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Abstract: *The purpose of this paper is to analyze the singularities of a well known benchmark problem "Andrews' squeezing mechanism". We show that for physically relevant parameter values this system admits singularities. The method is based on Gröbner bases computations and ideal decomposition. It is algorithmic and can thus be applied to study constraint singularities which arise in more general situations.*

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1 Introduction

The “Andrews’ squeezing system” was first described by Giles in [Gil78] and further studied in [Man81]. It is a planar multibody system whose topology consists of closed kinematic loops (see Figure 1). The Andrews’ system was promoted in [Sch90] as a benchmark problem to compare different multibody solvers. Nowadays it is a well-known benchmark problem [HW91, MI03] for numerical integration of differential-algebraic equations as well. The equations are of the Lagrangian form (or descriptor form, see also [Arp01])

$$\begin{cases} f(t, y, y', y'', \lambda) = 0 \\ g(y) = 0 \end{cases} \tag{1}$$

where the function f describes the dynamical equations and g gives the (holonomic) constraints. Here $y \in \mathbb{R}^n$ are the (generalized) position coordinates, y' and y'' are the first and second derivatives, respectively, and λ is the Lagrange multiplier.

It is well known that singularities of any kind hinder solving equations numerically [RS88, HW91, BA94, EH95]. Intuitively, a singularity is where the (generic) number of degrees of freedom of the system changes. Mathematically these are the points where the rank of the Jacobian of g drops. Hence in this paper we will not consider the actual dynamical equations and analyse only the constraints given by g.

Most differential equation solvers include a possibility to monitor singularities, and usually when proximity of a singularity is detected, the computation is best to be interrupted. But this kind of monitoring is local only, that is, it does not tell us a priori where the singularities lie but only alert us when it is too late to fix things, so to speak. Also, the monitoring is often a non-negligible part of computational cost. Therefore, it would be highly useful to know *a priori* where the singularities are, or to make sure that there are no singularities, or perhaps even remove them (for the latter approach, see [Arp01]). Locating singularities has been studied also in [McC00]. If we cannot avoid or remove the singularities, at least knowing where they are encountered is helpful (indeed, necessary) when planning the computation without interruptions. One can then tune the chosen integration algorithm such that the disturbing effect of the singularities is diminished, for example by compensating the singularity of the Kepler problem by a local change of variables as in [LR05] within the computation. Further techniques on compensating singularities in multibody systems are gathered and concisely compared in [BA94] and [EH95].

The paper is organized as follows: in the next Section we present the situation in detail and formulate the constraint equations in polynomial form. Section 3 gathers the necessary algebraic tools. Section 4 contains the actual analysis where we show that the mechanism indeed has singularities for certain parameter values. In Section 5 there are some numerical examples of singular configurations, and in Section 6 we summarize and discuss the results, and address possible future work.

2 Andrews' squeezing mechanism

The squeezing mechanism is given by the following equations.

$$g(y) = \begin{cases} a_1\cos(y_1) - a_2\cos(y_1+y_2) - a_3\sin(y_3) - b_1 \\ a_1\sin(y_1) - a_2\sin(y_1+y_2) + a_3\cos(y_3) - b_2 \\ a_1\cos(y_1) - a_2\cos(y_1+y_2) - a_4\sin(y_4+y_5) - a_5\cos(y_5) - w_1 \\ a_1\sin(y_1) - a_2\sin(y_1+y_2) + a_4\cos(y_4+y_5) - a_5\sin(y_5) - w_2 \\ a_1\cos(y_1) - a_2\cos(y_1+y_2) - a_6\cos(y_6+y_7) - a_7\sin(y_7) - w_1 \\ a_1\sin(y_1) - a_2\sin(y_1+y_2) - a_6\sin(y_6+y_7) + a_7\cos(y_7) - w_2 \end{cases} \tag{2}$$

Compared to the original articles mentioned above, we have chosen the following notation for the parameters and angles:

$$\begin{aligned} &a_1 = rr \quad a_2 = d \quad a_3 = ss \quad a_4 = e \quad a_5 = zt \quad a_6 = zf \quad a_7 = u \\ &b_1 = xb \quad b_2 = yb \quad w_1 = xa \quad w_2 = ya \\ &y_1 = \beta \quad y_2 = \Theta \quad y_3 = \gamma \quad y_4 = \Phi \quad y_5 = \delta \quad y_6 = \Omega \quad y_7 = \epsilon \end{aligned}$$

so the positions in Cartesian coordinates of the fixed nodes A and B are given by $b = (b_1, b_2)$ and $w = (w_1, w_2)$, and the lengths of the rods by $a = (a_1, \ldots, a_7)$, see Figures 1 and 2.

Fixing the parameters a, b, and w, we have a map $g : \mathbb{R}^7 \to \mathbb{R}^6$. Hence the set of possible configurations, which is the zeroset $M_g = g^{-1}(0)$, is in general a curve (or possibly empty). Our task is to analyse the singularities of M_g, so let us state more precisely what is meant by a singularity. As mentioned before, in a singularity the number of degrees of freedom changes. It is well known [RS88, BA94, McC00] that this corresponds to the situation where the rank of Jacobian drops.

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be any smooth map where $k < n$ and let df be its Jacobian matrix. Let $M = f^{-1}(0) \subset \mathbb{R}^n$ be the zeroset of f. A point $q \in M$ is a *singular point* of M, if df does not have maximal rank at q.

What in fact geometrically "happens" at a singular point may be quite complicated to determine. Typically the tangent space to M does not change continuously in the neighbourhood of a singular point, or possibly M intersects itself there. However, in all cases numerical problems occur, so it is important to try to find all singular points.

Note that the constraint equations (2) (and hence the elements of its Jacobian matrix) are *not* polynomials, yet our algebraic approach works only in a polynomial setting. However, this problem is circumvented by reformulating $g(y)$ as polynomials in the sines and cosines of y_i by using the trigonometric identities

$$\begin{aligned} &\cos(x)^2 + \sin(x)^2 = 1 \\ &\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \\ &\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \end{aligned}$$

Setting $c_i = \cos(y_i)$, $\quad s_i = \sin(y_i)$ we get the equations

$$p(c,s) = \begin{cases} a_1c_1 - a_2\big(c_1c_2 - s_1s_2\big) - a_3s_3 - b_1 = 0 \\ a_1s_1 - a_2\big(s_1c_2 + c_1s_2\big) + a_3c_3 - b_2 = 0 \\ a_1c_1 - a_2\big(c_1c_2 - s_1s_2\big) - a_4\big(s_4c_5 + c_4s_5\big) - a_5c_5 - w_1 = 0 \\ a_1s_1 - a_2\big(s_1c_2 + c_1s_2\big) + a_4\big(c_4c_5 - s_4s_5\big) - a_5s_5 - w_2 = 0 \\ a_1c_1 - a_2\big(c_1c_2 - s_1s_2\big) - a_6\big(c_6c_7 - s_6s_7\big) - a_7s_7 - w_1 = 0 \\ a_1s_1 - a_2\big(s_1c_2 + c_1s_2\big) - a_6\big(s_6c_7 + c_6s_7\big) + a_7c_7 - w_2 = 0 \\ c_i^2 + s_i^2 - 1 = 0, \qquad i = 1, \ldots, 7. \end{cases} \tag{3}$$

We have 13 polynomial equations ($p_i = 0$), 11 parameters ($a_1, \ldots, a_7$, b_1, b_2, w_1, w_2) and 14 variables ($c_1, s_1, \ldots, c_7, s_7$). Note that each p_i is of degree two in c_i, s_i. The equations $p_1 = 0, \ldots, p_6 = 0$ correspond directly to the 6 original equations $g(y) = 0$ with the simple substitutions above (for example $\cos(y_1 + y_2) = c_1c_2 - s_1s_2$) and the equations $p_7 = 0, \ldots, p_{13} = 0$ are the extra identities due to "forgetting" the angle variables y_i.

Note that this reformulation of the constraints as algebraic equations is not just a trick which happens to work in this special case; indeed most constraints appearing in the simulation of multibody systems are of this type.

Now the above equations define p as a map $p : \mathbb{R}^{14} \to \mathbb{R}^{13}$. Hence we expect that the zeroset $V = p^{-1}(0) \subset \mathbb{R}^{14}$ is a curve (or possibly empty). Singularities are then the points of this curve where the rank of dp is not maximal. To find these points we need now to introduce some tools from commutative algebra.

3 Background

In this section we present briefly the necessary definitions from commutative algebra and algebraic geometry. More details can be found in [CLO92], [GP02], [Nor76], and [Eis96]. These are roughly in the order of increasing difficulty, [CLO92] being the most accessible, but unfortunately not containing the necessary material on the Fitting ideals.

3.1 Ideals and varieties

Let $\mathbb{K}$ be an algebraic field and let $\mathbb{K}[x_1, \ldots, x_n]$ be the ring of polynomials in $x_1, \ldots, x_n$, with coefficients in $\mathbb{K}$. A subset $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is an *ideal* if it satisfies

(i) $0 \in I$.

(ii) If $f, g \in I$, then $f + g \in I$.

(iii) If $f \in I$ and $h \in \mathbb{K}[x_1, \ldots, x_n]$, then $hf \in I$.

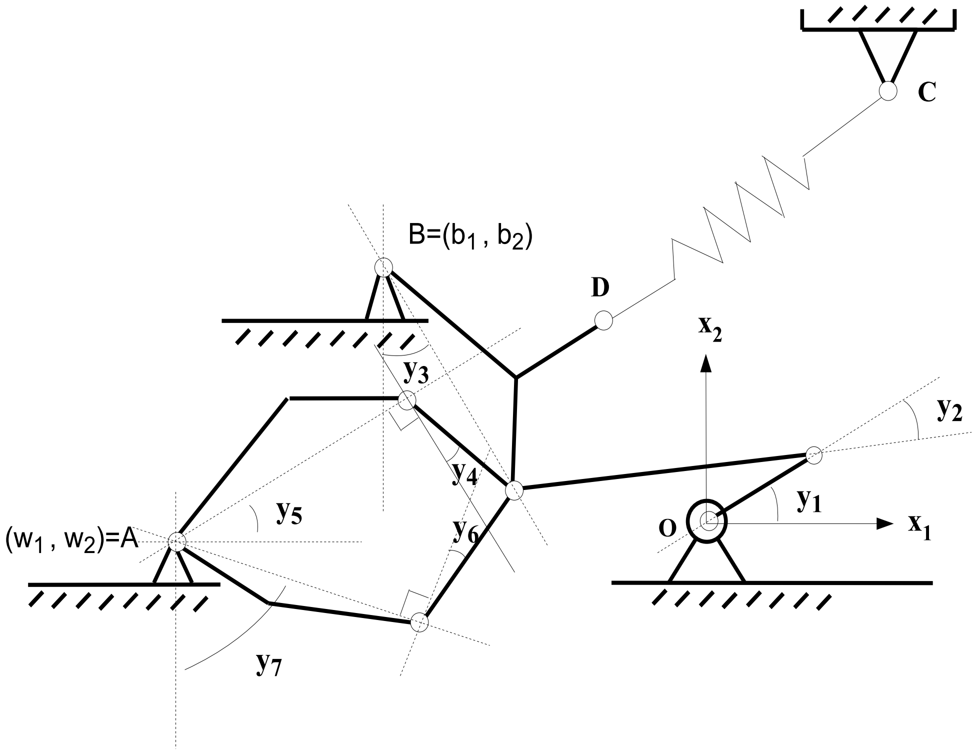

Figure 1: The angles y_i of the Andrews' system.

