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Mikko Parviainen

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Mikko Parviainen

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *We study the global higher integrability of the gradient of a parabolic quasiminimizer with quadratic growth conditions. Our objective is to show that the gradient belongs to a higher Sobolev space than assumed a priori if the lateral boundary satisfies a capacity density condition and boundary values are smooth enough. We derive estimates near the lateral and the initial boundary.*

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Correspondence

Mikko.Parviainen@tkk.fi

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set, $K \geq 1$. A function $u \in L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(\Omega))$ is a *parabolic quasiminimizer* if

$$-\int_{\operatorname{spt}\phi} u\frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\operatorname{spt}\phi} \frac{|\nabla u|^2}{2}\,\mathrm{d}x\,\mathrm{d}t \leq K\int_{\operatorname{spt}\phi} \frac{|\nabla (u-\phi)|^2}{2}\,\mathrm{d}x\,\mathrm{d}t$$

for all functions $\phi \in C_0^\infty(\Omega \times (0,T))$, see [Wie87]. A 1-minimizer, called a minimizer, is a weak solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

Being a weak solution to a partial differential equation is a local property, but being a quasiminimizer is not. Quasiminimizers do not provide a unique solution to the Dirichlet problem, and they do not obey the comparison principle. These facts indicate that the theory for quasiminimizers differs from the theory for minimizers, and unexpected phenomena occur. On the other hand, quasiminimizers provide a unifying approach in the calculus of variations, since the quasiminimizing condition applies to the whole class of variational integrals at the same time.

Our objective is to show that quasiminimizers belong to a slightly higher Sobolev space than assumed a priori and, in particular, that the gradient of a quasiminimizer satisfies a reverse Hölder inequality. This is always true locally, that is, in the interior of a domain as shown by Wieser in [Wie87], but here we study the question globally, that is, up to the boundary. In our case the regularity of the boundary and the regularity of boundary values play a role. We assume that the complement of a domain satisfies a capacity density condition. This condition is essentially sharp for our main results, but we point out that the results of this paper are interesting and new, as far as we know, already for smooth domains. The results are true also for systems of quasiminimizers, but we consider the scalar case for simplicity.

We derive a reverse Hölder inequality for the gradient near the lateral and the initial boundary. These cases are essentially different and therefore they are considered separately. Moreover, we obtain stronger results at the initial boundary. The proofs for the estimates are based on Caccioppoli and Poincaré type inequalities and the self-improving property of a reverse Hölder inequality. Higher integrability estimates play a decisive role in studying regularity questions, see [GM79], [GS82] and [Str80].

Elliptic quasiminimizers were first studied by Giaquinta and Giusti, see [GG82] and [GG84]. The concept of a quasiminimizer was extended to the parabolic case by Wieser in [Wie87]. Later the definition of a parabolic quasiminimizer and some of the local regularity results have been extended to a wider class of variational integrals by Zhou, see [Zho93] and [Zho94].

The local higher integrability of the gradient for nonlinear elliptic systems was observed by Elcrat and Meyers in [EM75] and for systems of parabolic equations with quadratic growth conditions by Giaquinta and Struwe in [GS82]. Recently Kinnunen and Lewis proved in [KL00] the local higher integrability for parabolic systems with more general growth conditions.

Granlund considered in [Gra82] the global higher integrability of the gradient in the elliptic case, when the complement of a domain satisfies a measure density condition, and later Kilpeläinen and Koskela generalized the elliptic results for the uniform capacity density condition in [KK94]. Arkhipova has studied the regularity of systems of parabolic partial differential equations for example in [Ark89], [Ark92] and [Ark95].

This work is organized as follows: In Section 2 we introduce the problem and the basic notation. In Section 3 we recall the concept of capacity and derive estimates near the lateral boundary. These estimates are crucial in Section 4, where we prove the integrability of the gradient to a higher power near the lateral boundary. Section 5 is devoted to estimates near the initial boundary. In the last section we prove the self-improving property for a modified reverse Hölder inequality and then complete the paper by proving the higher integrability of the gradient of a quasiminimizer near the initial boundary.

