* if for each $\varepsilon > 0$ there exists an open set U, $U \subset \Omega \subset B_{R'}$, such that $\operatorname{cap}_p(U, B_{2R'}) \le \varepsilon$, and the restriction of u to the set $\Omega \setminus U$ is finite valued and continuous.

The p-quasicontinuous functions are closely related to the Sobolev space $W^{1,p}(\Omega)$: For example, if $u \in W^{1,p}(\Omega)$, then u has a p-quasicontinuous representative. In addition, the capacity can be written in terms of quasicontinuous representatives.

From now on, we only consider the case $p \le n$ for simplicity. This restriction is only technical, but, in this way, we avoid repeating essentially the same proofs with more complicated powers emerging from the different versions of the Sobolev-Poincaré inequalities.

Lemma 3.14. *Suppose that $q \in (1,p)$ and that $u \in W^{1,q}(B_{2\rho})$ is q-quasicontinuous. Denote*

$$N_{B_\rho}(u) = \{\, x \in \overline{B}_\rho \,:\, u(x) = 0 \}$$

and choose $\tilde q \in [q, q^*]$, where $q^* = qn/(n-q)$. Then there exists a constant $c = c(n,q) > 0$ such that

$$\left(\fint_{B_{2\rho}} |u|^{\tilde q} \, \mathrm{d}x \right)^{1/\tilde q} \le \left(\frac{c}{\operatorname{cap}_q(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla u|^q \, \mathrm{d}x \right)^{1/q}.$$

Proof: First, assume that $u_{B_{2\rho}} = \fint_{B_{2\rho}} u(x)\,\mathrm{d}x \neq 0$. Then choose $\phi \in C_0^\infty(B_{2\rho})$ such that $\phi = 1$ in B_ρ and $|\nabla \phi| \le c/\rho$. We define

$$v = \phi(u_{B_{2\rho}} - u).$$

Clearly, $v \in W_0^{1,q}(B_{2\rho})$ is q-quasicontinuous and

$$v = u_{B_{2\rho}} - u \quad \text{in} \quad B_\rho.$$

Furthermore,

$$\int_{B_{2\rho}} |\nabla v|^q \, \mathrm{d}x \le c \int_{B_{2\rho}} |\nabla u|^q \, \mathrm{d}x \tag{3.15}$$

due to Poincaré's inequality.

The variational q-capacity of a set $E \subset \overline{B}_\rho$ can be written in the form

$$\operatorname{cap}_q(E, B_{2\rho}) = \inf_g \int_{B_{2\rho}} |\nabla g|^q \, \mathrm{d}x,$$

where $g \in W_0^{1,q}(B_{2\rho})$ is q-quasicontinuous and $g \geq 1$ in E, except on a set of q-capacity zero (see, for example, pages 75 and 66 of [MZ97]). It follows that

$$\int_{B_{2\rho}} \left|\nabla v/u_{B_{2\rho}}\right|^q \, \mathrm{d}x \geq \operatorname{cap}_q(N_{B_\rho}(u), B_{2\rho})$$

since $v/u_{B_{2\rho}} = 1$ in $N_{B_\rho}(u)$, and hence

$$\left|u_{B_{2\rho}}\right| \leq \left(\frac{1}{\operatorname{cap}_q(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla v|^q \, \mathrm{d}x \right)^{1/q}. \tag{3.16}$$

The triangle inequality, (3.16), Poincaré's inequality, and (3.15) lead to

$$\begin{aligned} \left(\fint_{B_{2\rho}} |u|^{\tilde{q}} \, \mathrm{d}x \right)^{1/\tilde{q}} &\leq \left(\fint_{B_{2\rho}} |u_{B_{2\rho}} - u|^{\tilde{q}} \, \mathrm{d}x \right)^{1/\tilde{q}} + \left|u_{B_{2\rho}}\right| \\ &\leq c \left(\rho^{q-n} \int_{B_{2\rho}} |\nabla u|^q \, \mathrm{d}x \right)^{1/q} + \left(\frac{1}{\operatorname{cap}_q(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla u|^q \, \mathrm{d}x \right)^{1/q}. \end{aligned}$$

Since $N_{B_\rho}(u) \subset \overline{B}_\rho$, estimate (3.6) implies

$$\operatorname{cap}_q(N_{B_\rho}(u), B_{2\rho}) \leq c\rho^{n-q},$$

and, consequently,

$$\left(\fint_{B_{2\rho}} |u|^{\tilde{q}} \, \mathrm{d}x \right)^{1/\tilde{q}} \leq \left(\frac{c}{\operatorname{cap}_q(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla u|^q \, \mathrm{d}x \right)^{1/q}.$$

If $u_{B_{2\rho}} = 0$, the claim follows immediately from Poincaré's inequality. □

In the same way, we could prove that the above estimate holds if the powers on both sides are replaced by p.

Lemma 3.17. *Suppose that $u \in W^{1,p}(B_{2\rho})$ is p-quasicontinuous and let $N_{B_\rho}(u)$ be as above. Then there exists a constant $c = c(n,p) > 0$ such that*

$$\left(\fint_{B_{2\rho}} |u|^p \, \mathrm{d}x \right)^{1/p} \leq \left(\frac{c}{\operatorname{cap}_p(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla u|^p \, \mathrm{d}x \right)^{1/p}.$$

In order to derive a reverse Hölder inequality, we estimate the right hand side of Caccioppoli's inequality in terms of the gradient. A natural idea is to use Sobolev's inequality, but there is a principal difficulty in the parabolic case: We assume little regularity for a weak solution u in the time direction, and Sobolev's inequality is not applicable in space-time cylinders as such. Nevertheless, weak solutions satisfy the following version of parabolic Sobolev's inequality.

Lemma 3.18 (parabolic Sobolev)**.** *Let u be a global solution with the boundary and initial conditions* (3.1)*. Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick. Let $\theta > 0$ and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0,t_0) \subset \mathbb{R}^{n+1}$ such that $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$. Further, choose M such that $\rho < M$. Then there exists a constant $c = c(n,p,M,\mu,\rho_0,\alpha,\beta) > 0$ so that*

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} \int_{B_\rho\cap\Omega} |u-\varphi|^2 \,\mathrm{d}x \\ \leq c\rho^{n+2}\left(\frac{1}{\left|Q_{4\rho,\theta(4\rho)^2}\right|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |\nabla(u-\varphi)|^p \,\mathrm{d}z\right)^{2/p} \\ + c\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |\nabla(u-\varphi)|^p \,\mathrm{d}z + c\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} f^p \,\mathrm{d}z,\end{aligned}$$

where $f = \left(|\varphi'|^{p/(p-1)} + |\nabla\varphi|^p\right)^{1/p}$.

Proof: In order to prove the claim, we estimate the right hand side of Caccioppoli's inequality by applying Lemma 3.17 and the uniform capacity density condition.

Lemma 3.2 provides the estimate

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} \int_{B_\rho\cap\Omega} |u-\varphi|^2 \,\mathrm{d}x \\ \leq \frac{c}{\theta\rho^2}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |u-\varphi|^2 \,\mathrm{d}z + \frac{c}{\rho^p}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |u-\varphi|^p \,\mathrm{d}z \\ + c\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} f^p \,\mathrm{d}z.\end{aligned} \tag{3.19}$$

We extend $u(\cdot,t)-\varphi(\cdot,t)$ by zero outside of Ω and use the same notation for the extension. For a given t, we denote

$$N_{B_{2\rho}}(u-\varphi) = \{\, x \in \overline{B}_{2\rho} \,:\, u(x,t)-\varphi(x,t) = 0\}.$$

We estimate the first term on the right side of (3.19) by using Hölder's inequality and Lemma 3.17. Consequently,

$$
\begin{aligned}
&\frac{c}{\theta\rho^2}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^2\,\mathrm{d}z\\
&\le \frac{c}{\theta\rho^2}\int_{\Lambda_{\theta(4\rho)^2}\cap(0,T)}\rho^n\left(\frac{1}{|B_{4\rho}|}\int_{B_{4\rho}}|u-\varphi|^p\,\mathrm{d}x\right)^{2/p}\mathrm{d}t\\
&\le \frac{c\rho^n}{\theta\rho^2}\int_{\Lambda_{\theta(4\rho)^2}\cap(0,T)}\left(\frac{1}{\mathrm{cap}_p(N_{B_{2\rho}}(u-\varphi),B_{4\rho})}\int_{B_{4\rho}}|\nabla(u-\varphi)|^p\,\mathrm{d}x\right)^{2/p}\mathrm{d}t.
\end{aligned}
$$

Since $\mathbb{R}^n\setminus\Omega$ is uniformly p-thick and $B_{\frac{4}{3}\rho}(x_0)\setminus\Omega\neq\emptyset$, we conclude by Lemma 3.8 and (3.6) that

$$
\mathrm{cap}_p(N_{B_{2\rho}}(u-\varphi),B_{4\rho}(x_0))\ge\tilde{\mu}\,\mathrm{cap}_p(\overline{B}_{2\rho}(x_0),B_{4\rho}(x_0))=c\rho^{n-p}
$$

for almost every $t\in[0,T]$. Notice that this estimate still holds true if we redefine $u(\cdot,t)-\varphi(\cdot,t)$ in a set of measure zero in Ω. Next we merge the estimates and obtain

$$
\begin{aligned}
&\frac{c}{\theta\rho^2}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^2\,\mathrm{d}z\\
&\le c\rho^{n+2}\left(\frac{1}{|Q_{4\rho,\theta(4\rho)^2}|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla(u-\varphi)|^p\,\mathrm{d}z\right)^{2/p}.
\end{aligned}
$$

A similar calculation can be repeated for the second term on the right hand side of (3.19), and thus the result follows. □

One of the difficulties in proving the main result is the fact that both powers 2 and p play a role in the above inequalities. For example, if we simply divide the term

$$
\frac{c}{\rho^p}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^p\,\mathrm{d}z
$$

into two parts, as in the quadratic case (see [GS82]), powers do not match. Therefore, we derive a Sobolev-type lemma that takes both powers into account. We again work out the proof in the case $p\le n$ for simplicity.

Lemma 3.20. *Let u be a global solution with the boundary and initial conditions* (3.1)*. Suppose that $\mathbb{R}^n\setminus\Omega$ is uniformly p-thick. Let $\theta>0$ and choose $Q_{\rho,\theta\rho^2}=Q_{\rho,\theta\rho^2}(x_0,t_0)\subset\mathbb{R}^{n+1}$ such that $B_{\frac{4}{3}\rho}(x_0)\setminus\Omega\neq\emptyset$. Then there exist constants $\tilde{q}=\tilde{q}(n,p,\mu)<p$ and $c=c(n,p,\mu,\rho_0)>0$*

such that

$$\begin{aligned}\frac{1}{|Q_{4\rho,\theta(4\rho)^2}|}&\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^p\,\mathrm{d}z\\ &\le\left(\frac{c}{|Q_{4\rho,\theta(4\rho)^2}|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla(u-\varphi)|^{\tilde q}\,\mathrm{d}z\right)^{q/\tilde q}\\ &\quad\cdot\left(\operatorname*{ess\,sup}_{t\in\Lambda_{\theta(4\rho)^2}\cap(0,T)}\int_{B_{4\rho}\cap\Omega}|u-\varphi|^2\,\mathrm{d}x\right)^{q/n},\end{aligned}$$

where $q=pn/(n+2)$.

Proof: The proof is based on Hölder's and Sobolev's inequalities. We set

$$v(x,t)=|u(x,t)-\varphi(x,t)|\,,$$

and employ Hölder's inequality to obtain

$$\begin{aligned}\int_{B_{4\rho}\cap\Omega}v^p\,\mathrm{d}x&=\int_{B_{4\rho}\cap\Omega}v^{2p/(2+n)}v^{p-2p/(2+n)}\,\mathrm{d}x\\ &\le\left(\int_{B_{4\rho}\cap\Omega}v^2\,\mathrm{d}x\right)^{q/n}\left(\int_{B_{4\rho}\cap\Omega}v^{q^*}\,\mathrm{d}x\right)^{q/q^*},\end{aligned}$$

where $q^*=qn/(n-q)=np/(n+2-p)$. Observe that q^* is well defined provided that $p<n+2$. This condition is satisfied since we assumed that $p\le n$.