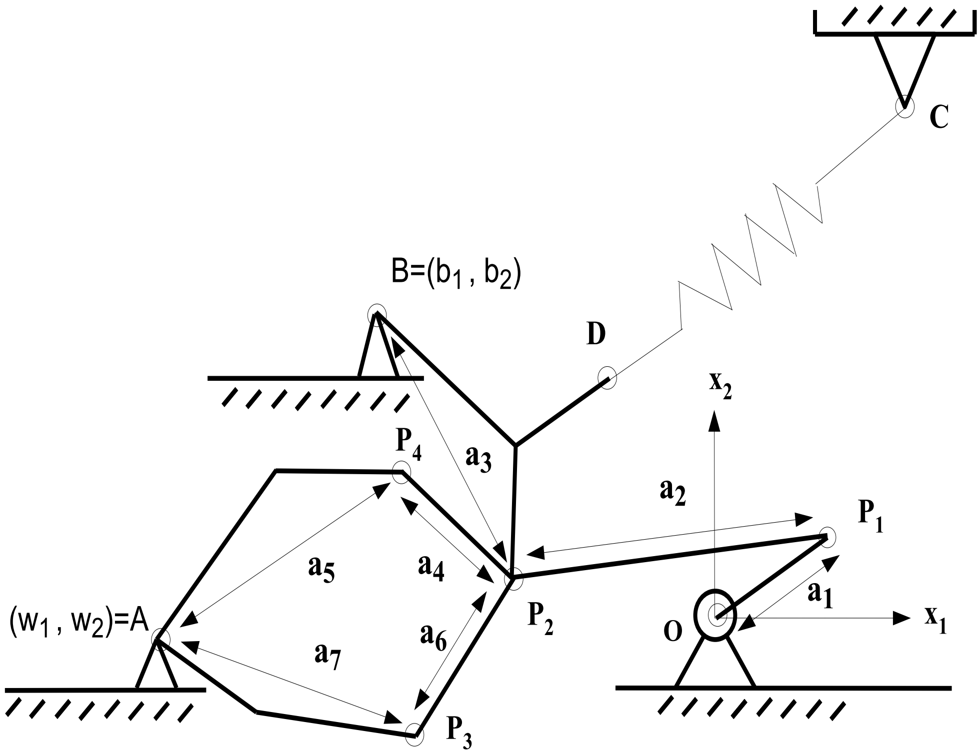

Figure 2: The lengths a_i and nodes of the Andrews' system.

Ideals are often given by *generators*. Let $f_1, \dots, f_s \in \mathbb{K}[x_1, \dots, x_n]$. Then the set

$$\langle f_1, \dots, f_s \rangle := \left\{ \sum_{i=1}^{s} h_i f_i \mid h_1, \dots, h_s \in \mathbb{K}[x_1, \dots, x_n] \right\}$$

is an *ideal generated by* $f_1, \dots, f_s$. Any set of generators is called a *basis*.

Ideals are purely algebraic objects. The geometrical counterpart of an ideal is its locus, or variety. Let I be an ideal in $\mathbb{K}[x_1, \dots, x_n]$. Its corresponding *variety* is

$$\mathsf{V}_{\mathbb{F}}(I) = \{(a_1, \dots, a_n) \in \mathbb{F}^n \mid f(a_1, \dots, a_n) = 0 \quad \forall f \in I\}$$

where $\mathbb{F}$ is some field extension of $\mathbb{K}$. Note that it is often natural to choose $\mathbb{F}$ different from $\mathbb{K}$. If the field is clear from context we will sometimes write simply $\mathsf{V}(I)$.

Now different ideals may have the same variety. However, if one is interested mainly in the variety then it is useful to define

$$\sqrt{I} = \big\{ f \in \mathbb{K}[x_1, \dots, x_n] \,|\, f^n \in I \ \text{ for some } \ n \geq 1 \big\}.$$

If I is an ideal, then $\sqrt{I}$ is the *radical* of I; it is the biggest ideal that has the same variety as I and all ideals having the same variety have the same radical. Also, always $I \subset \sqrt{I}$ and if $I = \sqrt{I}$ we say that I is a *radical ideal*. Some rudimentary properties among ideals and their varieties are in the following

Lemma 3.1. Let I and J be ideals. Then

1. $\mathsf{V}(I \cup J) = \mathsf{V}(I) \cap \mathsf{V}(J)$.
2. $\mathsf{V}(I \cap J) = \mathsf{V}(I) \cup \mathsf{V}(J)$.
3. $I \subset J$ if and only if $\mathsf{V}(I) \supset \mathsf{V}(J)$.

Next we have to express the rank condition algebraically. To this end we need

Definition 3.1. If $I = \langle f_1, \dots, f_s \rangle$, its *Fitting ideal* F_I is the ideal generated by all maximal minors of the Jacobian matrix of $(f_1, \dots, f_s)$.[1]

Now $\mathsf{V}(F_I)$ corresponds to the points where the rank is not maximal. However, the points are required also to be on $\mathsf{V}(I)$. Hence we conclude that the set of singular points, S, is given by

$$S = \mathsf{V}(I \cup F_I)$$

In analysing varieties it is often helpful to decompose them to simpler parts. Similarly one may try to decompose a given ideal to simpler parts. This leads to following notions.

[1]In general one can define Fitting ideals of minors of any given size. However, the above definition is sufficient for purposes of the present paper.

Definition 3.2. A variety V is *irreducible* if $V = V_1 \cup V_2$ implies $V = V_1$ or $V = V_2$.
An ideal I is *prime* if $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ and $fg \in I$ imply that either $f \in I$ or $g \in I$.

There is a very close connection between prime ideals and irreducible varieties. The precise nature of this depends on the chosen field. However, for our purposes the following is sufficient.

Lemma 3.2. If I is prime, then $\mathsf{V}(I)$ is irreducible.
Any radical ideal can be written uniquely as a finite intersection of prime ideals,

$$\sqrt{I} = I_1 \cap \cdots \cap I_r,$$

where $I_i \not\subset I_j$ for $i \neq j$.

This is known as the *prime decomposition of* $\sqrt{I}$ and the I_i's are called the minimal associated primes of I. The above Lemma then immediately gives:

Corollary 3.1.

$$\mathsf{V}(I) = \mathsf{V}(\sqrt{I}) = \mathsf{V}(I_1) \cup \cdots \cup \mathsf{V}(I_r),$$

where all $\mathsf{V}(I_i)$ are irreducible.

Hence our strategy in analysing varieties is to compute the minimal associated primes of the relevant ideal, and then examine each irreducible component separately.

3.2 Gröbner bases

An essential thing is that all the operations above, especially finding the radical and the prime decomposition can be computed *algorithmically* using the given generators of I. To do this we need to compute special bases for ideals, called Gröbner bases. We will only briefly indicate the relevant ideas and refer to [CLO92] and [GP02] for more details.

First we need to introduce *monomial orderings*. All the algorithms handling the ideals are based on some orderings among the terms of the generators of the ideal.

Intuitively, an ordering $\succ$ is such that given a set of monomials (e.g. terms of a given polynomial), $\succ$ puts them in order of importance: given any two monomials $x^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and x^β, where $\alpha \neq \beta$ are different multi-indices, then either $x^\alpha \succ x^\beta$ or $x^\beta \succ x^\alpha$. A common choice is to use *degree reversed lexicographic* ordering [CLO92]. In our analysis we shall frequently need *product orders*, which are formed as follows: if $\succ_A$ and $\succ_B$ are two orderings, we shall divide the variables x_i into two subsets, and use $\succ_A$ on the first subset and $\succ_B$ on the second. This is indicated with the following notation:

$$\mathbb{K}[(x_4, x_5, x_7), (x_1, x_2, x_3, x_6)].$$

This is the same set as $\mathbb{K}[x_1, \ldots, x_7]$ but now the parenthesis indicate that we will use $\succ_A$ among the variables (x_4, x_5, x_7), and $\succ_B$ among the variables (x_1, x_2, x_3, x_6), and moreover all monomials where variables of the first group appear are always bigger than monomials where there are only variables of the second group. We will see later why this is useful.

Finally, the aforementioned Gröbner basis is a special kind of generating set, with respect to some ordering. Given any set of generators and an ordering, the corresponding Gröbner basis exists and can be computed. The relevant algorithm is usually called the *Buchberger algorithm.* The drawback of this algorithm is that it has a very high complexity in the worst case, and in practice the complexity depends quite much on the chosen ordering.[2]

Anyway Gröbner bases have proved to be very useful in many different applications. Nowadays there exist many different implementations and improvements of the Buchberger algorithm. We chose to use the well-known program Singular [GPS05], [GP02] in all the computations in this paper.

4 Analysing singularities

4.1 Geometric description of the singularities

Now getting back to our system (3) we see that we can take the components of p to be elements of $\mathbb{Q}(a, b, w)[c, s]$ where $\mathbb{Q}(a, b, w)$ is the field of rational functions of a, b, and w. Hence we have an ideal $J = \langle p_1, \ldots, p_{13} \rangle \subset \mathbb{Q}(a, b, w)[c, s]$ and the corresponding Fitting ideal F_J. On the other hand we may view the "parameters" a, b, and w also as variables since they appear polynomially in the equations; hence we could also consider $J \subset \mathbb{Q}[a, b, w, c, s]$. Taking this point of view we can give an intuitive description of what kind of situations we can expect.

$$\begin{cases} J \subset \mathbb{Q}[a, b, w, c, s] \\ V_{\mathbb{R}}(J) \subset \mathbb{R}^{25}. \end{cases}$$

In this way $V_{\mathbb{R}}(J)$ should be 12 dimensional (recall J is generated by 13 equations), i.e. a curve depending on 11 parameters. On the other hand if we fix parameters a, b, and w we get a curve in $\mathbb{R}^{14}$ which will be denoted by $V_{a,b,w}$. In the same way we can view $V_{\mathbb{R}}(J \cup F_J)$ as a variety in $\mathbb{R}^{25}$, and fixing the parameters we get the singular points $V^S_{a,b,w}$. Obviously $V^S_{a,b,w} \subset V_{a,b,w} \subset \mathbb{R}^{14}$.

Then what kind of variety should $V_{\mathbb{R}}(J \cup F_J)$ be? Since the Jacobian of p is of size 13×14, *generically* we expect to get 2 independent conditions in order the rank to drop. That is, augmenting J with F_J should bring in 2 more equations. Hence we expect that $V_{\mathbb{R}}(J \cup F_J)$ is 10 dimensional; in other words we expect that if 11 parameters are chosen independently then $V^S_{a,b,w}$ should be empty. On the other hand if a single condition among parameters is satisfied, then $V^S_{a,b,w}$ should consist of isolated points.

[2]So far, no satisfactory theory of Gröbner basis complexity has been done.

Further, if there are 2 conditions among parameters (i.e. 9 parameters freely chosen), then it would be possible that $V^S_{a,b,w}$ were one dimensional. But then our original constraint equations would be redundant, i.e. there would be more than one degree of freedom.

Below we will in fact observe that if a certain condition on parameters is satisfied, $V^S_{a,b,w}$ is indeed a finite set of points.

4.2 Singular variety

To study $V_{\mathbb{R}}(J \cup F_J)$ we could in principle use Gröbner basis theory in a straightforward manner. Let G be the Gröbner basis of $J \cup F_J$ using the product order $\mathbb{Q}[(c,s),(a,b,w)]$. Let us denote by $g_1,\dots,g_r$ the elements of G which do not depend on c and s.

Definition 4.1. Let $S_J = \langle g_1,\dots,g_r\rangle$; then we say that $\mathsf{V}_{\mathbb{R}}(S_J) \subset \mathbb{R}^{11}$ is the singular variety associated to J.

It follows from the Gröbner basis theory that $V_{a,b,w}$ can have singularities *only if* $(a,b,w) \in \mathsf{V}_{\mathbb{R}}(S_J)$. Hence theoretically, we could now find the singularities of the Andrews' system in a straightforward manner by calculating the Gröbner basis of $J \cup F_J$. But this is an enormous task, due to F_J being generated by high degree polynomials, not to mention including the 11 parameters a,b,w. We could not get the solution in a finite time using our work station with 64GB memory.

Instead, something else needs to be done. Luckily there is another approach: noting that p_1,p_3,p_5 have common terms, as well as p_2,p_4,p_6, gives us motivation to study two subsystems. One spanned by $p_5 - p_3$ and $p_6 - p_4$, the other one spanned by $p_5 - p_1$ and $p_6 - p_2$ (along with the relevant trigonometric identities from $p_7,\dots,p_{13}$). These subsystems are handleable and give useful information for the whole system as well. Proceeding in this way we could at least determine that the singular variety is not empty and we could compute some subvarieties of it.