2 Preliminaries

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, $u : \Omega \times (0,T) \to \mathbb{R}$ and $K \geq 1$. A function u belonging to the parabolic space $L^2_{\mathrm{loc}}(0,T; W^{1,2}_{\mathrm{loc}}(\Omega))$ is a *parabolic quasiminimizer* if

$$-\int_{\operatorname{spt}\phi} u\frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\operatorname{spt}\phi} E(u)\,\mathrm{d}x\,\mathrm{d}t \leq K\int_{\operatorname{spt}\phi} E(u-\phi)\,\mathrm{d}x\,\mathrm{d}t, \tag{2.1}$$

for every $\phi \in C_0^\infty(\Omega \times (0,T))$, $E(u) = F(x,t,\nabla u)$ and $F : \Omega \times (0,T) \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumptions:

1. $x \mapsto F(x,t,\xi)$ and $t \mapsto F(x,t,\xi)$ are measurable for every ξ,
2. $\xi \mapsto F(x,t,\xi)$ is continuous for every (x,t),
3. there exist $0 < \alpha \leq \beta < \infty$ such that

$$\alpha\,|\xi|^2 \leq F(x,t,\xi) \leq \beta\,|\xi|^2\,. \tag{2.2}$$

There is a well-recognized difficulty in proving useful estimates for variational integrals: one often needs a test function depending on a solution u itself, but u is not admissible. For example the time derivative of the test function contains $\frac{\partial u}{\partial t}$ which does not necessarily exist as a function. There are two ways to treat this difficulty: the first option is to use the Steklov averages like for example in [DiB93] on pages 18 and 25, and the second option is to use a mollification of u in the time direction. Here we use the latter approach and have

$$-\int_{\operatorname{spt}(\phi)} u_\varepsilon \frac{\partial \phi}{\partial t} \, \mathrm{d}x \, \mathrm{d}t + \int_{\operatorname{spt}(\tilde{\phi})} E(u) - KE(u - \tilde{\phi}) \, \mathrm{d}x \, \mathrm{d}t \leq 0, \tag{2.3}$$

for every $\phi \in C_0^\infty(\Omega \times (0,T))$, where $\tilde{\phi}$ is a standard mollification of ϕ and u_ε a standard mollification of u in the time direction.

We finish this section with the notation used throughout the paper. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded open set and $D = \Omega \times (0,T)$ a space-time domain. We denote the points of the domain by $z = (x,t)$ and use a shorthand notation $\mathrm{d}z = \mathrm{d}x \, \mathrm{d}t$. Given $z_0 = (x_0, t_0) \in D$ and $\rho > 0$, let

$$B_\rho(x_0) = \{ x \in \mathbb{R}^n \, : \, |x - x_0| < \rho \},$$

denote an open ball in $\mathbb{R}^n$, and let

$$\Lambda_\rho(t_0) = (t_0 - \frac{1}{2}\rho^2, \, t_0 + \frac{1}{2}\rho^2),$$

denote an open interval in $\mathbb{R}$. A space-time cylinder in $\mathbb{R}^{n+1}$ is denoted by

$$Q_\rho(z_0) = Q_\rho = B_\rho(x_0) \times \Lambda_\rho(t_0).$$

If $|B_\rho|$ denotes the Lebesgue measure of B_ρ, then the integral average of u is denoted by

$$u_\rho(t) = \fint_{B_\rho} u(x,t) \, \mathrm{d}x = \frac{1}{|B_\rho|} \int_{B_\rho} u(x,t) \, \mathrm{d}x.$$

Finally, the time derivative of ϕ is denoted by ϕ' or $\frac{\partial \phi}{\partial t}$.

3 Estimates near the lateral boundary

In the following two sections we consider the higher integrability of the gradient of a quasiminimizer near the lateral boundary. The proof for the higher integrability contains the following intermediate stages: we derive a pre-Caccioppoli type estimate near the lateral boundary which implies Caccioppoli's estimate and parabolic Poincaré's inequality. Then we combine these estimates and apply the self-improving property of a reverse Hölder inequality together with capacity estimates.

We say that u is a global quasiminimizer if $u \in L^2(0,T; W^{1,2}(\Omega))$ satisfies (2.1) and the initial and boundary conditions

$$\begin{gathered} u(\bullet, t) - \varphi(\bullet, t) \in W_0^{1,2}(\Omega) \\ \text{and} \\ \frac{1}{h} \int_0^h \int_\Omega |u - \varphi|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0 \text{ as } h \to 0, \end{gathered} \tag{3.1}$$

for a given $\varphi \in W^{1,2}(0,T;W^{1,2}(\Omega))$.