We extend $v(\cdot,t)$ by zero outside of Ω and use the same notation for the extension. Let $\tilde q\ge q$ be fixed later and set $\tilde q^*=\tilde q n/(n-\tilde q)$. Furthermore, for a given t, denote

$$N_{B_{2\rho}}(v)=\{\,x\in\overline{B}_{2\rho}\,:\,v(x,t)=0\}.$$

According to Hölder's inequality and Lemma 3.14, we get

$$\begin{aligned}&\left(\int_{B_{4\rho}\cap\Omega}v^{q^*}\,\mathrm{d}x\right)^{q/q^*}\\ &\qquad\le c\rho^{nq/q^*}\left(\frac{1}{|B_{4\rho}|}\int_{B_{4\rho}}v^{\tilde q^*}\,\mathrm{d}x\right)^{q/\tilde q^*}\qquad(3.21)\\ &\qquad\le c\rho^{nq/q^*}\left(\frac{1}{\operatorname{cap}_{\tilde q}(N_{B_{2\rho}}(v),B_{4\rho})}\int_{B_{4\rho}}|\nabla v|^{\tilde q}\,\mathrm{d}x\right)^{q/\tilde q}.\end{aligned}$$

Notice that the assumption $\tilde q<p\le n$ is used here. In the case $\tilde q>n$, we should use a different version of Sobolev's inequality.

To continue, we would like to use the uniform capacity density condition, but this is not immediately possible since $\tilde{q} < p$ and since we only assumed that the complement of the domain is uniformly p-thick. Nevertheless, Theorem 3.12 asserts that the density condition satisfies the self-improving property. This, together with Lemma 3.8 and (3.6), implies

$$\operatorname{cap}_{\tilde{q}}(N_{B_{2\rho}}(u-\varphi), B_{4\rho}) \geq \tilde{\mu} \operatorname{cap}_{\tilde{q}}(\overline{B}_{2\rho}, B_{4\rho}) = c\rho^{n-\tilde{q}},$$

for almost every t and for large enough $\tilde{q} < p$. We combine this capacity estimate with (3.21) and conclude that

$$\left(\int_{B_{4\rho}\cap\Omega} v^{q^*} \,\mathrm{d}x \right)^{q/q^*} \leq c\rho^n \left(\fint_{B_{4\rho}} |\nabla v|^{\tilde{q}} \,\mathrm{d}x \right)^{q/\tilde{q}}.$$

Collecting the estimates, we arrive at

$$\begin{aligned} \frac{1}{|B_{4\rho}|} \int_{B_{4\rho}\cap D} v^p \,\mathrm{d}x & \\ \leq c \left(\int_{B_{4\rho}} v^2 \,\mathrm{d}x \right)^{q/n} & \left(\frac{1}{|B_{4\rho}|} \int_{B_{4\rho}\cap D} |\nabla v|^{\tilde{q}} \,\mathrm{d}x \right)^{q/\tilde{q}}. \end{aligned}$$

The claim follows by integrating this estimate with respect to time and using Hölder's inequality. □

4. Reverse Hölder inequalities near the lateral boundary

In this section, we derive a reverse Hölder inequality for the gradient of a solution near the lateral boundary and show that this inequality has a self-improving property. We first apply the estimates from the previous section in scaled space-time cylinders and later use covering arguments to extend the results to general cylinders. The scaling takes both the nonlinearity and the boundary effects into account.

Lemma 4.1 (reverse Hölder). *Let u be a global solution with the boundary and initial conditions* (3.1). *Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick. Let $\lambda > 0$, set $\theta = \lambda^{2-p}$, and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0,t_0) \subset \mathbb{R}^{n+1}$ such that $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$. Further, choose M such that $\rho < M$ and suppose that there exists a constant $c_1 \geq 1$ for which*

$$\begin{aligned} c_1^{-1}\lambda^p &\leq \frac{1}{|Q_{\rho,\theta\rho^2}|} \int_{Q_{\rho,\theta\rho^2}\cap D} \big(|\nabla u|^p + f^p \big) \,\mathrm{d}z \\ &\leq \frac{c_1}{|Q_{20\rho,\theta(20\rho)^2}|} \int_{Q_{20\rho,\theta(20\rho)^2}\cap D} \big(|\nabla u|^p + f^p \big) \,\mathrm{d}z \leq c_1^2 \lambda^p, \end{aligned} \tag{4.2}$$

where $f = \big(|\nabla \varphi|^p + |\varphi'|^{p/(p-1)} \big)^{1/p}$*. Then there exist constants* $c = c(n, p, c_1, \mu, \rho_0, M, \alpha, \beta) > 0$ *and* $\tilde{q} = \tilde{q}(n, p, \mu) < p$ *such that*

$$\begin{aligned} &\frac{1}{\left|Q_{20\rho,\theta(20\rho)^2}\right|} \int_{Q_{\rho,\theta\rho^2}\cap D} |\nabla u|^p \,\mathrm{d}z \\ &\qquad \le \left(\frac{c}{\left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |\nabla u|^{\tilde{q}} \,\mathrm{d}z \right)^{p/\tilde{q}} \\ &\qquad\quad + \frac{c}{\left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} f^p \,\mathrm{d}z. \end{aligned}$$

Proof: The idea in the proof is to estimate the terms on the right hand side of Caccioppoli's inequality with the gradient by using the parabolic and capacity versions of Sobolev's inequality. The scaling of the time direction is used in absorbing the additional terms into the left.

Recalling Lemma 3.2, we have

$$\begin{aligned} &\frac{1}{\left|Q_{\rho,\theta\rho^2}\right|} \int_{Q_{\rho,\theta\rho^2}\cap D} \big(|\nabla u|^p + f^p \big) \,\mathrm{d}z \\ &\qquad \le \frac{c}{\theta\rho^2 \left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |u-\varphi|^2 \,\mathrm{d}z \\ &\qquad\quad + \frac{c}{\rho^p \left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |u-\varphi|^p \,\mathrm{d}z \\ &\qquad\quad + \frac{c}{\left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} f^p \,\mathrm{d}z. \end{aligned} \tag{4.3}$$

Since $p \ge 2$ and $\theta = \lambda^{2-p}$, we may estimate the first term on the right in terms of the second by using Hölder's and Young's inequalities. We conclude that

$$\begin{aligned} &\frac{c}{\theta\rho^2 \left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |u-\varphi|^2 \,\mathrm{d}z \\ &\qquad \le c\lambda^{p-2} \left(\frac{1}{\rho^p \left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |u-\varphi|^p \,\mathrm{d}z \right)^{2/p} \\ &\qquad \le \lambda^p \varepsilon + \frac{c}{\rho^p \left|Q_{4\rho,\theta(4\rho)^2}\right|} \int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |u-\varphi|^p \,\mathrm{d}z, \end{aligned} \tag{4.4}$$

and hence it is enough to estimate the second term on the right hand side of (4.3).

In view of Lemma 3.20, there exists a constant $\tilde q < p$ such that

$$\begin{aligned}\frac{1}{\left|Q_{4\rho,\theta(4\rho)^2}\right|}&\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^p\,\mathrm{d}z\\&\leq\left(\frac{c}{\left|Q_{4\rho,\theta(4\rho)^2}\right|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla(u-\varphi)|^{\tilde q}\,\mathrm{d}z\right)^{q/\tilde q}\\&\quad\cdot\left(\operatorname*{ess\,sup}_{t\in\Lambda_{\theta(4\rho)^2}\cap(0,T)}\int_{B_{4\rho}}|u-\varphi|^2\,\mathrm{d}x\right)^{q/n}.\end{aligned}\tag{4.5}$$

Furthermore, Lemma 3.18 allows us to estimate

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in\Lambda_{\theta(4\rho)^2}\cap(0,T)}&\int_{B_{4\rho}\cap\Omega}|u-\varphi|^2\,\mathrm{d}x\\&\leq c\rho^{n+2}\left(\frac{1}{\left|Q_{16\rho,\theta(16\rho)^2}\right|}\int_{Q_{16\rho,\theta(16\rho)^2}\cap D}|\nabla(u-\varphi)|^p\,\mathrm{d}z\right)^{2/p}\\&\quad+c\int_{Q_{16\rho,\theta(16\rho)^2}\cap D}|\nabla(u-\varphi)|^p\,\mathrm{d}z+c\int_{Q_{16\rho,\theta(16\rho)^2}\cap D}f^p\,\mathrm{d}z\\&\leq c\rho^{n+2}\lambda^2,\end{aligned}\tag{4.6}$$

where we also used assumption (4.2) and the scaling $\theta=\lambda^{2-p}$.

Young's inequality, (4.5), and (4.6) imply

$$\begin{aligned}\frac{c}{\rho^p\left|Q_{4\rho,\theta(4\rho)^2}\right|}&\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|u-\varphi|^p\,\mathrm{d}z\\&\leq\frac{c}{\rho^p}\left(\frac{1}{\left|Q_{4\rho,\theta(4\rho)^2}\right|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla(u-\varphi)|^{\tilde q}\,\mathrm{d}z\right)^{q/\tilde q}\left(\rho^{n+2}\lambda^2\right)^{q/n}\\&\leq\left(\frac{c}{\left|Q_{4\rho,\theta(4\rho)^2}\right|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla(u-\varphi)|^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q}+\varepsilon\lambda^p\end{aligned}$$

since $\rho^{-p}=\rho^{-(n+2)q/n}$. We combine the previous estimate with (4.3) and (4.4). Thus, we deduce

$$\begin{aligned}\frac{1}{|Q_{\rho,\theta\rho^2}|}&\int_{Q_{\rho,\theta\rho^2}\cap D}\left(|\nabla u|^p+f^p\right)\mathrm{d}z\\&\leq 2\varepsilon\lambda^p+\left(\frac{c}{\left|Q_{4\rho,\theta(4\rho)^2}\right|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla(u-\varphi)|^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q}\\&\quad+\frac{c}{\left|Q_{4\rho,\theta(4\rho)^2}\right|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}f^p\,\mathrm{d}z.\end{aligned}$$

By assumption (4.2), we have

$$c_1^{-1}\lambda^p \le \frac{1}{|Q_{\rho,\theta\rho^2}|}\int_{Q_{\rho,\theta\rho^2}\cap D}\big(|\nabla u|^p + f^p\big)\,\mathrm{d}z,$$

and, as a consequence, we can choose $\varepsilon > 0$ small enough to absorb $2\varepsilon\lambda^p$ into the left hand side. Finally, since (4.2) implies

$$\begin{aligned}\frac{1}{\big|Q_{20\rho,\theta(20\rho)^2}\big|}&\int_{Q_{20\rho,\theta(20\rho)^2}\cap D}|\nabla u|^p\,\mathrm{d}z\\ &\le \frac{c}{|Q_{\rho,\theta\rho^2}|}\int_{Q_{\rho,\theta\rho^2}\cap D}\big(|\nabla u|^p + f^p\big)\,\mathrm{d}z,\end{aligned}$$

we have proven the claim. □

Next we prove that the reverse Hölder inequality has a self-improving property. In the case $p = 2$, we could use the well-known Giaquinta-Modica lemma, which can be found from [GM79] or [Gia83]. See also [Geh73], [Str80], and [GS82]. Since $p \ge 2$, we follow a different strategy: We split the space-time domain into scaled cylinders so that the reverse Hölder inequality holds in each of them.

We say that $Q_{4R,(4R)^2}(x_0,t_0)$ intersects the lateral boundary if

$$Q_{4R,(4R)^2}(x_0,t_0)\cap(\partial\Omega\times[0,T])\neq\emptyset,$$

and that $Q_{4R,(4R)^2}(x_0,t_0)$ intersects the initial boundary if

$$Q_{4R,(4R)^2}(x_0,t_0)\cap(\Omega\times\{0\})\neq\emptyset.$$

Furthermore, we denote

$$\begin{aligned}&\tilde V^p_\delta(0,T;\Omega)\\ &\quad=\{\varphi\in W^{1,2}(0,T;L^2(\Omega))\cap L^{p+\delta}(0,T;W^{1,p+\delta}(\Omega))\ :\\ &\qquad\qquad\qquad \varphi\in C([0,T];L^2(\Omega)),\ \varphi(\cdot,0)\in W^{1,q+\delta}(\Omega)\},\end{aligned}$$

where $\delta > 0$ and $q = pn/(n+2)$.