4.3 Subsystem 4567

Intuitively, the nodes and bars 4, 5, 6, 7 formulate a subsystem, see Figures 1 and 2. We suspect that when the lengths $a_4,\dots,a_7$ are such that the "4567" system is able to become one-dimensional, hence in some sense degenerated, there should be a singularity in the whole system (see also the net example in [Arp01]). We will shortly see that this is indeed the case.

Define

$$\begin{aligned}
q_1 &:= p_5 - p_3 = a_4\big(s_4c_5 + c_4s_5\big) + a_5c_5 - a_6\big(c_6c_7 - s_6s_7\big) - a_7s_7\\
q_2 &:= p_4 - p_6 = a_4\big(c_4c_5 - s_4s_5\big) - a_5s_5 + a_6\big(s_6c_7 + c_6s_7\big) - a_7c_7\\
q_i &:= p_{i+7} = c_{i+1}^2 + s_{i+1}^2 - 1, \quad i = 3,\dots,6.
\end{aligned}$$

Note that q_1, q_2 contain only angles c_i, s_i and parameters a_i for $i = 4, \ldots, 7$. That is why we do not need the other p_i's. Let J_{4567} be the ideal spanned by $q_1, \ldots, q_6$. Hence we have

$$J_{4567} \subset \mathbb{Q}[(c_4, s_4, c_5, s_5, c_6, s_6, c_7, s_7), (a_4, a_5, a_6, a_7)] \tag{4}$$

where we have indicated the relevant product order. The Gröbner basis G for $J_{4567} \cup F_{J_{4567}}$ with respect to this ordering contains 191 elements (denoted by $g_1, \ldots, g_{191}$), out of which 3 are especially enlightening:

$$\begin{aligned}
g_5 &= c_6 a_6 a_7, \\
g_{16} &= c_4 a_4 a_5, \qquad \text{and} \\
g_1 &= \prod_{i=1}^{8} t_i, \qquad \text{where} \\
t_1 &= a_4 - a_5 - a_6 - a_7 \\
t_2 &= a_4 - a_5 + a_6 + a_7 \\
t_3 &= a_4 + a_5 + a_6 + a_7 \\
t_4 &= a_4 + a_5 - a_6 - a_7 \\
t_5 &= a_4 - a_5 + a_6 - a_7 \\
t_6 &= a_4 - a_5 - a_6 + a_7 \\
t_7 &= a_4 + a_5 - a_6 + a_7 \\
t_8 &= a_4 + a_5 + a_6 - a_7.
\end{aligned}$$

Since g_1 is the only generator which does not contain any variables c_i and s_i we conclude that

Theorem 1. The singular variety of J_{4567} is

$$S_{J_{4567}} = \mathsf{V}(\langle g_1 \rangle).$$

Note that the factorization of g_1 gives us the prime decomposition of $\langle g_1 \rangle$ and hence decomposition of $\mathsf{V}(\langle g_1 \rangle)$ into 8 linear irreducible varieties.

Our next task is to show that at least some points of the singular variety extend to actual (physically relevant) singularities of the whole system. Recall that each generator g_i corresponds to an equation $g_i = 0$. Since $a_i > 0$ in physically relevant cases, generators g_5 and g_{16} imply that all the singularities of J_{4567} have necessarily $c_6 = c_4 = 0$ (conditions for the angles 4 and 6). In other words, in ideal-theoretic language, we can as well study the ideal

$$T := \langle J_{4567},\, F_{J_{4567}},\, c_4,\, c_6 \rangle.$$

Now the prime decomposition of $\sqrt{T}$ has 16 components:

$$\sqrt{T} = T_1 \cap \ldots \cap T_{16}. \tag{5}$$

Inspecting the generators of each of T_j, it is noticed that every T_j contains the t_i's or a_i's. Recall that a generator a_i in an ideal corresponds in the variety to a condition $a_i = 0$ which is non-physical. Moreover, t_3 is now a non-physical condition contradicting $a_i > 0\,\forall i$. Hence we discard (as in [Arp01]) those ideals which have a non-physical generator that would imply $a_i \leq 0$ for some i, and we are left with 7 ideals, whose generators are:

$$\begin{aligned}
T_1 &= \langle c_7^2 + s_7^2 - 1,\ t_1,\ s_6 + 1,\ s_5 - c_7,\ c_5 + s_7,\ s_4 + 1,\ c_4,\ c_6\rangle \\
T_2 &= \langle c_7^2 + s_7^2 - 1,\ t_2,\ s_6 + 1,\ s_5 + c_7,\ c_5 - s_7,\ s_4 + 1,\ c_4,\ c_6\rangle \\
T_3 &= \langle c_7^2 + s_7^2 - 1,\ t_4,\ s_6 + 1,\ s_5 + c_7,\ c_5 - s_7,\ s_4 - 1,\ c_4,\ c_6\rangle \\
T_4 &= \langle c_7^2 + s_7^2 - 1,\ t_5,\ s_6 - 1,\ s_5 - c_7,\ c_5 + s_7,\ s_4 + 1,\ c_4,\ c_6\rangle \\
T_5 &= \langle c_7^2 + s_7^2 - 1,\ t_6,\ s_6 - 1,\ s_5 + c_7,\ c_5 - s_7,\ s_4 + 1,\ c_4,\ c_6\rangle \\
T_6 &= \langle c_7^2 + s_7^2 - 1,\ t_7,\ s_6 - 1,\ s_5 - c_7,\ c_5 + s_7,\ s_4 - 1,\ c_4,\ c_6\rangle \\
T_7 &= \langle c_7^2 + s_7^2 - 1,\ t_8,\ s_6 - 1,\ s_5 + c_7,\ c_5 - s_7,\ s_4 - 1,\ c_4,\ c_6\rangle.
\end{aligned}$$

Especially, we see that $s_6 = \pm 1$, $s_5 = \pm c_7$, $c_5 = \pm s_7$, and $s_4 = \pm 1$. Now we are ready to continue with the original system $J \cup F_J$.

Remark 4.1. Mathematically speaking the analyses of all cases T_i are completely similar. However, on physical grounds the cases T_1, T_2, T_6 and T_7 are not so interesting. Indeed, in these cases the length of one of the rods corresponding to a_4, a_5, a_6 and a_7 is equal to the sum of the lengths of three others. Hence all four rods could be modelled as a single rod which would make the whole model significantly simpler. In the remaining cases no such reduction can be done, and we chose to examine the ideal T_5 in detail. See also remark 4.3.

The case T_5 gives us conditions $s_4 = -1$, $s_6 = 1$, $s_5 = -c_7$, $c_5 = s_7$, and $a_7 = a_5 + a_6 - a_4$ which we substitute into the original system. Next we will show that the resulting system has real solutions. These will be the required singular points.

The above substitutions simplify the generators of $J \cup F_J$ so that we get the following ideal:

$$\begin{aligned}
&K = \langle K_1 \cup K_2 \rangle, \\
&K_1 \quad : \quad \begin{cases} k_1 = a_2(-c_1c_2 + s_1s_2) + c_1a_1 - s_3a_3 - b_1 \\ k_2 = a_2(-s_1c_2 - c_1s_2) + s_1a_1 + c_3a_3 - b_2 \\ k_3 = c_1^2 + s_1^2 - 1 \\ k_4 = c_2^2 + s_2^2 - 1, \end{cases} \\
&K_2 \quad : \quad \begin{cases} k_5 = s_7(a_4 - a_5) + s_3a_3 + b_1 - w_1 \\ k_6 = c_7(a_5 - a_4) - c_3a_3 + b_2 - w_2 \\ k_7 = c_3^2 + s_3^2 - 1 \\ k_8 = c_7^2 + s_7^2 - 1. \end{cases}
\end{aligned} \tag{6}$$

In K_2 we have 4 equations for 4 unknowns c_3, s_3, c_7, and s_7; hence it appears reasonable that we can get a finite number of solutions. Then we can substitute the computed values to K_1 which then becomes also a system of 4 equations for 4 unknowns c_1, s_1, c_2, and s_2. By the same reasoning we again expect that it is possible to get some solutions for appropriate parameter values.

We could numerically solve the variables from these equations (and, indeed, we will, in the numerical examples), but to analyze the situation in more detail we need to study these further.

Then starting with the system K_2 we solve the angles 3 and 7 by the following trick. First we inspect the ideal generated by K_2 in the ring

$$\mathbb{Q}(b_1, b_2, w_1, w_2, a_3, a_4, a_5)[c_3, s_3, c_7, s_7].$$

Calculating the Gröbner basis $\tilde{G}$ of $\langle K_2 \rangle$ with respect to the lexicographic ordering we get 4 generators:

$$\begin{aligned} \tilde{g}_1 &= f_1 s_7^2 + f_2 s_7 - f_3 f_4 \\ \tilde{g}_2 &= 2(b_2 - w_2)(a_4 - a_5)c_7 - 2(b_1 - w_1)(a_4 - a_5)s_7 + f_5 = 0 \\ \tilde{g}_3 &= a_3 s_3 + (a_4 - a_5)s_7 + b_1 - w_1 = 0 \\ \tilde{g}_4 &= a_3 c_3 + (a_4 - a_5)c_7 + w_2 - b_2 = 0. \end{aligned} \tag{7}$$

where the auxiliary expressions f_i are lengthy combinations of the parameters a_i, b_i (see the appendix).[3]

Now $\tilde{g}_1$ contains only s_7 and parameters. Note that $f_1 = 0$ if and only if $a_4 = a_5$. Assuming $a_4 \neq a_5$ the equation $\tilde{g}_1 = 0$ is a polynomial in s_7 of degree 2, hence in order to have real solutions we need to impose the condition

$$f_2^2 + 4f_1 f_3 f_4 \geq 0. \tag{8}$$

This condition can easily be checked when the parameters a, b, w have been given numerical values. Once s_7 is known, c_7, s_3, c_3 can be solved from the linear equations of $\tilde{G}$, provided $a_4 \neq a_5$ and $w_2 \neq b_2$.

The cases $w_2 = b_2$ and/or $a_4 = a_5$ can be summarized as follows:

(i) If $w_2 = b_2$ but $a_4 \neq a_5$, we still get equations similar to $\tilde{G}$, but now s_3 has a quadratic equation instead of s_7.

(ii) If $a_4 = a_5$, the system typically does not have solutions. At least, a further condition among parameters, namely $|b - w| = a_3$, arises. We shall not elaborate this nongeneric behaviour further. In Section 4.5.2 we consider an example of this situation.

Remark 4.2. In general, when the inequality in (8) is strict, s_7 has 2 possible values. Therefore, the tuples (s_3, c_3, s_7, c_7) have in general 2 possible values because the other ones in the tuple are determined uniquely from s_7.

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[3] The algorithms actually give by default only sums of monomials instead of products like $2(b_2 - w_2)(a_4 - a_5)$ but we have simplified these by hand. Also Singular [GPS05] could be used to automatically factorize into products but would involve some more elaborate programming.

The only thing left to be done, in this J_{4567} subsystem case, is to solve c_1, s_1, c_2, s_2. This is done with the ideal $\langle K_1 \rangle$ given in (6).

Remark 4.3. Had we used any other T_i instead of T_5 above, we would have ended up with this same ideal $\langle K_1 \rangle$.