The next lemma is a pre-Caccioppoli type inequality.

Lemma 1 *Let u be a global quasiminizer with the boundary and initial conditions (3.1). Suppose that $0<\rho<\sigma<M$ for some $M>0$, and let $Q_\rho \subset Q_\sigma \subset \mathbb{R}^{n+1}$ be concentric cylinders. Then there exists a positive constant $c = c(n, M, \alpha, \beta, K)$ such that*

$$\begin{aligned}\int_{Q_\rho\cap D} |\nabla u|^2 \,\mathrm{d}z + \operatorname*{ess\,sup}_{t\in\Lambda_\rho\cap(0,T)} \int_{B_\rho\cap\Omega} |u-\varphi|^2 \,\mathrm{d}x \\ \le c\int_{(Q_\sigma\setminus Q_\rho)\cap D} |\nabla u|^2 \,\mathrm{d}z + \frac{c}{(\sigma-\rho)^2}\int_{Q_\sigma\cap D} |u-\varphi|^2 \,\mathrm{d}z \\ + c\int_{Q_\sigma\cap D} \left(|\varphi'|^2 + |\nabla\varphi|^2\right) \mathrm{d}z,\end{aligned}$$

where $D = \Omega \times (0,T)$.

Proof: We may assume that $Q_\rho \cap D \neq \emptyset$ since otherwise the claim is trivial. Let $\chi^h_{0,t_1}(t) \in C_0(0,T)$ be a piecewise linear approximation of a characteristic function such that $\chi^h_{0,t_1}(t) = 1$, when $t \in (h, t_1-h)$, and $\left|(\chi^h_{0,t_1}(t))'\right| \le c/h$. We denote by $\chi^{h,\varepsilon}_{0,t_1}(t)$, u_ε and φ_ε the standard mollifications in the time direction and extend $u(\bullet,t)-\varphi(\bullet,t) \in W^{1,2}_0(\Omega)$ by zero outside Ω. Then we choose a test function

$$\phi_\varepsilon(x,t) = \eta^2(x,t)(u(x,t)-\varphi(x,t))_\varepsilon \chi^{h,\varepsilon}_{0,t_1}(t),\ t_1 \in \Lambda_\rho \cap (0,T),$$

where $\eta \in C^\infty_0(Q_\sigma),\ 0 \le \eta \le 1$, is a cut-off function such that $\eta(x,t) = 1$ in Q_ρ, and

$$(\sigma-\rho)\,|\nabla\eta| + (\sigma-\rho)^2 \left|\frac{\partial\eta}{\partial t}\right| \le c. \tag{3.2}$$

Let us insert this test function into (2.3) and consider the first term. We add and subtract $\varphi_\varepsilon\phi'_\varepsilon$, integrate by parts and apply the initial condition. For almost all t_1, we obtain

$$\begin{aligned}-\int_D u_\varepsilon\phi'_\varepsilon \,\mathrm{d}z \to -\int_{\Omega\times(0,t_1)} |u-\varphi|^2\eta\eta' \,\mathrm{d}z \\ +\frac{1}{2}\int_\Omega |u(x,t_1)-\varphi(x,t_1)|^2\eta^2(x,t_1)\,\mathrm{d}x \\ +\int_{\Omega\times(0,t_1)} \varphi'\eta^2(u-\varphi)\,\mathrm{d}z,\end{aligned}$$

as first $\varepsilon \to 0$ and then $h \to 0$. Next, denote by $\tilde{\phi}_\varepsilon$ the mollification of ϕ_ε and