The proof of the following theorem quotes some initial boundary estimates from Section 6. We postpone the proofs of these estimates in order to provide the main result as early as possible.

Theorem 4.7. *Let u be a global solution to* (2.2)*, satisfying the boundary and initial conditions* (3.1) *for a boundary function*

$$\varphi\in\tilde V^p_\delta(0,T;\Omega),$$

where $\delta > 0$. Suppose that $\mathbb{R}^n\setminus\Omega$ is uniformly p-thick and that $R < M$ for some $M > 0$. Choose $Q_{R,R^2} = Q_{R,R^2}(x_0,t_0)\subset\mathbb{R}^{n+1}$ such that $Q_{4R,(4R)^2}$ intersects the lateral and initial boundaries. Then there exist *constants* $\varepsilon_0 = \varepsilon_0(n, p, M, \delta, \rho_0, \mu, \alpha, \beta) > 0$ *and* $c > 0$ *with the same dependencies such that for all* $0 \le \varepsilon < \varepsilon_0$*, we have*

$$\begin{aligned}\left(\frac{1}{|Q_{R,R^2}|}\int_{Q_{R,R^2}\cap D}|\nabla u|^{p+\varepsilon}\,\mathrm{d}z\right)^{1/(p+\varepsilon)} &\le \left(\frac{c}{|B_{4R}|}\int_{B_{4R}\cap\Omega}\tilde{f}^{q+\varepsilon}\,\mathrm{d}x\right)^{1/(q+\varepsilon)}\\ &+\left(\frac{c}{|Q_{4R,(4R)^2}|}\int_{Q_{4R,(4R)^2}\cap D}\left(|\nabla u|^p + f^{p+\varepsilon}\right)\mathrm{d}z\right)^{1/(p+\varepsilon)}\\ &+\left(\frac{c}{|Q_{4R,(4R)^2}|}\int_{Q_{4R,(4R)^2}\cap D}\left(|\nabla u|^p + f^{p}\right)\mathrm{d}z\right)^{\sigma},\end{aligned}$$

where $\sigma = (2+\varepsilon)/(2(p+\varepsilon))$, $q = pn/(n+2)$, $\tilde{f} = |\nabla\varphi(x,0)|$, *and* $f = \big(|\nabla\varphi|^p + |\varphi'|^{p/(p-1)}\big)^{1/p}$.

Proof: The proof consists of several steps. First, we cover the space-time cylinder with smaller Whitney-type cylinders. By using Whitney cylinders, we are able to derive estimates with constants independent of the location. Then we divide the space-time cylinder into a good and a bad set. In the good set, the function $|\nabla u|^p$ is in control by definition, and in the bad set, we can estimate the average of the gradient by using the reverse Hölder inequality. The Calderón-Zygmund decomposition is usually applied for this, but here we use a different strategy that seems to work better in the parabolic setting with general growth conditions. Finally, we obtain the higher integrability by using Fubini's theorem.

We denote $Q_0 = Q_{4R,(4R)^2}(z_0) = Q_{4R,(4R)^2}(x_0, t_0)$ and divide Q_0 into the Whitney-type cylinders

$$Q_i = Q_{r_i, r_i^2}(y_i, \tau_i),\ i = 1, 2, \ldots,$$

where r_i is comparable to the parabolic distance of Q_i to the ∂Q_0 (see, for example, page 15 of [Ste93]). *Parabolic distance* is defined to be

$$\operatorname{dist}_p(E, F) = \inf\left\{|x-\overline{x}| + |t-\overline{t}|^{1/2} : (x,t)\in E, (\overline{x},\overline{t})\in F\right\}.$$

In addition, cylinders Q_i are of bounded overlap, meaning that every z belongs, at most, to a fixed finite number of cylinders, and

$$Q_{5r_i,(5r_i)^2} \subset Q_0.$$

The next step is to divide Q_0 into a good and a bad set. We aim to choose the scaling $\lambda > 0$ so that condition (4.2) holds in the cylinders having a center point in the bad set. To this end, set

$$\lambda_0' = \left(\frac{1}{|Q_0|} \int_{Q_0 \cap D} (|\nabla u|^p + f^p) \, \mathrm{d}z \right)^{1/2},$$

and choose λ such that

$$\lambda > \max(\lambda_0', 1) = \lambda_0.$$

For $(x,t) \in Q_0 \cap D$, we define

$$h(x,t) = \frac{1}{c_2 \, |Q_0|^{1/2}} \min\{|Q_i|^{1/2} : (x,t) \in Q_i\} \, |\nabla u(x,t)| \, ,$$

where $c_2 \geq 1$ is fixed later. Further, choose $(\tilde{x}, \tilde{t}) \in D$ such that

$$h(\tilde{x}, \tilde{t}) > \lambda$$

and fix Q_i for which $(\tilde{x}, \tilde{t}) \in Q_i \cap D$. We define

$$\alpha = \alpha(\tilde{x}, \tilde{t}) = \frac{|Q_0|}{|Q_i|},$$

and

$$\theta = \lambda^{2-p} \alpha^{1-p/2}.$$

If $(\tilde{x}, \tilde{t})$ belongs to many Whitney cylinders, any of them will do.

Next we show that the second inequality in condition (4.2) is valid due to the definition of λ. For $Q_{r,\theta r^2} = Q_{r,\theta r^2}(\tilde{x}, \tilde{t})$, $r_i/20 \leq r \leq r_i$, we obtain

$$\begin{aligned} \frac{1}{|Q_{r,\theta r^2}|} & \int_{Q_{r,\theta r^2} \cap D} (|\nabla u|^p + f^p) \, \mathrm{d}z \\ & \leq \frac{c \, |Q_0|}{|Q_i| \theta} \frac{1}{|Q_0|} \int_{Q_0 \cap D} (|\nabla u|^p + f^p) \, \mathrm{d}z \\ & \leq {c_2}^p \alpha^{p/2} \lambda^p, \end{aligned}$$

where c_2 is chosen to be large enough. The first inequality in (4.2) will be valid for small cylinders due to Lebesgue's differentiation theorem. We arrive at

$$\lim_{r' \to 0} \frac{1}{|Q_{r',\theta r'^2}|} \int_{Q_{r',\theta r'^2}(\tilde{x}, \tilde{t})} (|\nabla u|^p + f^p) \, \mathrm{d}z > {c_2}^p \alpha^{p/2} \lambda^p,$$

which holds for almost every $(\tilde{x}, \tilde{t}) \in Q_i \cap D$ such that $h(\tilde{x}, \tilde{t}) > \lambda$. An appropriate version of Lebesgue's differentiation theorem is proven in [Zyg34].

Observe that the integral above is continuous with respect to r. Furthermore, the integral is less than or equal to ${c_2}^p \alpha^{p/2} \lambda^p$ for all r, $r_i/20 \leq r \leq r_i$, and greater than ${c_2}^p \alpha^{p/2} \lambda^p$ for r small enough. Thus, there exists ρ_1, $0 < \rho_1 \leq r_i/20$, such that the integral equals ${c_2}^p \alpha^{p/2} \lambda^p$ if $r = \rho_1$. Moreover, for all larger values of r, the integral is less than or equal to $c_2{}^p\alpha^{p/2}\lambda^p$. Consequently, there exists a constant $c \geq 1$, independent of the location, such that

$$\begin{aligned} & c^{-1}\alpha^{p/2}\lambda^p \\ & \leq \frac{1}{\left|Q_{\rho_1,\theta\rho_1^2}\right|}\int_{Q_{\rho_1,\theta\rho_1^2}\cap D}\left(|\nabla u|^p + f^p\right)\,\mathrm{d}z \\ & \leq \frac{c}{\left|Q_{20\rho_1,\theta(20\rho_1)^2}\right|}\int_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap D}\left(|\nabla u|^p + f^p\right)\,\mathrm{d}z \leq c^2\alpha^{p/2}\lambda^p. \end{aligned} \tag{4.8}$$

Similar reasoning implies that there exists $\rho_2 < \rho_1,\ 0 < \rho_2 \leq r_i/20$, such that

$$\begin{aligned} c^{-1}\alpha^{p/2}\lambda^p & \leq \frac{1}{\left|Q_{\rho_2,\theta\rho_2^2}\right|}\int_{Q_{\rho_2,\theta\rho_2^2}\cap D}|\nabla u|^p\,\mathrm{d}z \\ & \leq \frac{c}{\left|Q_{20\rho_2,\theta(20\rho_2)^2}\right|}\int_{Q_{20\rho_2,\theta(20\rho_2)^2}\cap D}|\nabla u|^p\,\mathrm{d}z \leq c^2\alpha^{p/2}\lambda^p. \end{aligned} \tag{4.9}$$

At this point, we remark that $\alpha, \lambda > 1$, and, therefore, $\theta < 1$ as well as $Q_{20\rho_1,\theta(20\rho_1)^2} \subset Q_0$.

If λ is replaced by $\alpha^{1/2}\lambda$, then (4.8) shows that condition (4.2) in Lemma 4.1 holds with ρ_1 whenever $h(\tilde{x},\tilde{t}) > \lambda$ and, further, $\theta\rho^2 = (\alpha^{1/2}\lambda)^{2-p}\rho^2$. If $B_{\frac{4}{3}\rho_1}(\tilde{x}) \setminus \Omega \neq \emptyset$, then Lemma 4.1 implies

$$\begin{aligned} & \frac{1}{\left|Q_{\rho_1,\theta\rho_1^2}\right|}\int_{Q_{\rho_1,\theta\rho_1^2}\cap D}|\nabla u|^p\,\mathrm{d}z \\ & \qquad \leq \left(\frac{c}{\left|Q_{4\rho_1,\theta(4\rho_1)^2}\right|}\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap D}|\nabla u|^{\tilde{q}}\,\mathrm{d}z\right)^{p/\tilde{q}} \\ & \qquad\quad + \frac{c}{\left|Q_{4\rho_1,\theta(4\rho_1)^2}\right|}\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap D}f^p\,\mathrm{d}z, \end{aligned} \tag{4.10}$$

for some $\tilde{q} < p$.

Assume then that $B_{\frac{4}{3}\rho_2}(\tilde{x}) \setminus \Omega = \emptyset$ and denote $q = pn/(n+2)$. If $Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}$ does not intersect the initial boundary, we obtain a local result

$$\begin{aligned}
&\frac{1}{\left|Q_{\rho_2,\theta\rho_2^2}\right|}\int_{Q_{\rho_2,\theta\rho_2^2}\cap D}|\nabla u|^p\,\mathrm{d}z\\
&\qquad\le c\left(\left|Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}\right|^{-1}\int_{Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}\cap D}|\nabla u|^q\,\mathrm{d}z\right)^{p/q}\\
&\qquad\le\left(\frac{c}{|Q_{4\rho_2,\theta(4\rho_2)^2}|}\int_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap D}|\nabla u|^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q},
\end{aligned}\tag{4.11}$$

where Hölder's inequality was applied in the last step. For the proof of the local result, we refer to Lemma 3.4 in [KL00].

Next we quote a result from Section 6. If $B_{\frac{4}{3}\rho_2}(\tilde x)\setminus\Omega=\emptyset$ and if $Q_{\frac{7}{6}\rho_2,\theta(\frac{7}{6}\rho_2)^2}$ intersects the initial boundary, then we obtain an initial boundary estimate

$$\begin{aligned}
&\frac{1}{\left|Q_{\rho_2,\theta\rho_2^2}\right|}\int_{Q_{\rho_2,\theta\rho_2^2}\cap D}|\nabla u|^p\,\mathrm{d}z\\
&\qquad\le\left(\frac{c}{|Q_{4\rho_2,\theta(4\rho_2)^2}|}\int_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap D}|\nabla u|^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q}\\
&\qquad\quad+\left(\frac{c}{|B_{4\rho_2}|}\int_{B_{4\rho_2}\cap\Omega}\tilde f^q\,\mathrm{d}x\right)^{p/q},
\end{aligned}\tag{4.12}$$

where $\tilde f=|\nabla\varphi(x,0)|$. Due to our assumptions on the initial values, the last term is well-defined. Actually, we derive this estimate in a slightly different form in Lemma 6.1. Nonetheless, the same proof applies here since $\varphi\in C([0,T];L^2(\Omega))$ and since (4.9) is available.