We calculate the Gröbner basis $\hat{G}$ of $\langle K_1 \rangle$, this time in the ring

$$\mathbb{Q}(a_1, a_2, a_3, b_1, b_2, c_3, s_3)[c_1, s_1, c_2, s_2].$$

Note especially that s_3, c_3 are here treated as parameters, due to being now known expressions in the parameters a, b, w. We again use lexicographic ordering and get 4 generators $\hat{g}_1, \ldots, \hat{g}_4$. Analogously to s_7 above, now for s_2 we get the second degree polynomial equation

$$\hat{g}_1 = (-4a_1^2a_2^2)s_2^2 - n_1n_2 = 0 \tag{9}$$

where

$$\begin{aligned} n_1 &= a_1^2 + 2a_1a_2 + a_2^2 - a_3^2 - 2a_3b_1s_3 + 2a_3b_2c_3 - b_1^2 - b_2^2 \\ n_2 &= a_1^2 - 2a_1a_2 + a_2^2 - a_3^2 - 2a_3b_1s_3 + 2a_3b_2c_3 - b_1^2 - b_2^2 \end{aligned}$$

and linear equations for c_2, s_1, c_1:

$$\begin{aligned} \hat{g}_2 &= d_1c_2 + d_2 + d_3 \\ \hat{g}_3 &= l_1s_1 + l_2 + l_3 \\ \hat{g}_4 &= (a_1^2 - a_2^2)c_1 + l_4 \end{aligned}$$

where the auxiliary expressions d_i, l_i are certain known (but lengthy) functions of a, b, apart from l_4 which depends on s_1, s_2, c_2 as well. (See the appendix.) In order to have real solutions for s_2, (9) implies the condition

$$E := n_1n_2 \leq 0. \tag{10}$$

These $\hat{g}_i$ determine s_2, c_2, s_1, c_1 provided $d_1 \neq 0$, $l_1 \neq 0$, $a_1 \neq a_2$. To analyse the cases $d_1 = 0$, $a_1 = a_2$, and/or $l_1 = 0$, it is helpful to define

$$d_0 := a_3^2 + 2a_3b_1s_3 - 2a_3b_2c_3 + b_1^2 + b_2^2.$$

It turns out that $l_1 = 0 \Leftrightarrow d_1 = 0 \Leftrightarrow d_0 = 0$. After rearranging the terms (see the appendix) it can be seen that the condition (10) is equivalent to

$$(a_1 - a_2)^2 \leq d_0 \leq (a_1 + a_2)^2.$$

Therefore, if $a_1 \neq a_2$ then $d_0 \neq 0$ and the equations above can be solved. The case $a_1 = a_2$, $d_0 \neq 0$ does not essentially change the situation: we still have a quadratic equation for s_2, and linear ones for the others, with a different coefficient for c_1.

The remaining case $a_1 = a_2$, $d_0 = 0$ corresponds to the situation where the centre node coincides with the origin. This gives another singularity (the angle y_1 remains arbitrary) but is a rather special case and will not be pursued further here.

Theorem 2. Let us suppose that the parameters a, b, w satisfy the following conditions: $a_4 \neq a_5$ and

$$n_1(4a_1a_2 - n_1) \geq 0 \tag{10}$$
$$f_2^2 + 16(a_4 - a_5)^2|b - w|^2 f_3 f_4 \geq 0 \tag{8}$$

Then $V_{a,b,w}$ contains at least 2 singular points. If the inequalities are strict we get in general at least 4 singular points.

It may appear that we also have at most 4 singular points. However, it is a priori possible that the other systems T_i yield more singular points with the same parameter values.

Proof. The first part of the theorem merely collects what we have shown above, with the simplifications $n_2 = n_1 - 4a_1a_2$ and $f_1 = 4(a_4 - a_5)^2|b - w|^2$. The conditions are due to univariate second degree polynomial equations, which have real solutions if and only if (8) and (10) (for s_7 and s_2, respectively) are fulfilled. The other variables are determined from linear equations: $s_4, c_4, \ldots, s_6, c_6$ from T_5; s_3, c_3, c_7 from K_1; s_1, c_1, c_2 from K_2.

For the number of singular configurations, note that we have second order equations for s_7, hence at most 2 values for the tuple (s_3, c_3, s_7, c_7), and s_2. So in general if there are two separate roots both for s_7 and s_2, we get four different singularities. □

Similar results can be presented for any T_i but we will not catalogue them here.

4.4 Subsystem 367

Comparing to examples in [Arp01] it was perhaps intuitively clear that subsystem J_{4567} produces singularities. It is a bit more surprising that there is another subsystem producing singularities: the one formed by the nodes 3, 6, and 7.

Define

$$\begin{aligned}
h_1 &:= -p_5 + p_1 = a_6\big(c_6c_7 - s_6s_7\big) + a_7s_7 - a_3s_3 + w_1 - b_1 \\
h_2 &:= -p_6 + p_2 = a_6\big(s_6c_7 + c_6s_7\big) - a_7c_7 + a_3c_3 + w_2 - b_2 \\
h_3 &:= p_9 = c_3^2 + s_3^2 - 1 \\
h_4 &:= p_{12} = c_6^2 + s_6^2 - 1 \\
h_5 &:= p_{13} = c_7^2 + s_7^2 - 1.
\end{aligned}$$

It is important to note that h_1, h_2 contain only angles 3,6, and 7, therefore only p_9, p_{12}, p_{13} are relevant to them. As parameters we now have not only the lengths a_3, a_6, a_7, but also $b_1, \ldots, w_2$ i.e. the positions of the fixed nodes A and B in Figure 2. Let J_{367} be the ideal generated by $h_1, \ldots, h_5$. We will proceed in a similar way as with the subsystem J_{4567}.

First we will consider the singularities of the subsystem J_{367} using the following product order:

$$J_{367} \cup F_{J_{367}} \subset \mathbb{Q}[(c_3, s_3, c_6, s_6, c_7, s_7), (a_3, a_6, a_7, b_1, b_2, w_1, w_2)] \tag{11}$$

The relevant Gröbner basis G contains 96 generators of which two are especially interesting:

$$\begin{aligned} g_{12} &= c_6 a_6 a_7 \\ g_1 &= \prod_{i=1}^{4} z_i \qquad \text{where} \\ z_1 &= (a_3 - a_6 + a_7)^2 - |b - w|^2 \\ z_2 &= (a_3 + a_6 + a_7)^2 - |b - w|^2 \\ z_3 &= (a_3 + a_6 - a_7)^2 - |b - w|^2 \\ z_4 &= (a_3 - a_6 - a_7)^2 - |b - w|^2. \end{aligned} \tag{12}$$

The latter one gives us the singular variety $S_{J_{367}}$.

Theorem 3. The singular variety of J_{367} is

$$S_{J_{367}} = \mathsf{V}(\langle g_1 \rangle).$$

Remark 4.4. It is worth noting that, contrary to the linear constraints t_i in Theorem 1 related to J_{4567}, the z_i in Theorem 3 give *quadratic* constraints $z_i = 0$ related to J_{367} and have the interpretation "$|a_3 \pm a_6 \pm a_7|$ = distance between the fixed points A and B". Furthermore, again the factors z_i give the irreducible decomposition of the singular variety.

Since $a_i > 0$, we get $c_6 = 0$ from $g_{12} = 0$. This simplifies computations considerably. Let us define

$$U := \langle J_{367},\, F_{J_{367}},\, c_6 \rangle.$$

The prime decomposition of U turns out to have 8 components:

$$\sqrt{U} = U_1 \cap \cdots \cap U_8.$$

Inspecting the generators of each of U_i, it is noticed that the ideals $U_k, \quad k = 5 \ldots 8$ contain generators which imply $a_i = 0$ for some i. Hence those are discarded as non-physical and we are left with 4 ideals:

$$\begin{aligned} U_1 &= \langle u_1,\, u_2,\, c_7^2 + s_7^2 - 1,\, c_6,\, s_6 - 1,\, s_3 + s_7,\, c_3 + c_7 \rangle \\ U_2 &= \langle u_1,\, u_2,\, c_7^2 + s_7^2 - 1,\, c_6,\, s_6 + 1,\, s_3 + s_7,\, c_3 + c_7 \rangle \\ U_3 &= \langle u_1,\, u_2,\, c_7^2 + s_7^2 - 1,\, c_6,\, s_6 + 1,\, s_3 - s_7,\, c_3 - c_7 \rangle \\ U_4 &= \langle u_1,\, u_2,\, c_7^2 + s_7^2 - 1,\, c_6,\, s_6 - 1,\, s_3 - s_7,\, c_3 - c_7 \rangle \end{aligned}$$

$$\text{where} \quad \begin{cases} u_1 = -s_6 c_7 a_6 - c_3 a_3 + c_7 a_7 + b_2 - w_2 \\ u_2 = s_6 s_7 a_6 + s_3 a_3 - s_7 a_7 + b_1 - w_1. \end{cases}$$

With these, we continue studying the whole system $J \cup F_J$. Each U_i will lead to a different case with $s_6 = \pm 1$, $s_3 = \pm s_7$, $c_3 = \pm c_7$. Let us look for example the ideal U_1.[4] This gives

$$\begin{aligned} s_6 &= 1, \\ c_7 &= \frac{b_2 - w_2}{a_6 - a_3 - a_7}, \\ s_7 &= \frac{b_1 - w_1}{a_3 - a_6 + a_7}, \\ c_3 &= -c_7, \\ s_3 &= -s_7. \end{aligned} \tag{13}$$

We should expect to run into an equation $z_i = 0$ for some i, where the expressions z_i are given in (12). Combined with $c_7^2 + s_7^2 - 1 = 0$ the equations (13) give $z_1 = 0$. Likewise, U_i implies $z_i = 0$ for $i = 2, 3, 4$.

Remark 4.5. The condition $z_2 = 0$ is physically a redundant case: it means that the system can barely reach from A to B when the subsystem of the rods a_3, a_6, a_7 is fully stretched, i.e. it has no room to move. Therefore also U_2 corresponds to a rather trivial case. See also Remark 4.1.

Using U_1 we can now eliminate the variables corresponding to angles 3, 6, and 7. Doing the substitutions in $J \cup F_J$ we are left with the following generators.

$$\begin{aligned} L &= \langle L_1 \cup L_2 \rangle, \\ L_1 \quad &: \quad \begin{cases} l_1 = a_2(-c_1c_2 + s_1s_2) + c_1a_1 + s_7a_3 - b_1 \\ l_2 = a_2(-s_1c_2 - c_1s_2) + s_1a_1 - c_7a_3 - b_2 \\ l_3 = c_1^2 + s_1^2 - 1 \\ l_4 = c_2^2 + s_2^2 - 1, \end{cases} \\ L_2 \quad &: \quad \begin{cases} l_5 = a_4(s_4c_5 + c_4s_5) + c_5a_5 + s_7(a_6 - a_7) \\ l_6 = a_4(c_4c_5 - s_4s_5) - s_5a_5 + c_7(a_6 - a_7) \\ l_7 = c_4^2 + s_4^2 - 1 \\ l_8 = c_5^2 + s_5^2 - 1, \end{cases} \end{aligned} \tag{14}$$

where the s_7, c_7 are no longer variables, but known expressions from (13) and kept here only for clarity of notation.

Remark 4.6. Before working on L_1 and L_2 we comment briefly on the other

[4] As with J_{4567} and T_5, the other cases are completely similar and we will comment them shortly.

U_i cases. Introduce L_3 and L_4:

$$L_3 : \begin{cases} a_2(-c_1c_2 + s_1s_2) + c_1a_1 - s_7a_3 - b_1 = 0 \\ a_2(-s_1c_2 - c_1s_2) + s_1a_1 + c_7a_3 - b_2 = 0 \\ c_1^2 + s_1^2 - 1 = 0 \\ c_2^2 + s_2^2 - 1 = 0 \end{cases}$$

$$L_4 : \begin{cases} a_4(s_4c_5 + c_4s_5) + c_5a_5 - s_7(a_6 + a_7) = 0 \\ a_4(c_4c_5 - s_4s_5) - s_5a_5 - c_7(a_6 + c_7) = 0 \\ c_4^2 + s_4^2 - 1 = 0 \\ c_5^2 + s_5^2 - 1 = 0. \end{cases}$$

Had we used U_2 instead of U_1, we would end up with the system L_1, L_4. Likewise, U_3 would give the system L_3, L_2, and U_4 would give the system L_3, L_4. Yet another point of view is, that $s_6 = \pm 1$ picks between L_2 and L_4, while $(c_3, s_3) = \pm(c_7, s_7)$ picks between L_1 and L_3. More precisely, $s_6 = 1$ ($s_6 = -1$) gives L_2 (L_4), and $(c_3, s_3) = (-c_7, -s_7)$ gives L_1. The choice $(c_3, s_3) = (c_7, s_7)$ would give L_3.

Continuing with L_1 and L_2, we notice that L_2 contains only the variables c_5, s_5, c_4, s_4 (angles 4 and 5), has 4 equations and 4 variables hence is expected to have a finite solution set and will be handled analogously to the ideal K_2 in (6). Calculating its Gröbner basis G in the ring

$$\mathbb{Q}(a_4, a_5, a_6, a_7)[(c_4, c_5, s_5, c_7, s_7), (s_4)]$$

we obtain 12 generators, the first one being

$$g_1 = 2a_4a_5s_4 + a_4^2 + a_5^2 - a_6^2 + 2a_6a_7 - a_7^2.$$

Hence s_4 can be explicitly solved:

$$s_4 = \frac{a_4^2 + a_5^2 - a_6^2 + 2a_6a_7 - a_7^2}{-2a_4a_5}. \tag{15}$$

The other generators are too messy to be of much use. Then using the formula $c_4^2 = 1 - s_4^2$ we get

$$\begin{aligned} c_4^2 &= -\frac{1}{4a_4^2a_5^2}[(a_4 + a_5 - a_6 + a_7)(a_4 - a_5 + a_6 - a_7)\cdot \\ &\qquad \cdot (a_4 - a_5 - a_6 + a_7)(a_4 + a_5 + a_6 - a_7)] \\ &= -\frac{t_7t_5t_6t_8}{4a_4^2a_5^2}. \end{aligned} \tag{16}$$

The product term in the numerator has to be nonpositive, in order to have any real solutions:

$$t_5t_6t_7t_8 \leq 0. \tag{17}$$

After solving s_4, c_4 we can proceed to solve s_5 and c_5. For this we use the ordering

$$\mathbb{Q}(a_4, a_5, a_6, a_7)[c_5, s_5, c_4, s_4, c_7, s_7]$$

and pick the two relevant equations from the corresponding Gröbner basis:

$$\begin{aligned}(-a_6 + a_7)s_5 - a_4c_4s_7 + a_4s_4c_7 + a_5c_7 &= 0\\ (-a_6 + a_7)c_5 - a_4c_4c_7 - a_4s_4s_7 - a_5s_7 &= 0,\end{aligned}$$

which are linear equations for s_5, c_5, provided $a_6 \neq a_7$.

Remark 4.7. In the case $a_6 = a_7$ the situation is different: L_2 then decomposes into 3 prime ideals, of which only one is physically feasible and gives a singularity only if $a_4 = a_5$. Thence this is a rather special case and will not be considered further here.

The subsystem L_2 is now fully solved. Moving on to L_1, we will see that the analysis is very similar to that of K_1 from (6). Therefore we will skip some details. After forming the Gröbner basis of L_1 in the ring

$$\mathbb{Q}(b_1, b_2, a_1, a_2, a_3, c_7, s_7)[c_1, s_1, c_2, s_2]$$

with respect to the lexicographic ordering, we get for s_2, after simplifications, the relation

$$s_2^2 = \frac{n_3(4a_1a_2 - n_3)}{4a_1^2a_2^2}, \tag{18}$$
$$\text{where} \quad n_3 = |b|^2 + 2a_3(b_2c_7 - b_1s_7) - (a_1 - a_2)^2 + a_3^2$$

Again for the real solutions the numerator has to be nonnegative

$$n_3(4a_1a_2 - n_3) \geq 0 \tag{19}$$

We can now solve c_2, s_1 and c_1, provided their coefficients are nonzero, from the linear equations

$$\begin{aligned}&2a_1a_2n_4c_2 - 4a_1^2a_2^2s_2^2 + r_1 = 0,\\ &-2a_1n_4s_1 + r_2 + r_3 = 0,\\ &(a_1^2 - a_2^2)c_1 + r_4 = 0.\end{aligned}$$

where

$$n_4 = |b|^2 + a_3^2 + 2a_3(b_2c_7 - b_1s_7)$$

and r_i are lengthy, yet polynomial, expressions in the parameters, apart from r_4 which depends on s_1, s_2, c_2 as well. (See the appendix.)

What about the cases $n_4 = 0$ and/or $a_1 = a_2$? It can be shown, as with d_0, that the condition $n_3(4a_1a_2 - n_3) \geq 0$ is equivalent to

$$(a_1 - a_2)^2 \leq n_4 \leq (a_1 + a_2)^2.$$

Therefore, if $a_1 \neq a_2$ then $n_4 \neq 0$ and the equations above are sufficient. The case $a_1 = a_2$, $n_4 \neq 0$ does not essentially change the situation: we still have a quadratic equation for s_2, and linear ones for the others, with a different coefficient for c_1.

The remaining case $a_1 = a_2$, $n_4 = 0$ is analogous to the $n_2 = 0$ case within J_{4567} and likewise will not be pursued further.

Theorem 4. Let us suppose that the parameters a,b,w satisfy the following conditions:

$$\begin{aligned}
&a_6 \neq a_7 \\
&n_4 \neq 0 \\
&n_3(4a_1a_2 - n_3) \geq 0 \qquad (20)\\
&t_7t_5t_6t_8 \leq 0. \qquad (21)
\end{aligned}$$

Then $V_{a,b}$ contains at least 2 singular points. If the inequalities are strict we get in general at least 4 singular points.

Similar results can be represented for any $\mathsf{V}(U_i)$ but we will not catalogue them here.

Proof. The last two conditions are due to univariate second degree polynomial equations, which have real solutions if and only if (20) (for s_2) and (21) (for c_4) are fulfilled. The first condition is needed for the other variables to be determined uniquely: $s_3, c_3, s_6, c_6, s_7, c_7$ from $\mathsf{V}(U_1)$, s_4, s_5, c_5 from L_2, and s_1, c_1, c_2 from L_1.

For the number of singular configurations, note that we have second order equations, hence at most 2 values, for c_4 and s_2. So in general if there are two separate roots both for c_4 and s_2, we get four different singularities. □

4.5 Two special cases with symmetry

Let us look more closely at two special cases: $a_4 = a_6$, $a_5 = a_7$, and either $a_4 = a_5$ or $a_4 \neq a_5$.

4.5.1 The case $a_4 \neq a_5$

Motivated by the original benchmark values [Sch90] we give the following

Lemma 4.1. When $a_4 = a_6$ and $a_5 = a_7$, there is a relation between the angles 4 and 6: either $y_6 = -y_4$ or $y_6 = y_4 + \pi$. Furthermore, if also $a_4 \neq a_5$, the angle y_7 variables, i.e. c_7, s_7, are uniquely determined from c_4, s_4, c_5, s_5.

Proof. Looking for relations between solely angles 4 and 6, we substitute $a_4 = a_6$ and $a_5 = a_7$ to the subsystem J_{4567} and formulate a suitable elimination ideal. In ideal-theoretic language, we define

$$\begin{aligned}
r_1 &:= a_4\big(s_4c_5 + c_4s_5\big) + a_5c_5 - a_4\big(c_6c_7 - s_6s_7\big) - a_5s_7 \\
r_2 &:= a_4\big(c_4c_5 - s_4s_5\big) - a_5s_5 + a_4\big(s_6c_7 + c_6s_7\big) - a_5c_7 \\
r_{i+2} &= c_{i+3}^2 + s_{i+3}^2 - 1, \quad i = 1, \dots, 4,
\end{aligned}$$

where $r_i = q_i$ with substitutions $a_4 = a_6$ and $a_5 = a_7$, and investigate the ideal $I := \langle r_1, \ldots, r_6\rangle$ in the ring

$$\mathbb{Q}(a_4, a_5, a_6, a_7)[(c_5, s_5, c_7, s_7), (c_4, s_4, c_6, s_6)].$$

Calculating the elimination ideal $I_{4,6} := I \cap \mathbb{Q}[c_4, s_4, c_6, s_6]$ we get

$$I_{4,6} = \langle s_4 + s_6, c_6^2 + s_6^2 - 1, c_4^2 + s_4^2 - 1\rangle.$$

Calculating the prime decomposition of $\sqrt{I_{4,6}}$ we get

$$\sqrt{I_{4,6}} = \langle c_6^2 + s_6^2 - 1, c_4 - c_6, s_4 + s_6\rangle \cap \langle c_6^2 + s_6^2 - 1, c_4 + c_6, s_4 + s_6\rangle.$$

Since $I_{4,6} \subset I \subset J \subset J \cup F_J$, we have

$$\mathsf{V}(I_{4,6}) \supset \mathsf{V}(J \cup F_J).$$

From these prime ideals we can see that everywhere in $\mathsf{V}(I_{4,6})$, and therefore in the variety of the singularities of the whole system as well, $s_6 = -s_4$ and either $c_6 = c_4$ or $c_6 = -c_4$. These translate into two possible relations between the angles y_4 and y_6.

$$\begin{array}{lcl} (c_6, s_6) = (c_4, -s_4) & \Leftrightarrow & y_6 = -y_4, \\ (c_6, s_6) = (-c_4, -s_4) & \Leftrightarrow & y_6 = y_4 + \pi. \end{array} \tag{22}$$

This proves the first claim. If we take into account either one of the prime ideals of $\sqrt{I_{4,6}}$ in I and calculate the Gröbner bases we get ideals where c_7 and s_7 depend linearly on c_4, s_4, c_5 and s_5, and can be explicitely solved, as we will show next to prove the latter claim of the lemma. For the case $(s_6, c_6) = (-s_4, -c_4)$ we get

$$\begin{cases} c_7 & = -s_5 \\ s_7 & = c_5 \end{cases} \quad \text{which imply} \quad y_7 = y_5 + \frac{\pi}{2}. \tag{23}$$

For the case $(s_6, c_6) = (-s_4, c_4)$ the expressions are, albeit linear, slightly more complicated:

$$\begin{aligned} 0 \;&=\; c_7\big(a_4^2(s_4^2 - c_4^2) - a_5(2a_4 s_4 + a_5)\big) + \\ &\quad + s_7\big(2a_4(a_5 + a_4 c_4 s_4)\big) - s_5\big((a_4^2 + a_5^2) - 2a_4 a_5 s_4\big) \\ 0 \;&=\; c_7\big(2a_4^2 c_4 s_4\big) + s_7\big(a_4^2(c_4^2 - s_4^2) + a_5^2)\big) + (a_4^2 - a_5^2)c_5 - 2a_4 a_5 s_5 c_4. \end{aligned}$$

We prove that these indeed determine c_7, s_7: all we need to do is check that the determinant of the coefficient matrix A of the linear equations does not equal zero:

$$A := \begin{pmatrix} a_4^2(s_4^2 - c_4^2) - a_5(2a_4 s_4 + a_5) & 2a_4(a_5 + a_4 c_4 s_4) \\ -2a_4^2 c_4 s_4 & a_4^2(c_4^2 - s_4^2) + a_5^2 \end{pmatrix}, \text{ prove } \det(A) \neq 0.$$

Now $\det(A)$ simplifies due to $c_4^2 + s_4^2 = 1$, resulting in

$$\det(A) = 2a_4a_5(a_4 + a_5)(a_4 - a_5)s_4 + (a_4 - a_5)(a_4 + a_5)(a_4^2 + a_5^2)$$

Let us then consider $\det(A)$ as a function of s_4. Since $s_4 \in [-1, 1]$, $\det(A) : [-1, 1] \mapsto \mathbb{R}$. Clearly if $a_4 = a_5$, $\det(A) \equiv 0$ so we need to assume $a_4 \neq a_5$. Set

$$h(s_4) := \frac{\det(A)}{(a_4 + a_5)(a_4 - a_5)} = 2a_4a_5s_4 + (a_4^2 + a_5^2)$$

and inspect when $h = 0$. Since $a_4 > 0$ and $a_5 > 0$ the linear function h has its minimum at -1.

$$h(-1) = a_4^2 + a_5^2 - 2a_4a_5 = (a_5 - a_4)^2 > 0.$$

This proves $h \neq 0$ always, therefore under the assumption $a_4 \neq a_5$ also $\det(A) \neq 0$ as claimed. □

4.5.2 The case $a_4 = a_5$

We study the special case $a_4 = a_5 = a_6 = a_7$, whence the 4567-subsystem is capable of "buckling" in more complicated ways, thereby producing further interesting configurations. This resembles then the net example in [Arp01].