Let us now return to the case $B_{\frac{4}{3}\rho_1}(\tilde x)\setminus\Omega\neq\emptyset$. From (4.8), we obtain

$$\begin{aligned}
&c^{-1}\lambda^p\\
&\le\frac{1}{\left|Q_{\rho_1,\theta\rho_1^2}\right|}\int_{Q_{\rho_1,\theta\rho_1^2}\cap D}\left(h^p+\alpha^{-p/2}f^p\right)\,\mathrm{d}z\\
&\le\frac{c}{|Q_{20\rho_1,\theta(20\rho_1)^2}|}\int_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap D}\left(h^p+\alpha^{-p/2}f^p\right)\,\mathrm{d}z\le c^2\lambda^p
\end{aligned}\tag{4.13}$$

since the volumes of all the Whitney cylinders intersecting $Q_{20\rho_1,\theta(20\rho_1)^2}$ are comparable. In view of (4.10) and (4.13), we have

$$\begin{aligned}\frac{1}{\left|Q_{20\rho_1,\theta(20\rho_1)^2}\right|}&\int_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap D} h^p+\alpha^{-p/2}f^p\,\mathrm{d}z\\&\le\left(\frac{c}{\left|Q_{4\rho_1,\theta(4\rho_1)^2}\right|}\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap D} h^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q}\\&\quad+\frac{c}{\left|Q_{4\rho_1,\theta(4\rho_1)^2}\right|}\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap D}\alpha^{-p/2}f^p\,\mathrm{d}z.\end{aligned}\tag{4.14}$$

We used (4.13) above and thus obtained a larger cylinder on the left hand side. The cylinder $Q_{20\rho_1,\theta(20\rho_1)^2}$ might intersect the boundary of D even if $Q_{4\rho_1,\theta(4\rho_1)^2}$ does not. On the other hand, the first term on the right hand side also depends, in a sense, on the values near the boundary since $h(\tilde x,\tilde t)>\lambda$.

Next we decompose Q_0 into level sets. We define

$$G(\lambda)=\{(x,t)\in Q_0\cap D\,:\,h(x,t)>\lambda\}$$

and

$$\tilde G(\lambda)=\{(x,t)\in Q_0\cap D\,:\,f(x,t)>\lambda\}.$$

Since $h(x,t)>\lambda$ in $G(\lambda)$, we can use the previous estimates in $G(\lambda)$.

Observe that

$$h(x,t)\le\eta\lambda\quad\text{whenever}\quad(x,t)\in(Q_{4\rho_1,\theta(4\rho_1)^2}\cap D)\setminus G(\eta\lambda),$$

and

$$f(x,t)\le\eta\lambda\quad\text{whenever}\quad(x,t)\in(Q_{4\rho_1,\theta(4\rho_1)^2}\cap D)\setminus\tilde G(\eta\lambda).$$

Furthermore, since

$$\alpha^{-p/2}=(|Q_i|\,/\,|Q_0|)^{p/2}\le 1,$$

we obtain by (4.14) that

$$\begin{aligned}\frac{1}{\left|Q_{20\rho_1,\theta(20\rho_1)^2}\right|}&\int_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap D}\left(h^p+\alpha^{-p/2}f^p\right)\,\mathrm{d}z\\&\le c\eta^p\lambda^p+\left(\frac{c}{\left|Q_{4\rho_1,\theta(4\rho_1)^2}\right|}\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap G(\eta\lambda)}h^{\tilde q}\,\mathrm{d}z\right)^{p/\tilde q}\\&\quad+\frac{c}{\left|Q_{4\rho_1,\theta(4\rho_1)^2}\right|}\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap\tilde G(\eta\lambda)}f^p\,\mathrm{d}z.\end{aligned}\tag{4.15}$$

By Hölder's inequality and (4.13), there exists a constant $c \geq 1$ such that

$$\left(\frac{1}{\left|Q_{4\rho_1,\theta(4\rho_1)^2}\right|}\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap D} h^{\tilde q}\,\mathrm{d}z\right)^{(p-\tilde q)/\tilde q} \leq c\lambda^{p-\tilde q}. \tag{4.16}$$

To continue, we choose $\eta > 0$ small enough to absorb the first term on the right hand side of (4.15) into the left. This is possible due to (4.13). We combine the result with (4.16), multiply by $\left|Q_{20\rho_1,\theta(20\rho_1)^2}\right|$, and get

$$\begin{aligned}\int_{Q_{20\rho_1,\theta(20\rho_1)^2}\cap D} h^p\,\mathrm{d}z \leq c\lambda^{p-\tilde q}&\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z\\ &+ c\int_{Q_{4\rho_1,\theta(4\rho_1)^2}\cap \tilde G(\eta\lambda)} f^p\,\mathrm{d}z.\end{aligned} \tag{4.17}$$

If $B_{\frac43\rho_2}(\tilde x)\setminus\Omega = \emptyset$ and if $Q_{\frac76\rho_2,\theta(\frac76\rho_2)^2}$ does not intersect the initial boundary, then we obtain a local version of the above estimate by using (4.9) and (4.11). Consequently,

$$\begin{aligned}\int_{Q_{20\rho_2,\theta(20\rho_2)^2}\cap D} h^p\,\mathrm{d}z &\leq c\lambda^{p-\tilde q}\int_{Q_{\frac76\rho_2,\theta(\frac76\rho_2)^2}\cap G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z\\ &\leq c\lambda^{p-\tilde q}\int_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z.\end{aligned} \tag{4.18}$$

Finally, if $B_{\frac43\rho_2}(\tilde x)\setminus\Omega = \emptyset$ and if $Q_{\frac76\rho_2,\theta(\frac76\rho_2)^2}$ intersects the initial boundary, then we obtain an initial boundary version by using (4.9) and (4.12). Since $\left|Q_{20\rho_2,\theta(20\rho_2)^2}\right|^{-1} = c\left|B_{4\rho_2}\right|^{-p/q}$, we deduce

$$\begin{aligned}\int_{Q_{20\rho_2,\theta(20\rho_2)^2}\cap D} h^p\,\mathrm{d}z \leq c\lambda^{p-\tilde q}&\int_{Q_{4\rho_2,\theta(4\rho_2)^2}\cap G(\eta\lambda)} h^{\tilde q}\,\mathrm{d}z\\ &+\left(c\int_{B_{4\rho_2}\cap\overline{G}(\eta\lambda)} \tilde f^q\,\mathrm{d}x\right)^{p/q},\end{aligned} \tag{4.19}$$

where

$$\overline{G}(\eta\lambda) = \{x\in B_{4R}(x_0)\cap\Omega \,:\, \tilde f(x) > \eta\lambda\}.$$

We consider this case in more detail in the proof of Theorem 6.6. Since $\rho_2 < \rho_1$, either $B_{\frac43\rho_2}(\tilde x)\setminus\Omega = \emptyset$ and (4.9) holds, or $B_{\frac43\rho_1}(\tilde x)\setminus\Omega \neq \emptyset$ and (4.8) holds. Thus, one of the above estimates is always available.

As a next step, we use a covering argument to extend the estimates to the whole of $G(\lambda)$. By Vitali's covering theorem, we have a disjoint set of cylinders

$$\{Q_{4\rho_i',\theta(4\rho_i')}(\tilde z_i)\}_{i=1}^{\infty}, \quad \tilde z_i \in G(\lambda), \quad \tilde z_i = (\tilde x_i, \tilde t_i) \tag{4.20}$$

such that almost everywhere

$$G(\lambda) \subset \bigcup_{i=1}^{\infty} Q_{20\rho_i',\theta(20\rho_i')^2}(\tilde z_i) \subset Q_0,$$

and either (4.17), (4.18) or (4.19) holds in each of the cylinders. Then we sum over i and obtain

$$\begin{aligned}
\int_{G(\lambda)} h^p \,\mathrm{d}z &\le \sum_{i=1}^{\infty} \int_{Q_{20\rho_i,\theta(20\rho_i)^2}(\tilde z_i)\cap D} h^p \,\mathrm{d}z \\
&\le c \sum_{i=1}^{\infty} \left(\lambda^{p-\tilde q} \int_{Q_{4\rho_i,\theta(4\rho_i)^2}(\tilde z_i)\cap G(\eta\lambda)} h^{\tilde q} \,\mathrm{d}z + b_i \right) \\
&\le c\lambda^{p-\tilde q} \int_{G(\eta\lambda)} h^{\tilde q} \,\mathrm{d}z + c \int_{\tilde G(\eta\lambda)} f^p \,\mathrm{d}z + c \left(\int_{\overline{G}(\eta\lambda)} \tilde f^q \,\mathrm{d}x \right)^{p/q},
\end{aligned} \tag{4.21}$$

where b_i is either the lateral boundary term, initial boundary term, or zero, depending on the corresponding estimate. When summing over the initial boundary terms, we used the fact that $p/q > 1$.

The higher integrability result is now a consequence of (4.21) and Fubini's theorem. To see this, we integrate over $G(\lambda_0)$ and use (4.21) together with Fubini's theorem. Thus,

$$\begin{aligned}
\int_{G(\lambda_0)} h^{p+\varepsilon} \,\mathrm{d}z &= \int_{G(\lambda_0)} \left(\int_{\lambda_0}^{h} \varepsilon \lambda^{\varepsilon-1} \,\mathrm{d}\lambda + (\lambda_0)^\varepsilon \right) h^p \,\mathrm{d}z \\
&= \varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \int_{G(\lambda)} h^p \,\mathrm{d}z \,\mathrm{d}\lambda + (\lambda_0)^\varepsilon \int_{G(\lambda_0)} h^p \,\mathrm{d}z \\
&\le c \int_{\lambda_0}^{\infty} \Bigg(\varepsilon \lambda^{\varepsilon-1+p-\tilde q} \int_{G(\eta\lambda)} h^{\tilde q} \,\mathrm{d}z + \varepsilon\lambda^{\varepsilon-1} \int_{\tilde G(\eta\lambda)} f^p \,\mathrm{d}z \\
&\qquad + \varepsilon\lambda^{\varepsilon-1} \Big(\int_{\overline{G}(\eta\lambda)} \tilde f^q \,\mathrm{d}x \Big)^{p/q} \Bigg) \,\mathrm{d}\lambda + (\lambda_0)^\varepsilon \int_{G(\lambda_0)} h^p \,\mathrm{d}z.
\end{aligned} \tag{4.22}$$

We estimate the right hand side in three parts. First, by Fubini's theorem, we see that

$$\begin{aligned}
&\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1+p-\tilde q} \int_{G(\eta\lambda)} h^{\tilde q} \,\mathrm{d}z \,\mathrm{d}\lambda + (\lambda_0)^\varepsilon \int_{G(\lambda_0)} h^p \,\mathrm{d}z \\
&= c\varepsilon \int_{G(\eta\lambda_0)} \int_{\lambda_0}^{h/\eta} \lambda^{\varepsilon-1+p-\tilde q} h^{\tilde q} \,\mathrm{d}\lambda \,\mathrm{d}z + (\lambda_0)^\varepsilon \int_{G(\lambda_0)} h^p \,\mathrm{d}z \\
&\le \frac{c\varepsilon}{\varepsilon+p-\tilde q} \int_{G(\eta\lambda_0)} \left(h^{\varepsilon+p}\eta^{\tilde q-p-\varepsilon} - (\lambda_0)^{\varepsilon+p-\tilde q} h^{\tilde q} \right) \,\mathrm{d}z + (\lambda_0)^\varepsilon \int_{G(\lambda_0)} h^p \,\mathrm{d}z.
\end{aligned}$$