Let us see how J_{4567} simplifies with substitutions $a_4 = a_5 = a_6 = a_7$. Note that the assumptions of Lemma 4.1 considering y_7 are no longer valid. Let

$$I := J_{4567} \text{ with } a_4 = a_5 = a_6 = a_7 \text{ and } s_6 = -s_4$$

and compute its prime decomposition. This results in

$$\sqrt{I} = I_1 \cap I_2 \cap I_3 \quad \text{with generators}$$

$$I_1 = \begin{cases} s_4^2 + c_6^2 - 1, \\ c_4 - c_6, \\ c_7^2 + s_7^2 - 1, \\ s_5 + c_7s_4 - s_7c_6, \\ c_5 - c_7c_6 - s_7s_4 \end{cases} \quad I_2 = \begin{cases} c_6, \\ s_4 + 1, \\ c_4, \\ c_7^2 + s_7^2 - 1, \\ c_5^2 + s_5^2 - 1 \end{cases} \quad I_3 = \begin{cases} s_4^2 + c_6^2 - 1, \\ c_4 + c_6, \\ c_7^2 + s_7^2 - 1, \\ s_5 + c_7, \\ c_5 - s_7 \end{cases} \tag{24}$$

Each of these has a geometrical interpretation, see Figure 3. I_2 corresponds to $y_4 = -\pi/2, y_6 = \pi/2$ which means that nodes A and P_2 coincide. This is like the T_5 situation. Indeed, the ideal $J \cup F_J \cup I_2$ turns out to be exactly T_5 with the extra condition $a_4 = a_5$. Although it is not immediately apparent but in that situation there also arises a new condition among the parameters: $a_3 = |b - w|$, i.e. "a_3 equals the distance between A and B". Note that here the Fitting ideal $F_{J_{4567}}$ has not been used at all, contrary to the T_5 calculations.

I_3 corresponds to $y_6 = y_4 + \pi$ and $y_5 = y_7 - \pi/2$ so that now nodes P_3 and P_4 coincide. Then again, I_1 corresponds to $y_6 = -y_4$ and $y_5 = y_6 + y_7$, which interestingly is *not a singularity* but merely expressing a symmetry in the system due to $a_4 = a_5 = a_6 = a_7$.

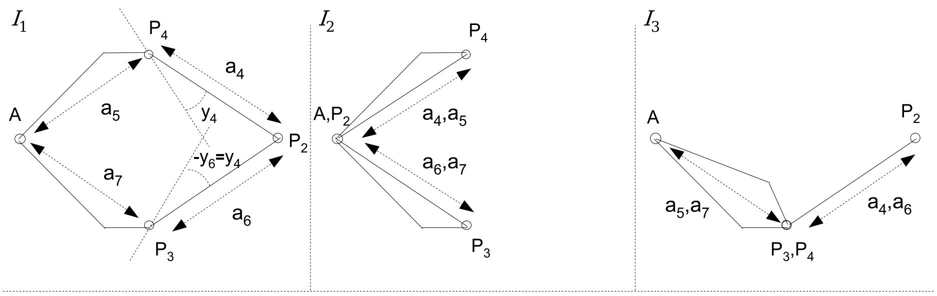

Figure 3: The configurations corresponding to I_1, I_2, I_3 in the case $a_4 = a_5 = a_6 = a_7$.

4.6 Other subsystems

Now contemplating Figure 2 we see that it would be possible to find other singularities by analysing still other subsystems. For example the subsystem corresponding to rods 3, 4 and 5 is by symmetry similar to subsystem 367: we simply exchange the roles of variables and parameters associated to rods 4 and 6, and 5 and 7. Further we could consider other subsystems formed from different "paths" between the nodes A, B, O: i.e. subsystems $J_{123}, J_{1245}, J_{1267}$. Again by symmetry the system J_{1267} is completely similar to J_{1245}, but cases J_{123} and J_{1245} give new singularities. We checked that in these cases the singular variety is not empty, and that at least for some parameter values we get singular points.

We did not analyse these cases in detail because computations are quite similar to those given above for subsystems J_{4567} and J_{367}. Hence we did not feel including these would give significant additional value and therefore left them out to avoid expanding this quite a long presentation further.

5 Numerical examples

In this section we will calculate numerical examples for both types of singularities. Interestingly, the explicit expressions within $\tilde{G}$, $\hat{G}$, as well as in the Gröbner bases of L_1 and L_2, are unstable for numerical computations. It is better to use the original defining equations of K_1, K_2, L_1, L_2 in the computations. We shall not explore this stability issue here due to its non-relevance for the present context.

We present 4 examples:

1. The original benchmark parameter values, see [MI03]. We show that then the system is avoiding singularities.[5]

2. We explore how should a_1, a_2 be changed in order to have J_{4567} type singularities in the system. Here we have an interpretation for the

[5]Thereby validating its benchmark status. That is, the numerical difficulties encountered there are indeed due to the "numerical stiffness" of the problem, not to a nearby singularity.

result: the lengths a_1, a_2 must be such that the subsystem 4567 has room for a certain kind of "buckled" configuration.

3. We explore how should b_1, a_1, a_2 be changed in order to have J_{367} type singularities in the system.

4. A special case which shows a rational solution, that is $c_i, s_i \in \mathbb{Q}$ for all i. This shows unambiguously that we can find singular points because in this case there are no numerical errors related to floating point computations.

5.1 Original values

In this example, we will use the original values for the parameters a_i, b_i and show that the system then has no singularities. The original parameters used in the benchmark tests [Sch90, HW91, MI03] are

$$\begin{array}{llll} a_1 = 0.007, & a_2 = 0.028, & a_3 = 0.035, & a_4 = 0.020, \\ a_5 = 0.040, & a_6 = 0.020, & a_7 = 0.040, & \\ b_1 = -0.03635, & b_2 = 0.03273, & & \\ w_1 = -0.06934, & w_2 = -0.00227. & & \end{array} . \tag{25}$$

Since $a_7 = a_5$ and $a_6 = a_4$, we have $t_4 = t_6 = 0$ (and $t_1 < 0$, $t_5 < 0$) so we could have an J_{4567} singularity: T_3 or T_5.

Remark 5.1. Interpretation: both T_3 and T_5 describe a situation where the 4567 system has 'collapsed' into a 1-dimensional object. The ideal K_2 tells us how a_3 restricts the possible attitudes of 4567. In T_5 the centre node P_2 has been pushed in, in T_3 it has been pulled out.

Let us look more closely first at T_5, say, and check the conditions (8) and (10). The first one is fulfilled. For E we first need to solve c_3, s_3 from $\mathsf{V}(K_2)$. Their solutions are

$$\begin{aligned} &(c_3, s_3, c_7, s_7) \in \\ &\quad \{\ (0.4299535996, -0.9028509856, -0.9975812008, 0.06951077517)\ , \\ &\quad (0.9266735994, -0.3758670513, -0.1283212011, 0.9917326602)\ \} \end{aligned} \tag{26}$$

With these c_3, s_3 we can compute E. Both sets in (26) give $E = \mathcal{O}(10^{-5}) > 0$ and the condition (10) is violated, hence there are no ($J_{4567}-$)singularities. What about other singularities? This is answered by the following

Theorem 5. With the original benchmark parameter values (25), the Andrews' squeezing system has no singularities.

Proof. We now have $a_4 = a_6$, $a_5 = a_7$ and $a_4 \neq a_5$. Lemma 4.1 implies variables c_6, s_6, c_7, s_7, and so y_6 and y_7 can be explicitely solved in terms of

c_4, s_4, c_5, and s_5. It is then possible to reduce the original system of constraint equations, by forgetting the last two equations from (2), and consider

$$\begin{cases} a_1\cos(y_1) - a_2\cos(y_1+y_2) - a_3\sin(y_3) - b_1 & = 0 \\ a_1\sin(y_1) - a_2\sin(y_1+y_2) + a_3\cos(y_3) - b_2 & = 0 \\ a_1\cos(y_1) - a_2\cos(y_1+y_2) - a_4\sin(y_4+y_5) - a_5\cos(y_5) - w_1 & = 0 \\ a_1\sin(y_1) - a_2\sin(y_1+y_2) + a_4\cos(y_4+y_5) - a_5\sin(y_5) - w_2 & = 0. \end{cases}$$

These are equivalent to

$$\begin{cases} a_1\cos(y_1) - a_2\cos(y_1+y_2) - a_3\sin(y_3) - b_1 & = 0 \\ a_1\sin(y_1) - a_2\sin(y_1+y_2) + a_3\cos(y_3) - b_2 & = 0 \\ -a_4\sin(y_4+y_5) - a_5\cos(y_5) + a_3\sin(y_3) + (b_1 - w_1) & = 0 \\ a_4\cos(y_4+y_5) - a_5\sin(y_5) - a_3\cos(y_3) + (b_2 - w_2) & = 0 \end{cases}$$

These can be again represented as polynomials.

$$\begin{aligned} m_1 &:= a_1c_1 - a_2\big(c_1c_2 - s_1s_2\big) - a_3s_3 - b_1 = 0 \\ m_2 &:= a_1s_1 - a_2\big(s_1c_2 + c_1s_2\big) + a_3c_3 - b_2 = 0 \\ m_3 &:= a_1c_1 - a_2\big(c_1c_2 - s_1s_2\big) - a_4\big(s_4c_5 + c_4s_5\big) - a_5c_5 - w_1 = 0 \\ m_4 &:= a_1s_1 - a_2\big(s_1c_2 + c_1s_2\big) + a_4\big(c_4c_5 - s_4s_5\big) - a_5s_5 - w_2 = 0 \\ m_{i+4} &:= c_i^2 + s_i^2 - 1 = 0, \quad i = 1, \ldots, 5. \end{aligned}$$

Substituting the original parameter values (25), as rational numbers, into the polynomials m_i we form an ideal $I := \langle m_1, \ldots, m_9 \rangle$. Let $K := I \cup F_I$, where F_I is the Fitting ideal of I, and inspect K in the ring

$$\mathbb{Q}[(c_1, s_1, c_2, s_2), (c_3, s_3, c_4, s_4, c_5, s_5)].$$

Now it is possible to compute the Gröbner basis G_K for K explicitly (unlike for $J \cup F_J$ in the introduction) and results in

$$G_K = \langle 1 \rangle.$$

This implies $\mathsf{V}(K) = \emptyset$, proving that with these original parameter values there are no singularities. □

5.2 J_{4567} singularity: original values, apart from a_1, a_2

Let us see how changing a_1 and/or a_2 might produce J_{4567} type singularities. Our analysis reveals that by suitable combinations of a_1 and a_2 we can get between zero and four singularities (of type J_{4567}, that is). The number of singularities is determined by c_3, s_3, and E.

Considering E as a function of a_1, a_2 we plot the area where $E \leq 0$. Recall that E depends on c_3 as well, and c_3 has two possible values so we get two functions: $E = E_1(a_1, a_2)$ (resp. $E = E_2(a_1, a_2)$) corresponding to the first (resp. second) value of c_3 from (26). See Figure 4 where the areas inside the rectangular areas are $E_i < 0$.

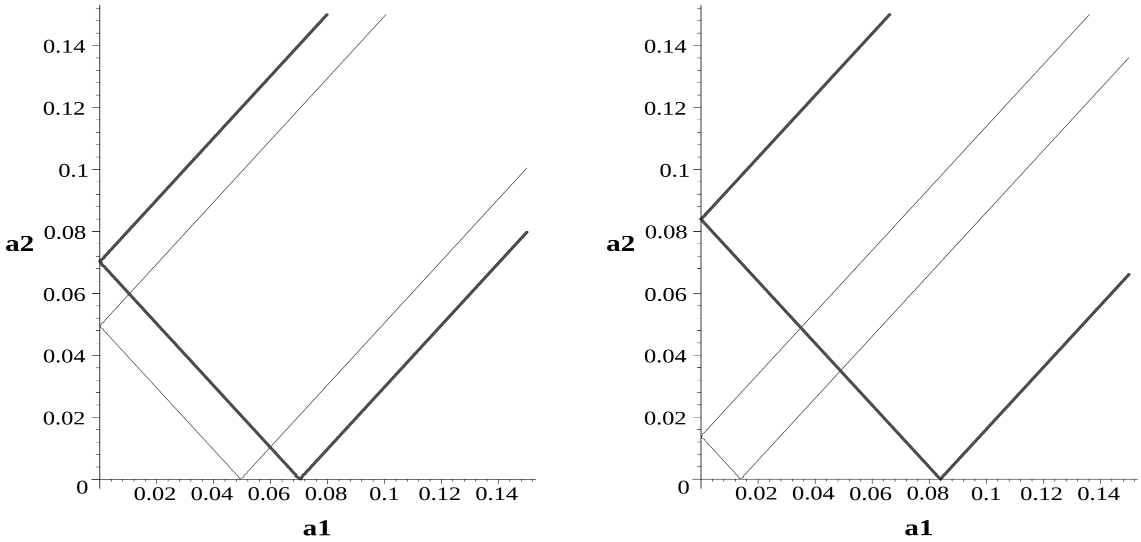

Figure 4: The rectangular lines are $E_1 = 0$ (thick line) and $E_2 = 0$ (thin line), the areas inside each $E_i = 0$ line are where $E_i < 0$. Left panel: T_5 case, right panel: T_3 case.