Since the second term in the first integral is negative, it follows that

$$\begin{aligned}
&\frac{c\varepsilon}{\varepsilon+p-\tilde q}\int_{G(\eta\lambda_0)}\left(h^{\varepsilon+p}\eta^{\tilde q-p-\varepsilon}-{\lambda_0}^{\varepsilon+p-\tilde q}h^{\tilde q}\right)\mathrm{d}z\\
&\le \frac{c\varepsilon}{\varepsilon+p-\tilde q}\left(\int_{G(\lambda_0)}h^{\varepsilon+p}\eta^{\tilde q-p-\varepsilon}\,\mathrm{d}z+(\lambda_0)^{\varepsilon}\int_{G(\eta\lambda_0)\setminus G(\lambda_0)}h^{p}\eta^{\tilde q-p-\varepsilon}\,\mathrm{d}z\right)\\
&\le \frac{c\varepsilon}{\varepsilon+p-\tilde q}\int_{G(\lambda_0)}h^{\varepsilon+p}\,\mathrm{d}z+c\,(\lambda_0)^{\varepsilon}\int_{G(\eta\lambda_0)}h^{p}\,\mathrm{d}z,
\end{aligned}$$

where we also used the fact that $\lambda_0\ge h$ in $G(\eta\lambda_0)\setminus G(\lambda_0)$. We end up with

$$\begin{aligned}
\varepsilon\int_{\lambda_0}^{\infty}\lambda^{\varepsilon-1+p-\tilde q}\int_{G(\eta\lambda)}h^{\tilde q}\,\mathrm{d}z\,\mathrm{d}\lambda\;+(\lambda_0)^{\varepsilon}\int_{G(\lambda_0)}h^{p}\,\mathrm{d}z&\\
\le \frac{c\varepsilon}{\varepsilon+p-\tilde q}\int_{G(\lambda_0)}h^{\varepsilon+p}\,\mathrm{d}z+c\,(\lambda_0)^{\varepsilon}\int_{G(\eta\lambda_0)}h^{p}\,\mathrm{d}z.&
\end{aligned}\tag{4.23}$$

Let us now estimate the lateral boundary term in (4.22). We utilize Fubini's theorem and obtain

$$\begin{aligned}
\varepsilon\int_{\lambda_0}^{\infty}\lambda^{\varepsilon-1}\int_{\tilde G(\eta\lambda)}f^{p}\,\mathrm{d}z\,\mathrm{d}\lambda\;&=\int_{\tilde G(\eta\lambda_0)}\left((f/\eta)^{\varepsilon}-(\lambda_0)^{\varepsilon}\right)f^{p}\,\mathrm{d}z\\
&\le c\int_{\tilde G(\eta\lambda_0)}f^{\varepsilon+p}\,\mathrm{d}z.
\end{aligned}\tag{4.24}$$

The initial boundary term in (4.22) can be estimated as

$$\begin{aligned}
&\varepsilon\int_{\lambda_0}^{\infty}\lambda^{\varepsilon-1}\left(\int_{\overline{G}(\eta\lambda)}\tilde f^{q}\,\mathrm{d}x\right)^{p/q}\mathrm{d}\lambda\\
&\qquad\le cR^{2\varepsilon/(q+\varepsilon)}\left(\int_{\overline{G}(\eta\lambda_0)}\tilde f^{q+\varepsilon}\,\mathrm{d}x\right)^{(p+\varepsilon)/(q+\varepsilon)}.
\end{aligned}\tag{4.25}$$

A detailed calculation for the initial boundary term is presented in (6.12).

Now we are ready to collect the estimates. We combine (4.23), (4.24), and (4.25) with (4.22). Then we choose $\varepsilon>0$ small enough to absorb the term containing $h^{p+\varepsilon}$ into the left hand side and get

$$\begin{aligned}
\int_{G(\lambda_0)}h^{p+\varepsilon}\,\mathrm{d}z\le c\,(\lambda_0)^{\varepsilon}\int_{G(\eta\lambda_0)}h^{p}\,\mathrm{d}z+c\int_{\tilde G(\eta\lambda_0)}f^{p+\varepsilon}\,\mathrm{d}z&\\
+\,cR^{2\varepsilon/(q+\varepsilon)}\left(\int_{\overline{G}(\eta\lambda_0)}\tilde f^{q+\varepsilon}\,\mathrm{d}x\right)^{(p+\varepsilon)/(q+\varepsilon)}.&
\end{aligned}\tag{4.26}$$

Notice that if the term we would like to absorb is infinite, then we can replace h by $h_k=\min\{h,k\}$, $k>\lambda_0$. Indeed, estimate (4.21) continues to hold in the form

$$\int_{\{h_k>\lambda\}} h_k^{p-\tilde q}\,\mathrm{d}\mu \le c\lambda^{p-\tilde q}\int_{\{h_k>\eta\lambda\}} \mathrm{d}\mu + c\int_{\tilde G(\eta\lambda)} f^p\,\mathrm{d}z + c\left(\int_{\overline{G}(\eta\lambda)} \tilde f^q\,\mathrm{d}x\right)^{p/q}, \tag{4.27}$$

where $\mathrm{d}\mu = h^{\tilde q}\,\mathrm{d}z$. Then we use this estimate in the calculations starting from (4.22) and end up with

$$\begin{aligned}\int_{\{h_k>\lambda_0\}} h_k^{p-\tilde q+\varepsilon}\,\mathrm{d}\mu \le{}& \frac{c\varepsilon}{\varepsilon+p-\tilde q}\int_{\{h_k>\lambda_0\}} h_k^{p-\tilde q+\varepsilon}\,\mathrm{d}\mu\\ &+ c\,(\lambda_0)^\varepsilon \int_{\{h_k>\eta\lambda_0\}} h_k^{p-\tilde q}\,\mathrm{d}\mu + c\int_{\tilde G(\eta\lambda_0)} f^{p+\varepsilon}\,\mathrm{d}z\\ &+ cR^{2\varepsilon/(q+\varepsilon)}\left(\int_{\overline{G}(\eta\lambda_0)} \tilde f^{q+\varepsilon}\,\mathrm{d}x\right)^{(p+\varepsilon)/(q+\varepsilon)}.\end{aligned}$$

As a result, we can absorb the first term on the right hand side into the left and then employ Lebesgue's dominated convergence theorem to let $k\to\infty$. Thus, we obtain (4.26).

Since $h\le\lambda_0$ in $(Q_0\cap D)\setminus G(\lambda_0)$, estimate (4.26) extends to the whole of $Q_{R,R^2}\cap D$. Indeed,

$$\begin{aligned}\int_{Q_{R,R^2}\cap D} h^{p+\varepsilon}\,\mathrm{d}z &\le (\lambda_0)^\varepsilon\int_{(Q_0\cap D)\setminus G(\lambda_0)} h^p\,\mathrm{d}z + \int_{G(\lambda_0)} h^{p+\varepsilon}\,\mathrm{d}z\\ &\le c\,(\lambda_0)^\varepsilon\int_{Q_0\cap D} h^p\,\mathrm{d}z + c\int_{Q_0\cap D} f^{p+\varepsilon}\,\mathrm{d}z\\ &\quad + cR^{2\varepsilon/(q+\varepsilon)}\left(\int_{B_0\cap\Omega}\tilde f^{q+\varepsilon}\,\mathrm{d}x\right)^{(p+\varepsilon)/(q+\varepsilon)}.\end{aligned}$$

We divide the estimate by $|Q_0|$ and apply the definition of $h(z)$. Since Q_{R,R^2} lies far away from the boundary of $Q_0 = Q_{4R,(4R)^2}$, there exists $c>0$, independent of R, such that

$$\min\{|Q_i|^{1/2} : (x,t)\in Q_i\}/\,|Q_0|^{1/2} > c$$

for every $(x,t)\in Q_{R,R^2}\cap D$. On the right hand side, we estimate

$$\min\{|Q_i|^{1/2} : (x,t)\in Q_i\}/\,|Q_0|^{1/2} \le 1,$$

and, consequently,

$$\begin{aligned}\frac{1}{|Q_0|}\int_{Q_{R,R^2}\cap D} |\nabla u|^{p+\varepsilon}\,\mathrm{d}z \le{}& \frac{c\,(\lambda_0)^\varepsilon}{|Q_0|}\int_{Q_0\cap D}|\nabla u|^p\,\mathrm{d}z\\ &+ \frac{c}{|Q_0|}\int_{Q_0\cap D} f^{p+\varepsilon}\,\mathrm{d}z + \left(\frac{c}{|B_0|}\int_{B_0\cap\Omega}\tilde f^{q+\varepsilon}\,\mathrm{d}x\right)^{(p+\varepsilon)/(q+\varepsilon)}.\end{aligned}$$

Next we take the cut-off level into account. Remember that either

$$\lambda_0 = 1 \quad \text{or} \quad \lambda_0 = \lambda_0'.$$

The first case is clear. Moreover, if $\lambda_0 = \lambda_0'$, then Young's inequality and the definition of λ_0' leads to

$$\begin{aligned}\frac{1}{|Q_{R,R^2}|}\int_{Q_{R,R^2}\cap D} |\nabla u|^{p+\varepsilon}\,\mathrm{d}z \le \left(\frac{c}{|Q_0|}\int_{Q_0\cap D}|\nabla u|^p\,\mathrm{d}z\right)^{(\varepsilon+2)/2} \\ + \left(\frac{c}{|Q_0|}\int_{Q_0\cap D} f^p\,\mathrm{d}z\right)^{(\varepsilon+2)/2} + \frac{c}{|Q_0|}\int_{Q_0\cap D} f^{p+\varepsilon}\,\mathrm{d}z \\ + \left(\frac{c}{|B_0|}\int_{B_0\cap\Omega} \tilde{f}^{q+\varepsilon}\,\mathrm{d}x\right)^{(p+\varepsilon)/(q+\varepsilon)}.\end{aligned}$$

This finishes the proof. □

5. Estimates near the initial boundary

This section provides estimates near the initial boundary $\Omega\times\{0\}$. Here we compare the solution with its average instead of the boundary function, and the estimates become somewhat different. Furthermore, the regularity of the lateral boundary does not play a role in the proofs, and weaker assumptions on the initial data can be used.

We say that u is a weak solution to an initial value problem if u, belonging to the parabolic space $L^p(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$, satisfies (2.2) and the initial condition

$$\frac{1}{h}\int_0^h\int_C |u(x,t)-\varphi(x)|^2\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as} \quad h\to 0, \tag{5.1}$$

for all compact sets $C \subset \Omega$ and for a given initial value function $\varphi \in W^{1,2_*}_{\mathrm{loc}}(\Omega)$. Here $2_* = 2n/(n+2)$.

The proof uses the weighted mean

$$u^{\eta}_{2\rho}(t) = \frac{\int_{B_{2\rho}} \eta^p(x,t)u(x,t)\,\mathrm{d}x}{\int_{B_{2\rho}} \eta^p(x,t)\,\mathrm{d}x},$$

instead of the standard mean

$$u_{2\rho}(t) = \int_{B_{2\rho}} u(x,t)\,\mathrm{d}x.$$

The weighted mean should be close to the standard mean, and therefore the weight $\eta \in C_0^\infty(\mathbb{R}^{n+1})$ is defined to be a cut-off function such that

$$\operatorname{spt}\eta \subset Q_{2\rho,\theta(2\rho)^2}(x_0,t_0), \quad 0\le\eta\le 1, \quad \text{and} \quad \eta = 1 \text{ in } Q_{\rho,\theta\rho^2}(x_0,t_0),$$

where $\theta > 0$. In addition,

$$\sup_{x\in B_{2\rho}} \eta(x,t) \le \tilde{c} \fint_{B_{2\rho}} \eta(x,t)\,\mathrm{d}x, \qquad t \in \Lambda_{\theta(2\rho)^2}(t_0), \tag{5.2}$$

where

$$\Lambda_{\theta(2\rho)^2}(t_0) = (t_0 - \frac{1}{2}\theta(2\rho)^2,\, t_0 + \frac{1}{2}\theta(2\rho)^2).$$

The following lemma gives a useful connection between the standard mean and the weighted mean.

Lemma 5.3. *Suppose that $B_{2\rho} \Subset \Omega$, let $u(\cdot,t) \in L^p_{loc}(\Omega)$, where $p > 1$, and let η, $u^\eta_{2\rho}(t)$, $u_{2\rho}(t)$ be as above. Then there exists a constant $c = c(p,\tilde{c}) > 0$ such that*

$$\int_{B_{2\rho}} |u - u_{2\rho}(t)|^p\,\mathrm{d}x \le c \int_{B_{2\rho}} |u - u^\eta_{2\rho}(t)|^p\,\mathrm{d}x$$

and

$$\int_{B_{2\rho}} |u - u^\eta_{2\rho}(t)|^p\,\mathrm{d}x \le c \int_{B_{2\rho}} |u - u_{2\rho}(t)|^p\,\mathrm{d}x.$$

Here $\tilde{c}$ is the constant in (5.2).

Proof: Let us begin with the first inequality. We add and subtract $u^\eta_{2\rho}(t)$, which leads to

$$\begin{aligned}\int_{B_{2\rho}} &|u - u^\eta_{2\rho}(t) + u^\eta_{2\rho}(t) - u_{2\rho}(t)|^p\,\mathrm{d}x \\ &\le c \int_{B_{2\rho}} |u - u^\eta_{2\rho}(t)|^p\,\mathrm{d}x + c\,|B_{2\rho}|\,\big|u^\eta_{2\rho}(t) - u_{2\rho}(t)\big|^p\end{aligned}$$

since $p > 1$. This implies the desired estimate since

$$|B_{2\rho}|\,\big|u^\eta_{2\rho}(t) - u_{2\rho}(t)\big|^p \le \int_{B_{2\rho}} |u^\eta_{2\rho}(t) - u|^p\,\mathrm{d}x$$

due to Hölder's inequality.

To obtain the second inequality of the claim, we add and subtract $u_{2\rho}(t)$. It follows that

$$\begin{aligned}\fint_{B_{2\rho}} &|u - u^\eta_{2\rho}(t)|^p\,\mathrm{d}x \\ &\le c \fint_{B_{2\rho}} |u - u_{2\rho}(t)|^p\,\mathrm{d}x + c|u_{2\rho}(t) - u^\eta_{2\rho}(t)|^p.\end{aligned}$$

Then we estimate the last terms on the right hand side by using the definition of $u^\eta_{2\rho}(t)$, Hölder's inequality, and assumption (5.2). We conclude that

$$\begin{aligned}
|u^\eta_{2\rho}(t) - u_{2\rho}(t)| &\le \frac{⨍_{B_{2\rho}} |u - u_{2\rho}(t)|\eta^p \,\mathrm{d}x}{⨍_{B_{2\rho}} \eta^p \,\mathrm{d}x} \\
&\le \left(\frac{\sup_{x\in B_{2\rho}} \eta}{⨍_{B_{2\rho}} \eta \,\mathrm{d}x} \right)^p ⨍_{B_{2\rho}} |u - u_{2\rho}(t)| \,\mathrm{d}x \\
&\le \tilde{c}^p \left(⨍_{B_{2\rho}} |u - u_{2\rho}(t)|^p \,\mathrm{d}x \right)^{1/p},
\end{aligned}$$

which completes the proof. □

When we employ Lemma 5.3 in deriving estimates, the constants in the final estimates depend on $\tilde{c}$ in condition (5.2). Since this constant is fixed as soon as the weight is fixed, we do not write out this dependency explicitly.

From now on, we assume that the cut-off function η, defined at the beginning of the section, also satisfies

$$\rho\,|\nabla \eta| + \theta\rho^2 \left| \frac{\partial \eta}{\partial t} \right| \le c. \tag{5.4}$$

Next we derive a Caccioppoli-type inequality near the initial boundary.

Lemma 5.5 (Caccioppoli). *Let u be a solution to an initial value problem with the initial condition (5.1). Let $\theta > 0$ and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0,t_0) \subset \mathbb{R}^{n+1}$ such that $B_{4\rho}(x_0) \subset \Omega$ and $0 \in \Lambda_{\theta(2\rho)^2}(t_0)$. Then there exists a constant $c = c(n,p,\alpha,\beta) > 0$ such that*

$$\begin{aligned}
&\int_{Q_{\rho,\theta\rho^2}\cap D} |\nabla u|^p \,\mathrm{d}z + \operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} \int_{B_\rho} |u - u^\eta_{2\rho}(t)|^2 \,\mathrm{d}x \\
&\le \frac{c}{\theta\rho^2} \int_{Q_{2\rho,\theta(2\rho)^2}\cap D} |u - u_{2\rho}(t)|^2 \,\mathrm{d}z + \frac{c}{\rho^p} \int_{Q_{2\rho,\theta(2\rho)^2}\cap D} |u - u_{2\rho}(t)|^p \,\mathrm{d}z \\
&\quad + c \left(\int_{B_{2\rho}} |\nabla \varphi|^{2_*} \,\mathrm{d}x \right)^{2/2_*},
\end{aligned}$$

where $2_* = 2n/(n+2)$ and $D = \Omega \times (0,T)$.

Proof: We may assume that $Q_{\rho,\theta\rho^2} \cap D \neq \emptyset$ since otherwise the claim is trivial. We choose a test function

$$\phi_\varepsilon(x,t) = \eta^p(x,t)(u_\varepsilon(x,t) - u^\eta_{2\rho,\varepsilon}(t))\chi^{h,\varepsilon}_{0,t_1}(t), \qquad t_1 \in \Lambda_{\theta\rho^2} \cap (0,T),$$

where $u^{\eta}_{2\rho,\varepsilon}(t)$ is the weighted average of u_ε and otherwise the notation is the same as in Lemma 3.2.

Next we aim to estimate the first term of (2.3). To accomplish this, we add and subtract $u^{\eta}_{2\rho,\varepsilon}(t)\phi'_\varepsilon$ to obtain

$$-\int_D u_\varepsilon \phi'_\varepsilon \,\mathrm{d}z = -\int_D (u_\varepsilon - u^{\eta}_{2\rho,\varepsilon}(t))\phi'_\varepsilon \,\mathrm{d}z - \int_D u^{\eta}_{2\rho,\varepsilon}(t)\phi'_\varepsilon \,\mathrm{d}z. \tag{5.6}$$

The last term in the above expression vanishes. To see this, we integrate by parts, use the definition of $u^{\eta}_{2\rho,\varepsilon}(t)$, and have

$$\begin{aligned}
&-\int_D u^{\eta}_{2\rho,\varepsilon}(t)\phi'_\varepsilon \,\mathrm{d}z \\
&= \int_0^{t_1} \chi^{h,\varepsilon}_{0,t_1}(t)\left(\int_{B_{2\rho}} u_\varepsilon \eta^p \,\mathrm{d}x - \frac{\int_{B_{2\rho}} \eta^p \,\mathrm{d}x \int_{B_{2\rho}} \eta^p u_\varepsilon \,\mathrm{d}x}{\int_{B_{2\rho}} \eta^p \,\mathrm{d}x}\right)(u^{\eta}_{2\rho,\varepsilon}(t))' \,\mathrm{d}t \\
&= 0.
\end{aligned}$$

Let us then integrate the first term of (5.6) by parts, take limits, apply the initial condition, and deduce for almost every t_1 that

$$\begin{aligned}
&-\int_D u_\varepsilon \phi'_\varepsilon \,\mathrm{d}z \\
&\to -\frac{1}{2}\int_{\Omega\times(0,t_1)} |u - u^{\eta}_{2\rho}(t)|^2 p\eta^{p-1}\eta' \,\mathrm{d}z - \frac{1}{2}\int_{B_{2\rho}} |\varphi - \varphi^{\eta}_{2\rho}|^2 \eta^p(x,0)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{B_{2\rho}} |u(x,t_1) - u^{\eta}_{2\rho}(t_1)|^2 \eta^p(x,t_1)\,\mathrm{d}x,
\end{aligned}$$

as first $\varepsilon \to 0$ and then $h \to 0$. See also the proof of Lemma 3.2. We may now choose $t_1 \in \Lambda_{\theta\rho^2} \cap (0,T)$ such that

$$\begin{aligned}
\frac{1}{2} \operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)} &\int_{B_\rho} |u - u^{\eta}_{2\rho}(t)|^2 \eta^p \,\mathrm{d}x \\
&\le \int_{B_{2\rho}} |u(x,t_1) - u^{\eta}_{2\rho}(t_1)|^2 \eta^p(x,t_1)\,\mathrm{d}x.
\end{aligned}$$

Furthermore, we use Lemma 5.3 and Poincaré's inequality to estimate

$$\int_{B_{2\rho}} |\varphi - \varphi^{\eta}_{2\rho}|^2 \,\mathrm{d}x \le c\left(\int_{B_{2\rho}} |\nabla\varphi|^{2_*} \,\mathrm{d}x\right)^{2/2_*}.$$

To continue, we estimate the second term of (2.3). We see that

$$\begin{aligned}
&\int_D A(x,t,\nabla u)_\varepsilon \cdot \nabla\phi_\varepsilon \,\mathrm{d}z \\
&\qquad \to \int_{\Omega\times(0,t_1)} A(x,t,\nabla u)\cdot\left[p\eta^{p-1}\nabla\eta(u - u^{\eta}_{2\rho}(t)) + \eta^p\nabla u\right]\,\mathrm{d}z,
\end{aligned}$$

as first $\varepsilon \to 0$ and then $h \to 0$.

Collecting the estimates, we end up with

$$\int_{\Omega\times(0,t_1)} A(x,t,\nabla u)\cdot\eta^p\nabla u\,\mathrm{d}z + \frac{1}{2}\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)}\int_{B_\rho}\eta^p|u-u_{2\rho}^\eta(t)|^2\,\mathrm{d}x$$
$$\le c\int_{\Omega\times(0,t_1)}|u-u_{2\rho}^\eta(t)|^2\eta\,|\eta'|\,\mathrm{d}z + c\left(\int_{B_{2\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{2/2_*}$$
$$+\int_{\Omega\times(0,t_1)}|A(x,t,\nabla u)|\,p\eta^{p-1}\,|\nabla\eta|\,\left|u-u_{2\rho}^\eta(t)\right|\,\mathrm{d}z.$$

Finally, we complete the argument in a way similar to that used in the proof of Lemma 3.2: We apply the growth bounds, Young's inequality, and assumption (5.4). Due to Lemma 5.3, we can replace the weighted means by the standard means on the right hand side of the resulting estimate. □

Next we prove a parabolic Poincaré-type inequality.

Lemma 5.7 (PARABOLIC POINCARÉ). *Let u be a solution to an initial value problem with the initial condition (5.1). Let $\theta > 0$ and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0,t_0) \subset \mathbb{R}^{n+1}$ such that $B_{4\rho}(x_0)\subset\Omega$ and $0\in\Lambda_{\theta(2\rho)^2}(t_0)$. Then there exists a constant $c = c(n,p,\alpha,\beta) > 0$ such that*

$$\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)}\int_{B_\rho}|u-u_{2\rho}^\eta(t)|^2\,\mathrm{d}x \le \frac{c}{\theta}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|\nabla u|^2\,\mathrm{d}z$$
$$+ c\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|\nabla u|^p\,\mathrm{d}z + c\left(\int_{B_{2\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{2/2_*},$$

where $2_* = 2n/(n+2)$.