- no singularities: $E_1 > 0, E_2 > 0$.
- 1 singularity: $E_1 = 0, E_2 = 0$, which leads (with T_5) to two possible values:

$$(a_1 = 0.05986,\ a_2 = 0.01035), \quad (a_1 = 0.01035,\ a_2 = 0.05986)$$

- 2 singularities: one of E_1, E_2 is < 0, the other one > 0.
- 3 singularities: one of E_1, E_2 is < 0, the other one $= 0$.
- 4 singularities: $E_1 < 0,\ E_2 < 0$.

For example, let us concentrate on T_5 and choose $a_1 = 0.03$, $a_2 = 0.055$, say, whence the system is able to reach four singular configurations (see the left panel of Figure 4). Now s_i, c_i for $i = 1, 2, 3, 7$ are determined by $\mathsf{V}(K)$. The other values, for angles 4,5,6, are determined by $\mathsf{V}(T_5)$. The results are in the Table 1. The corresponding configurations are visualized in Figure 5.

Doing similar tests with T_3 instead of T_5 yields the E_i areas in the right hand panel of Figure 4. Singular configurations implied by T_3, with choices $a_1 = 0.06$, $a_2 = 0.06$ which imply 4 singularities, are in Figure 6. To save space we have not tabulated the actual values of the angles in T_3 case.

5.3 J_{367} singularity: original values, apart from b_1, a_1, a_2

A necessary condition to have a J_{367} type singularity is at least one of the z_i's vanishes (12). Substituting the original parameter values we notice that none of these is zero. Let us then investigate how we should change some of the parameters in order to have J_{367} type singularities. Take b_1 and U_1, say,

variable	singularity 1	singularity 2	singularity 3	singularity 4
c_1	-0.8322	-0.4564	-0.1157	-0.1038
s_1	-0.5544	0.8898	-0.9933	0.9946
c_2	-0.3045	-0.3045	0.4467	0.4467
s_2	0.9525	-0.9525	0.8947	-0.8947
c_3	0.4300	0.4300	0.9267	0.9267
s_3	-0.9029	-0.9029	-0.3759	-0.3759
c_4	0	0	0	0
s_4	-1	-1	-1	-1
c_5	0.0695	0.0695	0.9917	0.9917
s_5	0.9976	0.9976	0.1283	0.1283
c_6	0	0	0	0
s_6	1	1	1	1
c_7	-0.9976	-0.9976	-0.1283	-0.1283
s_7	0.0695	0.0695	0.9917	0.9917

Calculating the corresponding angles we get the following values.

Angle	singularity 1	singularity 2	singularity 3	singularity 4
y_1	-2.5539	2.0448	-1.6867	1.6747
y_2	1.8802	-1.8802	1.1077	-1.1077
y_3	-1.1264	-1.1264	-0.3853	-0.3853
y_4	-1.5708	-1.5708	-1.5708	-1.5708
y_5	1.5012	1.5012	0.1287	0.1287
y_6	1.5708	1.5708	1.5708	1.5708
y_7	3.0720	3.0720	1.6995	1.6995

Table 1: The singularities of J_{4567} type, original values apart from a_1, a_2. The values are presented only with 4 decimals but were computed with 16 decimals.

and choose $b_1 := -0.026913593$ so that $z_1 = 0$. [6] We seek to further fulfil the sufficient requirements by U_1:

$$n_3(4a_1a_2 - n_3) \geq 0 \tag{20}$$
$$t_7t_5t_6t_8 \leq 0, \tag{21}$$

and use L_1, L_2 to find the actual singular configurations. With the original parameter values $t_6 = 0$, therefore (21) is fulfilled. Therefore we only need to study (20). For that, we proceed analogously to Example 5.2: treat the expression $n_3(4a_1a_2 - n_3)$ as a function of a_1, a_2. For that, we first need c_7, s_7.

[6] This corresponds to moving B slightly to left.

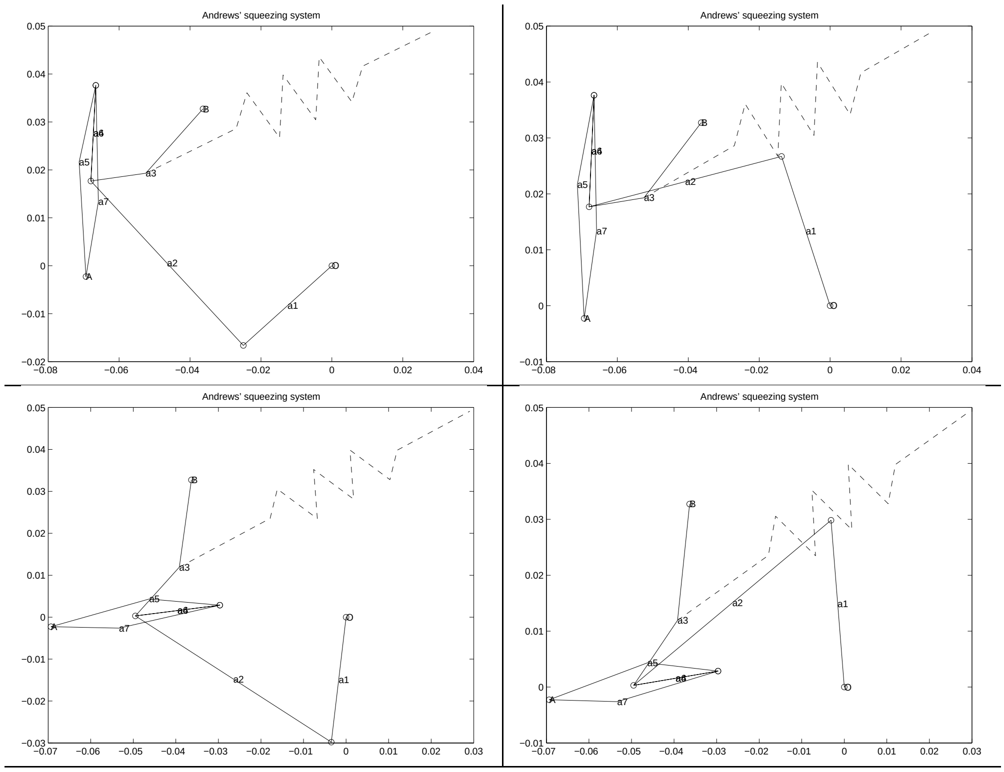

Figure 5: Singular positions (according to J_{4567}, T_5) when $a_1 = 0.03$, $a_2 = 0.055$ and $a_3, \ldots, a_7$ have the original values. One can see a physical explanation to the singularity: the centre node P_2 is 'pushed in' so that nodes P_3 and P_4 coincide.

Them we get from (13)

$$c_7 = \frac{b_2 - w_2}{a_6 - a_3 - a_7} = -0.6364$$
$$s_7 = \frac{b_1 - w_1}{a_3 + a_7 - a_6} = 0.7714.$$

The region of a_1, a_2 plane where $n_3(4a_1a_2 - n_3) \geq 0$ is shown in Figure 7. We pick a value inside the "allowed" annulus, say $a_1 = 0.02$ and $a_2 = 0.055$ in order to get singularities. Then let us find the actual singular configurations: since $t_6 = 0$, from (16) we get $c_4 = 0$ and from (15) $s_4 = -1$. The other angles are found as follows: 3 and 6 from (13) and the remaining ones 1,2,5 from L. The results are in Table 2. The corresponding singular configurations are drawn in Figure 8. Note that there are only two singular configurations, instead of four, since (16) has only one (double) root $c_4 = 0$ instead of two separate roots.

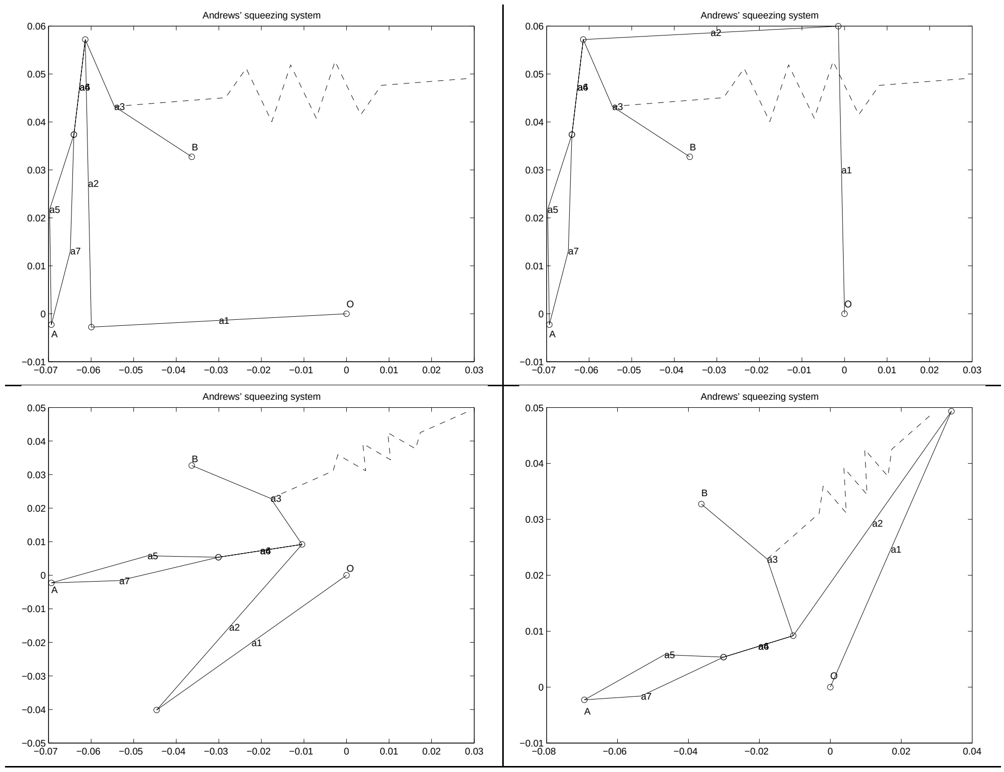

Figure 6: Singular positions (according to J_{4567}, T_3) when $a_1 = 0.06$, $a_2 = 0.06$ and $a_3, \ldots, a_7$ have the original values. One can see a physical explanation to the singularity: the centre node P_2 is now 'pulled out' so that nodes P_3 and P_4 coincide.