Proof: We leave out the first term on the left hand side in Lemma 5.5 and have

$$\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)}\int_{B_\rho}|u-u_{2\rho}^\eta(t)|^2\,\mathrm{d}x$$
$$\le\frac{c}{\theta\rho^2}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|u-u_{2\rho}(t)|^2\,\mathrm{d}z$$
$$+\frac{c}{\rho^p}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|u-u_{2\rho}(t)|^p\,\mathrm{d}z + c\left(\int_{B_{2\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{2/2_*}.$$

We may estimate the first term on the right hand side as

$$\begin{aligned}
&\frac{c}{\theta\rho^2}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|u-u_{2\rho}(t)|^2\,\mathrm{d}z\\
&\qquad\le \frac{c}{\theta\rho^2}\int_{\Lambda_{\theta(2\rho)^2}\cap(0,T)}\int_{B_{2\rho}}|u-u_{2\rho}(t)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\le \frac{c}{\theta}\int_{\Lambda_{\theta(2\rho)^2}\cap(0,T)}\int_{B_{2\rho}}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

by Poincaré's inequality. Similar reasoning also implies

$$\frac{c}{\rho^p}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|u-u_{2\rho}(t)|^p\,\mathrm{d}z\le c\int_{\Lambda_{\theta(2\rho)^2}\cap(0,T)}\int_{B_{2\rho}}|\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t,$$

which completes the proof. □

The following lemma helps us to combine Caccioppoli's inequality with parabolic Poincaré's inequality. The proof is a straightforward application of Hölder's and Poincaré's inequalities.

Lemma 5.8. *Let $u\in L^q(0,T;W^{1,q}_{loc}(\Omega))$, where $q=pn/(2+n)$. Let $\theta>0$ and choose $Q_{\rho,\theta\rho^2}=Q_{\rho,\theta\rho^2}(x_0,t_0)\subset\mathbb{R}^{n+1}$ such that $B_{4\rho}(x_0)\subset\Omega$ and $0\in\Lambda_{\theta(2\rho)^2}(t_0)$. Then there exists a constant $c=c(n,p)>0$ such that*

$$\begin{aligned}
&\int_{Q_{\rho,\theta\rho^2}\cap D}|u-u_\rho(t)|^p\,\mathrm{d}z\\
&\qquad\le c\int_{Q_{\rho,\theta\rho^2}\cap D}|\nabla u|^q\,\mathrm{d}z\left(\operatorname*{ess\,sup}_{t\in\Lambda_{\theta\rho^2}\cap(0,T)}\int_{B_\rho}|u-u_{2\rho}(t)|^2\,\mathrm{d}x\right)^{q/n}.
\end{aligned}$$

Proof: First, we apply Hölder's inequality and obtain

$$\begin{aligned}
&\int_{B_\rho}|u-u_\rho(t)|^p\,\mathrm{d}x\\
&\qquad=\int_{B_\rho}|u-u_\rho(t)|^{2p/(2+n)}\,|u-u_\rho(t)|^{p-2p/(2+n)}\,\mathrm{d}x\\
&\qquad\le\left(\int_{B_\rho}|u-u_\rho(t)|^2\,\mathrm{d}x\right)^{q/n}\left(\int_{B_\rho}|u-u_\rho(t)|^{q^*}\,\mathrm{d}x\right)^{q/q^*},
\end{aligned}$$

where $q^*=qn/(n-q)=np/(n+2-p)$ is well-defined only when $q<n$, that is, $p<n+2$. The claim, nevertheless, is true for large values of p as well. To prove the result for $p\ge n+2$, we should use Poincaré's inequality with the exponent 1 instead of q and then use Hölder's inequality once more.

The previous estimate and Poincaré's inequality lead to

$$\int_{B_\rho} |u - u_\rho(t)|^p \,\mathrm{d}x \le c \int_{B_\rho} |\nabla u|^q \,\mathrm{d}x \left(\int_{B_\rho} |u - u_\rho(t)|^2 \,\mathrm{d}x \right)^{q/n}.$$

We would like to have $u_{2\rho}(t)$ on the right hand side instead of $u_\rho(t)$ when applying this lemma. This can be obtained since

$$\begin{aligned} \int_{B_\rho} |u - u_\rho(t)|^2 \,\mathrm{d}x &\le \int_{B_\rho} |u - u_{2\rho}(t) + u_{2\rho}(t) - u_\rho(t)|^2 \,\mathrm{d}x \\ &\le c \int_{B_\rho} |u - u_{2\rho}(t)|^2 \,\mathrm{d}x + c\,|B_\rho|\,|u_{2\rho}(t) - u_\rho(t)|^2 \end{aligned}$$

and the last term can be estimated as

$$|u_{2\rho}(t) - u_\rho(t)|^2 \le \left| \fint_{B_\rho} (u_{2\rho}(t) - u) \,\mathrm{d}x \right|^2 \le \fint_{B_\rho} |u_{2\rho}(t) - u|^2 \,\mathrm{d}x.$$

Combining the estimates, we have

$$\int_{B_\rho} |u - u_\rho(t)|^p \,\mathrm{d}x \le c \int_{B_\rho} |\nabla u|^q \,\mathrm{d}x \left(\int_{B_\rho} |u - u_{2\rho}(t)|^2 \,\mathrm{d}x \right)^{q/n}.$$

The claim follows by integrating this inequality with respect to time. □

6. Reverse Hölder inequalities near the initial boundary

In this section, we show that the gradient is integrable to a higher power near the initial boundary. First, we derive a reverse Hölder inequality and then show that it has a self-improving property. We already have used some estimates from this section in the proof of Theorem 4.7, but now we work out the details. Observe that cylinders are not scaled with respect to initial values, since this would not simplify the proof here.

Lemma 6.1 (reverse Hölder). *Let u be a solution to an initial value problem with the initial condition* (5.1). *Let $\lambda > 0$, set $\theta = \lambda^{2-p}$, and choose $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(x_0, t_0) \subset \mathbb{R}^{n+1}$ such that $B_{40\rho}(x_0) \subset \Omega$ and $0 \in \Lambda_{\theta(4\rho)^2}(t_0)$. Suppose that there exists a constant $c_1 \ge 1$ for which*

$$\begin{aligned} c_1^{-1}\lambda^p &\le \frac{1}{|Q_{\rho,\theta\rho^2}|} \int_{Q_{\rho,\theta\rho^2} \cap D} |\nabla u|^p \,\mathrm{d}z \\ &\le \frac{c_1}{|Q_{20\rho,\theta(20\rho)^2}|} \int_{Q_{20\rho,\theta(20\rho)^2} \cap D} |\nabla u|^p \,\mathrm{d}z \le c_1^2 \lambda^p. \end{aligned} \tag{6.2}$$

Then there exists a constant $c = c(n, p, c_1, \alpha, \beta) > 0$ *such that*

$$\frac{1}{|Q_{\rho,\theta\rho^2}|}\int_{Q_{\rho,\theta\rho^2}\cap D} |\nabla u|^p \,\mathrm{d}z \le \left(\frac{c}{|Q_{4\rho,\theta(4\rho)^2}|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} |\nabla u|^q \,\mathrm{d}z\right)^{p/q} + c\left(\fint_{B_{4\rho}} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q},$$

where $q = np/(n+2)$.

Proof: In view of Lemma 5.5, we have

$$\begin{aligned}\frac{1}{|Q_{\rho,\theta\rho^2}|}&\int_{Q_{\rho,\theta\rho^2}\cap D} |\nabla u|^p \,\mathrm{d}z \\ &\le \frac{c}{\theta\rho^2\,|Q_{2\rho,\theta(2\rho)^2}|}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D} |u - u_{2\rho}(t)|^2 \,\mathrm{d}z \\ &\quad + \frac{c}{\rho^p\,|Q_{2\rho,\theta(2\rho)^2}|}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D} |u - u_{2\rho}(t)|^p \,\mathrm{d}z \\ &\quad + \frac{c}{\theta}\left(\fint_{B_{2\rho}} |\nabla\varphi|^{2_*} \,\mathrm{d}x\right)^{2/2_*}.\end{aligned} \tag{6.3}$$

Since $p \ge 2$ and $\theta = \lambda^{2-p}$, we can estimate the first term on the right hand side in terms of the second in the same way as in (4.4). Thus, we can concentrate on the second term of (6.3).

Recalling Lemma 5.8, we see that

$$\begin{aligned}\frac{1}{\rho^p\,|Q_{2\rho,\theta(2\rho)^2}|}&\int_{Q_{2\rho,\theta(2\rho)^2}\cap D} |u - u_{2\rho}(t)|^p \,\mathrm{d}z \\ &\le \frac{c}{\rho^p\,|Q_{2\rho,\theta(2\rho)^2}|}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D} |\nabla u|^q \,\mathrm{d}z \\ &\quad \cdot \operatorname*{ess\,sup}_{t\in\Lambda_{\theta(2\rho)^2}\cap(0,T)}\left(\int_{B_{2\rho}} |u - u^{\eta}_{4\rho}(t)|^2 \,\mathrm{d}x\right)^{q/n}.\end{aligned}$$

We also applied Lemma 5.3 to manipulate the last part. The first integral is of the correct form, but the second integral should be estimated from above by the gradient. To accomplish this, we apply Lemma 5.7, Hölder's inequality, and assumption (6.2). This leads to

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in\Lambda_{\theta(2\rho)^2}\cap(0,T)}\int_{B_{2\rho}}|u-u^\eta_{4\rho}(t)|^2\,\mathrm{d}x\\&\le\frac{c}{\theta}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla u|^2\,\mathrm{d}z+c\int_{Q_{4\rho,\theta(4\rho)^2}\cap D}|\nabla u|^p\,\mathrm{d}z\\&\quad+c\left(\int_{B_{4\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{2/2_*}\le c\rho^{n+2}\lambda^2+c\left(\int_{B_{4\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{2/2_*}\end{aligned}$$

since $\theta=\lambda^{2-p}$ and $\left|Q_{4\rho,\theta(4\rho)^2}\right|=c\,\theta\rho^{n+2}$.

Collecting the facts, we end up with

$$\begin{aligned}&\frac{1}{\rho^p\left|Q_{2\rho,\theta(2\rho)^2}\right|}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|u-u^\eta_{2\rho}(t)|^p\,\mathrm{d}z\\&\qquad\le\frac{c}{\rho^p\left|Q_{2\rho,\theta(2\rho)^2}\right|}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|\nabla u|^q\,\mathrm{d}z\\&\qquad\quad\cdot\left(\rho^{n+2}\lambda^2+\left(\int_{B_{4\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{2/2_*}\right)^{q/n}.\end{aligned}$$

Observe that $\rho^{-p}=\rho^{-(n+2)q/n}$ and, on the other hand, $\rho^{-p}=(\rho^{-n})^{2q/(2_*n)}$. Young's inequality now implies

$$\begin{aligned}&\frac{1}{\rho^p\left|Q_{2\rho,\theta(2\rho)^2}\right|}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|u-u_{2\rho}(t)|^p\,\mathrm{d}z\\&\qquad\le\left(\frac{c}{\left|Q_{2\rho,\theta(2\rho)^2}\right|}\int_{Q_{2\rho,\theta(2\rho)^2}\cap D}|\nabla u|^q\,\mathrm{d}z\right)^{p/q}\\&\qquad\quad+\varepsilon\left(\fint_{B_{4\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{p/2_*}+\varepsilon\lambda^p.\end{aligned}\tag{6.4}$$

Next we estimate the last term in (6.3). Since $\theta=\lambda^{2-p}$, we see by Young's and Hölder's inequalities that

$$\frac{c}{\theta}\left(\fint_{B_{2\rho}}|\nabla\varphi|^{2_*}\,\mathrm{d}x\right)^{2/2_*}\le\varepsilon\lambda^p+c\left(\fint_{B_{4\rho}}|\nabla\varphi|^{q}\,\mathrm{d}x\right)^{p/q}.\tag{6.5}$$

Moreover, the second term on the right hand side of (6.4) can be estimated by the last term of (6.5) due to Hölder's inequality.

Finally, we combine (6.3), (6.4), and (6.5), as well as recall the remark after (6.3). Furthermore, we absorb the terms containing λ^p into the left by choosing $\varepsilon > 0$ small enough. This is possible due to assumption (6.2). □

The previous lemma makes sense if the gradient of the initial value function is integrable to the power $pn/(n+2)$ instead of p. Below, we show that the reverse Hölder inequality has a self-improving property in this case as well.

Theorem 6.6. *Let u be a solution to an initial value problem with the initial condition* (5.1) *and a given initial value function*

$$\varphi \in W^{1,q+\delta}_{loc}(\Omega),$$

where $\delta > 0$ and $q = pn/(n+2)$. Let $Q_{R,R^2}(x_0,t_0) \subset \mathbb{R}^{n+1}$ be such that $B_{8R}(x_0) \subset \Omega$ and $0 \in \Lambda_{(4R)^2}(t_0)$. Then there exist constants $\varepsilon_0 = \varepsilon_0(n,p,\alpha,\beta,\delta)$ and $c > 0$ with the same dependencies such that for all $0 \le \varepsilon < \varepsilon_0$, we have

$$\begin{aligned}\left(\frac{1}{|Q_{R,R^2}|}\int_{Q_{R,R^2}\cap D} |\nabla u|^{p+\varepsilon}\,\mathrm{d}z\right)^{1/(p+\varepsilon)} &\le c\left(\fint_{B_{4R}} |\nabla\varphi|^{q+\varepsilon}\,\mathrm{d}x\right)^{1/q+\varepsilon}\\ &\quad+\left(\frac{c}{|Q_{4R,(4R)^2}|}\int_{Q_{4R,(4R)^2}\cap D} |\nabla u|^{p}\,\mathrm{d}z\right)^{\sigma}\\ &\quad+\left(\frac{c}{|Q_{4R,(4R)^2}|}\int_{Q_{4R,(4R)^2}\cap D} |\nabla u|^{p}\,\mathrm{d}z\right)^{1/(p+\varepsilon)},\end{aligned}$$

where $\sigma = (2+\varepsilon)/(2(p+\varepsilon))$.

Proof: The proof is partly similar to the proof of Theorem 4.7. The main difference is in the choice of the level sets and in the calculations concerning the boundary term. We shall focus our attention on the changes.

We denote $Q_0 = Q_{4R,(4R)^2}(z_0) = Q_{4R,(4R)^2}(x_0,t_0)$ and divide Q_0 into the Whitney-type cylinders

$$Q_i = Q_{r_i,r_i^2}(y_i,\tau_i),\ i = 1,2,\dots,$$

in the same way as in the proof of Theorem 4.7.

We choose

$$\lambda_0' = \left(\frac{1}{|Q_0|}\int_{Q_0\cap D} |\nabla u|^{p}\,\mathrm{d}z\right)^{1/2}$$

and

$$\lambda > \max(\lambda_0', 1) = \lambda_0.$$

Let $h(x,t)$, α and θ be defined as in the proof of Theorem 4.7. Again, we consider $(\tilde{x},\tilde{t}) \in Q_i \cap D$ such that

$$h(\tilde{x},\tilde{t}) > \lambda.$$

Reasoning similar to that in Theorem 4.7 implies that, for almost all such $(\tilde{x},\tilde{t})$ and for $Q_{\rho,\theta\rho^2} = Q_{\rho,\theta\rho^2}(\tilde{x},\tilde{t})$, we obtain

$$\begin{aligned} c^{-1}\lambda^p &\le \frac{1}{|Q_{\rho,\theta\rho^2}|}\int_{Q_{\rho,\theta\rho^2}\cap D} h^p \,\mathrm{d}z \\ &\le \frac{c}{|Q_{20\rho,\theta(20\rho)^2}|}\int_{Q_{20\rho,\theta(20\rho)^2}\cap D} h^p \,\mathrm{d}z \le c^2\lambda^p. \end{aligned} \tag{6.7}$$

Notice also that $\theta < 1$ and that

$$Q_{20\rho,\theta(20\rho)^2} \subset Q_0.$$

Since the volumes of all the Whitney cylinders intersecting $Q_{20\rho,\theta(20\rho)^2}$ are comparable and since

$$\alpha^{-p/2} \le (|Q_i| \,/\, |Q_0|)^{p/2} \le 1,$$

we conclude by Lemma 6.1 that

$$\begin{aligned} &\frac{1}{|Q_{20\rho,\theta(20\rho)^2}|}\int_{Q_{20\rho,\theta(20\rho)^2}\cap D} h^p \,\mathrm{d}z \\ &\qquad \le \left(\frac{c}{|Q_{4\rho,\theta(4\rho)^2}|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap D} h^q \,\mathrm{d}z\right)^{p/q} \\ &\qquad\quad + c\left(\fint_{B_{4\rho}} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q}, \end{aligned} \tag{6.8}$$

if $0 \in \Lambda_{\theta(4\rho)^2}(\tilde{t})$. On the other hand, if $0 \notin \Lambda_{\theta(4\rho)^2}(\tilde{t})$, then we obtain a local result without the boundary term by using similar methods as in the proof of Lemma 6.1. See also (4.11).

Let us then consider the case $0 \in \Lambda_{\theta(4\rho)^2}(\tilde{t})$. We define the level sets

$$G(\lambda) = \{(x,t) \in Q_0 \cap D \,:\, h(x,t) > \lambda\},$$

and

$$\overline{G}(\lambda) = \{x \in B_{4R}(x_0) \,:\, |\nabla\varphi(x)| > \lambda\}.$$

Estimates (6.7) and (6.8) now imply

$$\begin{aligned} \frac{1}{|Q_{20\rho,\theta(20\rho)^2}|}\int_{Q_{20\rho,\theta(20\rho)^2}\cap D} h^p \,\mathrm{d}z &\le \frac{c\lambda^{p-q}}{|Q_{4\rho,\theta(4\rho)^2}|}\int_{Q_{4\rho,\theta(4\rho)^2}\cap G(\eta\lambda)} h^q \,\mathrm{d}z \\ &\quad + \left(\frac{c}{|B_{4\rho}|}\int_{B_{4\rho}\cap\overline{G}(\eta\lambda)} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q}. \end{aligned}$$

For further details, we refer to (4.17).

Since $q = pn/(n+2)$ and $\theta < 1$, we have

$$\left|Q_{4\rho,\theta(4\rho)^2}\right| / |B_{4\rho}|^{p/q} \le c.$$

Thus, if we multiply the previous estimate by $\left|Q_{20\rho,\theta(20\rho)^2}\right|$, it follows that

$$\begin{aligned}\int_{Q_{20\rho,\theta(20\rho)^2}\cap D} h^p \,\mathrm{d}z \le c\lambda^{p-q} &\int_{Q_{4\rho,\theta(4\rho)^2}\cap G(\eta\lambda)} h^q \,\mathrm{d}z \\ &+ c\left(\int_{B_{4\rho}\cap\overline{G}(\eta\lambda)} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q}.\end{aligned} \tag{6.9}$$

Let us now construct a Vitali covering for $G(\lambda)$ (see (4.20)) such that either (6.9) or the local estimate holds in each of the cylinders. Similarly to (4.21), a summation over the covering implies

$$\int_{G(\lambda)} h^p \,\mathrm{d}z \le c\lambda^{p-q} \int_{G(\eta\lambda)} h^q \,\mathrm{d}z + c\left(\int_{\overline{G}(\eta\lambda)} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q}. \tag{6.10}$$

Here we also used the fact that $p/q > 1$.

In much the same way as in the calculations starting from (4.22), we obtain

$$\begin{aligned}\int_{G(\lambda_0)} h^{p+\varepsilon} \,\mathrm{d}z \le \frac{c\varepsilon}{\varepsilon+p-q} &\int_{G(\lambda_0)} h^{\varepsilon+p} \,\mathrm{d}z + c\,(\lambda_0)^{\varepsilon} \int_{G(\eta\lambda_0)} h^p \,\mathrm{d}z \\ &+ c\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \left(\int_{\overline{G}(\eta\lambda)} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q} \mathrm{d}\lambda.\end{aligned} \tag{6.11}$$

Next we estimate the initial boundary term in (6.11). First, we divide the term into two parts as

$$\begin{aligned}&\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \left(\int_{\overline{G}(\eta\lambda)} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q} \mathrm{d}\lambda \\ &\quad\le \left(\int_{\overline{G}(\eta\lambda_0)} |\nabla\varphi|^q \,\mathrm{d}x\right)^{p/q-1} \int_{\lambda_0}^{\infty} \varepsilon\lambda^{\varepsilon-1} \int_{\overline{G}(\eta\lambda)} |\nabla\varphi|^q \,\mathrm{d}x \,\mathrm{d}\lambda.\end{aligned}$$

Now we can apply Fubini's theorem to estimate the second part. The first part can be estimated by using Hölder's inequality. It follows that

$$
\begin{aligned}
&\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \left(\int_{\overline{G}(\eta\lambda)} |\nabla \varphi|^q \,\mathrm{d}x \right)^{p/q} \mathrm{d}\lambda \\
&\leq \left(\int_{\overline{G}(\eta\lambda_0)} |\nabla \varphi|^q \,\mathrm{d}x \right)^{p/q-1} \int_{\overline{G}(\eta\lambda_0)} \int_{\lambda_0}^{|\nabla\varphi|/\eta} \varepsilon \lambda^{\varepsilon-1} |\nabla \varphi|^q \,\mathrm{d}\lambda \,\mathrm{d}x \qquad (6.12)\\
&\leq cR^{2\varepsilon/(q+\varepsilon)} \left(\int_{\overline{G}(\eta\lambda_0)} |\nabla \varphi|^{q+\varepsilon} \,\mathrm{d}x \right)^{(p+\varepsilon)/(q+\varepsilon)} .
\end{aligned}
$$

To continue, we merge (6.11) with (6.12) and choose $\varepsilon > 0$ small enough to absorb the term containing $h^{p+\varepsilon}$ into the left. If the term containing $h^{p+\varepsilon}$ is infinite, then we can consider the truncated functions instead, as in (4.27).

So far we have considered $G(\lambda_0)$, but the estimate extends to the whole of $Q_0 \cap D$. Indeed,

$$
h^{p+\varepsilon} \leq \lambda_0^{\varepsilon} h^p \quad \text{in} \quad (Q_0 \cap D) \setminus G(\lambda_0),
$$

and, consequently,

$$
\begin{aligned}
\int_{Q_0 \cap D} h^{p+\varepsilon} \,\mathrm{d}z \leq c\,(\lambda_0)^{\varepsilon} &\int_{Q_0 \cap D} h^p \,\mathrm{d}z \\
&+ cR^{2\varepsilon/(q+\varepsilon)} \left(\int_{B_{4R}} |\nabla \varphi|^{q+\varepsilon} \,\mathrm{d}x \right)^{(p+\varepsilon)/(q+\varepsilon)} .
\end{aligned}
\qquad (6.13)
$$

Taking a smaller cylinder on the left side, we can estimate h by $|\nabla u|$. Since $q = pn/(n+2)$, it follows that

$$
|Q_0|^{-1} R^{2\varepsilon/(q+\varepsilon)} = c\,|B_0|^{-(p+\varepsilon)/(q+\varepsilon)} .
$$

Thus, (6.13) implies

$$
\begin{aligned}
\frac{1}{|Q_0|} \int_{Q_0 \cap D} |\nabla u|^{p+\varepsilon} \,\mathrm{d}z \leq \frac{c\,(\lambda_0)^{\varepsilon}}{|Q_0|} &\int_{Q_0 \cap D} |\nabla u|^p \,\mathrm{d}z \\
&+ c \left(\fint_{B_{4R}} |\nabla \varphi|^{q+\varepsilon} \,\mathrm{d}x \right)^{(p+\varepsilon)/(q+\varepsilon)} .
\end{aligned}
\qquad (6.14)
$$

The final step is to take the cut-off level into account. The case $\lambda_0 = 1$ is again clear. Moreover, if $\lambda_0 = \lambda_0'$, then Young's inequality implies

$$\begin{aligned}\frac{1}{|Q_{R,R^2}|} \int_{Q_{R,R^2}\cap D} |\nabla u|^{p+\varepsilon} \,\mathrm{d}z \\ &\le \left(\frac{c}{|Q_{4R,(4R)^2}|} \int_{Q_{4R,(4R)^2}\cap D} |\nabla u|^p \,\mathrm{d}z \right)^{(\varepsilon+2)/2} \\ &\quad + c \left(⨍_{B_{4R}} |\nabla \varphi|^{q+\varepsilon} \,\mathrm{d}x \right)^{(p+\varepsilon)/(q+\varepsilon)}.\end{aligned}$$

This proves the claim. □

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