5.4 A rational case

Finally, let us show a rational valued singularity, that is $c_i, s_i \in \mathbb{Q}$. Choose

$$a_4 = a_5 = a_6 = a_7 = 3/20 \quad a_1 = 1/10 \quad a_2 = a_3 = 1/2$$
$$b_1 = -1/10 \quad b_2 = 1/5 \quad w_1 = -2/5 \quad w_2 = -1/5$$

and solve c, s from the generators of $I_2 \cup J \cup F_J$ in (24). Now c_5, s_5, c_7, s_7 are arbitrary (apart from $c_5^2 + s_5^2 = 1$, $c_7^2 + s_7^2 = 1$) and the chosen result is (see also Figure 9)

$$c = (0, 3/5, 4/5, 0, 3/5, 0, 4/5)$$
$$s = (1, -4/5, -3/5, -1, 4/5, 1, 3/5).$$

6 Conclusion

We have studied singularities of the multibody system "Andrews' squeezing system" which is a well-known benchmark problem both for multibody solvers and differential-algebraic equation solvers. Using our tools we have shown in

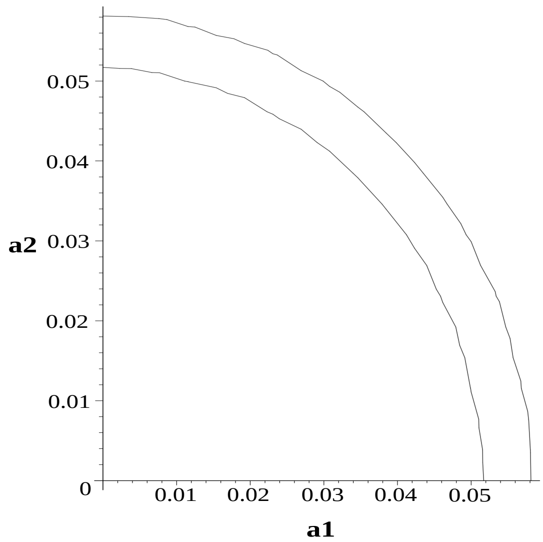

Figure 7: J_{367}, U_1 case: the region inside the annulus is where $n_3(4a_1a_2 - n_3) \geq 0$.

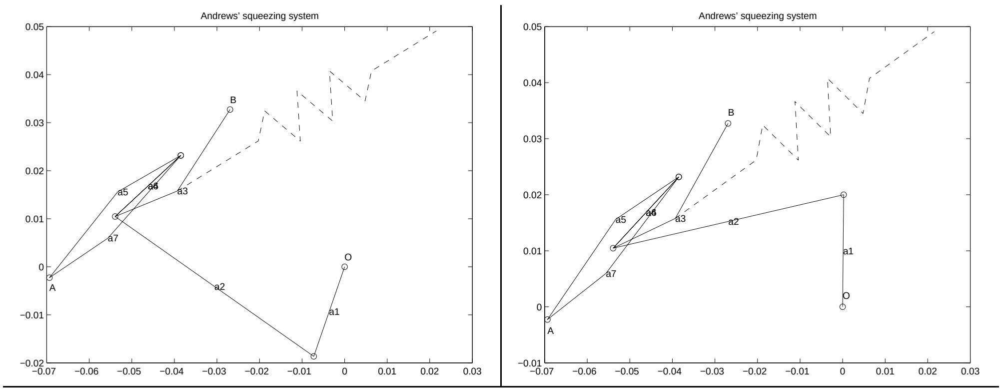

Figure 8: Singular positions (according to J_{367}, U_1) when $b_1 = -0.02691$, $a_1 = 0.02$, $a_2 = 0.055$ and $a_3, \ldots, a_7$ have the original values. The physical interpretation is as in Figure 5.

Theorem 5 that the original benchmark problem is indeed void of singularities, thereby assuring that whatever numerical problems in the benchmark tests are met, they are indeed due to something else than a nearby singularity of the system. Apparently, this non-singularity of the problem has not been rigorously proven in the literature.

However, we have shown that with suitably chosen parameters (a, b, w), this system can exhibit singular configurations. In fact, there are families of values (a, b, w) that produce singularities, see Theorems 2 and 4. We provide examples of singularities, calculated using the original benchmark parameter values apart from b_1, a_1, a_2. Considering a_1, a_2 as freely chosen parameters, Figures 4 and 7 show the areas of a_1, a_2 plane where the system exhibits singularities. For example, choosing the point (a_1, a_2) within the intersection of the three areas in Figures 4 (both panels) and 7 would give a system with 10 singular configurations.

variables	singularity 1	singularity 2
c_1	-0.3621	0.0127
s_1	-0.9322	0.9999
c_2	0.1860	0.1860
s_2	0.9862	-0.9826
c_3	0.6364	0.6364
s_3	-0.7714	-0.7714
c_4	0	0
s_4	-1	-1
c_5	0.7714	0.7714
s_5	0.6364	0.6364
c_6	0	0
s_6	1	1
c_7	-0.6364	-0.6364
s_7	0.7714	0.7714

Expressed in angles, these are

Angles	singularity 1	singularity 2
y_1	-1.9413	1.5581
y_2	1.3837	-1.3837
y_3	-0.8810	-0.8810
y_4	1.5708	1.5708
y_5	0.6898	0.6898
y_6	1.5708	1.5708
y_7	2.2606	2.2606

Table 2: The singularities of J_{367} type, original values apart from b_1, a_1, a_2. The values are presented only with 4 decimals but were computed with 16 decimals.

A natural question that remains is, if these presented singularities are the only possible ones? In other words are there singularities which do not come from the singularities of some subsystem? While the Gröbner bases techniques *in principle* provide a way to answer this question directly, we could not do so in practice due to complexity problems.

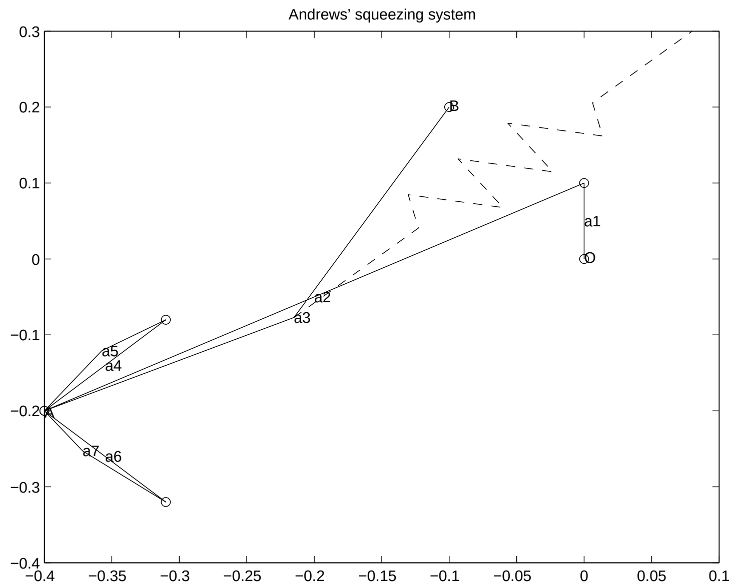

Figure 9: A singular configuration with rational c_i, s_i, a_i, b_i.

6.1 Appendix

The coefficients f_i: The coefficients $f_1, \dots, f_5$ in the context of T_5 are

$$
\begin{aligned}
f_1 &= 4(a_5-a_4)^2(b_1^2-2b_1w_1+b_2^2-2b_2w_2+w_1^2+w_2^2)\\
&= 4(a_5-a_4)^2|b-w|^2,\\
f_2 &= 4(w_1-b_1)(a_4-a_5)\\
&\quad (-b_1^2+2b_1w_1-b_2^2+2b_2w_2-w_1^2-w_2^2+a_3^2-a_4^2+2a_4a_5-a_5^2)\\
&= 4(w_1-b_1)(a_4-a_5)\big(a_3^2-(a_4-a_5)^2-|b-w|^2\big),\\
f_3 &= b_1^2-2b_1w_1+b_2^2-2b_2w_2+2b_2a_4-2b_2a_5+\\
&\quad +w_1^2+w_2^2-2w_2a_4+2w_2a_5-a_3^2+a_4^2-2a_4a_5+a_5^2\\
&= |b-w|^2+2(b_2-w_2)(a_4-a_5)-a_3^2+(a_4-a_5)^2,\\
f_4 &= b_1^2-2b_1w_1+b_2^2-2b_2w_2-2b_2a_4+2b_2a_5+\\
&\quad +w_1^2+w_2^2+2w_2a_4-2w_2a_5-a_3^2+a_4^2-2a_4a_5+a_5^2\\
&= |b-w|^2-2(b_2-w_2)(a_4-a_5)-a_3^2+(a_4-a_5)^2,\\
f_5 &= a_3^2-a_4^2+2a_4a_5-a_5^2-b_1^2+2b_1w_1-b_2^2+2b_2w_2-w_1^2-w_2^2\\
&= a_3^2-(a_4-a_5)^2-|b-w|^2.
\end{aligned}
$$

The coefficients d_i, l_i: The coefficients d_i, l_i in the context of K_2 are

$$
\begin{aligned}
d_1 &= 2a_1a_2(a_3^2 + 2a_3b_1s_3 - 2a_3b_2c_3 + b_1^2 + b_2^2) \\
d_2 &= -4a_1^2a_2^2s_2^2 \\
d_3 &= -a_1^4 + 2a_1^2a_2^2 + a_1^2a_3^2 + 2a_1^2a_3b_1s_3 - 2a_1^2a_3b_2c_3 + a_1^2b_1^2 \\
&\quad +a_1^2b_2^2 - a_2^4 + a_2^2a_3^2 + 2a_2^2a_3b_1s_3 - 2a_2^2a_3b_2c_3 + a_2^2b_1^2 + a_2^2b_2^2
\end{aligned}
$$

$$
\begin{aligned}
l_1 &= -2a_1a_2(a_3^2 + 2a_3b_1s_3 - 2a_3b_2c_3 + b_1^2 + b_2^2) \\
l_2 &= 2a_1a_2(a_3s_3 + b_1) \\
l_3 &= -(a_3c_3 - b_2)(a_1^2 - a_2^2 + a_3^2 + 2a_3b_1s_3 - 2a_3b_2c_3 + b_1^2 + b_2^2) \\
l_4 &= 2a_1a_2s_1s_2 - (a_3s_3 + b_1)a_2c_2 + (a_3c_3 - b_2)a_2s_2 - (a_3s_3 + b_1)a_1.
\end{aligned}
$$

We can also simplify these expressions:

$$
\begin{aligned}
d_0 &= a_3^2 + |b|^2 + 2a_3(b_1s_3 - b_2c_3) \\
d_1 &= 2a_1a_2d_0 \\
d_2 &= n_1n_2 \\
d_3 &= (a_1^2 + a_2^2)d_0 - (a_1^2 - a_2^2)^2 \\
n_1 &= (a_1 + a_2)^2 - d_0 \\
n_2 &= (a_1 - a_2)^2 - d_0 = 4a_1a_2 - n_1 \\
l_1 &= -d_1 \\
l_3 &= -(a_3c_3 - b_2)(a_1^2 - a_2^2 + d_0) \\
l_4 &= -(a_3s_3 + b_1)(a_2c_2 + a_1) + a_2s_2(a_3c_3 - b_2 + 2a_1s_1) \\
\hat{g}_1 &= -4a_1^2a_2^2s_2^2 + n_1(4a_1a_2 - n_1) \\
\hat{g}_2 &= 2a_1a_2d_0c_2 + (a_1^2 + a_2^2)d_0 - (a_1^2 - a_2^2)^2 \\
\hat{g}_3 &= -2a_1a_2d_0s_1 + 2a_1a_2(a_3s_3 + b_1) - (a_3c_3 - b_2)(a_1^2 - a_2^2 + d_0) \\
\hat{g}_4 &= (a_1^2 - a_2^2)c_1 + l_4
\end{aligned}
$$

The coefficients r_i: The coefficients r_i in the context of L_1 are

$$
\begin{aligned}
r_1 &= (a_1^2 + a_2^2)|b|^2 - 2b_1a_1^2a_3s_7 - 2b_1a_2^2a_3s_7 + 2b_2a_1^2a_3c_7 \\
&\quad +2b_2a_2^2a_3c_7 - (a_1^2 - a_2^2)^2 + (a_1^2 + a_2^2)a_3^2 \\
r_2 &= 2a_1(b_1a_2 - a_2a_3s_7)s_2 \\
r_3 &= b_1^2b_2 + b_1^2a_3c_7 - 2b_1b_2a_3s_7 - 2b_1a_3^2c_7s_7 + b_2^3 + 3b_2^2a_3c_7 \\
&\quad +b_2a_1^2 - b_2a_2^2 + 3b_2a_3^2c_7^2 + b_2a_3^2s_7^2 + a_1^2a_3c_7 - a_2^2a_3c_7 + a_3^3c_7 \\
r_4 &= (2a_1a_2)s_1s_2 + (-b_1a_2 + a_2a_3s_7)c_2 \\
&\quad +(-b_2a_2 - a_2a_3c_7)s_2 + (-b_1a_1 + a_1a_3s_7)
\end{aligned}
$$

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