$\phi = \eta^2(u-\varphi)\chi_{0,t_1}$. For the second term of (2.3), we obtain

$$\begin{aligned}
&\int_{\mathrm{spt}(\tilde{\phi_\varepsilon})} \Big[E(u) - KE(u-\tilde{\phi_\varepsilon})\Big]\,\mathrm{d}z \\
&\quad \to \int_{\mathrm{spt}(\phi)} \big[E(u) - KE(u-\eta^2(u-\varphi))\big]\,\mathrm{d}z \\
&\quad = \int_{\mathrm{spt}(\phi)} E(u)\,\mathrm{d}z - K\int_{\mathrm{spt}(\phi)\setminus Q_\rho} E(u-\eta^2(u-\varphi))\,\mathrm{d}z \\
&\qquad - K\int_{\mathrm{spt}(\phi)\cap Q_\rho} E(\varphi)\,\mathrm{d}z,
\end{aligned} \tag{3.3}$$

as first $\varepsilon \to 0$ and then $h \to 0$. Collecting the facts, we arrive at

$$\begin{aligned}
&\int_{\mathrm{spt}(\phi)} E(u)\,\mathrm{d}z + \frac{1}{2}\int_\Omega |u(x,t_1)-\varphi(x,t_1)|^2\eta^2(x,t_1)\,\mathrm{d}x \\
&\quad \le K\int_{\mathrm{spt}(\phi)\setminus Q_\rho} E(u-\eta^2(u-\varphi))\,\mathrm{d}z + K\int_{\mathrm{spt}(\phi)\cap Q_\rho} E(\varphi)\,\mathrm{d}z \\
&\quad + \int_{\Omega\times(0,t_1)} |\varphi|'\,\eta^2\,|u-\varphi|\,\mathrm{d}z + \int_{\Omega\times(0,t_1)} |u-\varphi|^2\eta\,|\eta'|\,\mathrm{d}z.
\end{aligned} \tag{3.4}$$

Since $\sigma < M$, by Young's inequality there exists a positive constant $c = c(M,\varepsilon)$ such that

$$\begin{aligned}
&\int_{\Omega\times(0,t_1)} |\varphi'|\,\eta^2\,|u-\varphi|\,\mathrm{d}z \\
&\quad \le \varepsilon\int_{\Omega\times(0,t_1)} \eta^2\,|\varphi'|^2\,\mathrm{d}z + \frac{c}{(\sigma-\rho)^2}\int_D \eta^2\,|u-\varphi|^2\,\mathrm{d}z.
\end{aligned}$$

Then we choose $t_1 \in \Lambda_\rho \cap (0,T)$ such that

$$\frac{1}{2}\operatorname*{ess\,sup}_{t\in\Lambda_\rho\cap(0,T)} \int_{B_\rho\cap\Omega} |u-\varphi|^2\,\mathrm{d}x \le \int_{B_\rho\cap\Omega} |u(x,t_1)-\varphi(x,t_1)|^2\,\eta^2(x,t_1)\,\mathrm{d}x.$$

These estimates together with (2.2) and (3.4) imply the result. □

The next lemma is Caccioppoli's inequality. In the proof we use an iteration technique to get rid of the term containing $|\nabla u|^2$ on the right hand side in Lemma 1.

Lemma 2 (Caccioppoli) *Let u be a global quasiminizer with the boundary and initial conditions* (3.1). *Suppose that $0 < \rho < M$ for some $M > 0$, and let $Q_\rho \subset \mathbb{R}^{n+1}$. Then there exists a positive constant $c = c(n,\alpha,\beta,M,K)$ such that*

$$\int_{Q_\rho\cap D} |\nabla u|^2\,\mathrm{d}z \le \frac{c}{\rho^2}\int_{Q_{2\rho}\cap D} |u-\varphi|^2\,\mathrm{d}z + c\int_{Q_{2\rho}\cap D} (|\varphi'|^2 + |\nabla\varphi|^2)\,\mathrm{d}z.$$

Proof: We start with Lemma 1 and denote the constant of the first term on the right by $\widehat{c}$. We add $\widehat{c}\int_{Q_\rho\cap D}|\nabla u|^2\,\mathrm{d}z$ on both sides, divide by $\widehat{c}+1$ and obtain

$$\begin{aligned}\int_{Q_\rho\cap D}&|\nabla u|^2\,\mathrm{d}z\\ &\le \frac{\widehat{c}}{1+\widehat{c}}\int_{Q_\sigma\cap D}|\nabla u|^2\,\mathrm{d}z+\frac{c}{(1+\widehat{c})(\sigma-\rho)^2}\int_{Q_\sigma\cap D}|u-\varphi|^2\,\mathrm{d}z\\ &+\frac{c}{1+\widehat{c}}\int_{Q_\sigma\cap D}\left(|\varphi'|^2+|\nabla\varphi|^2\right)\,\mathrm{d}z.\end{aligned}$$

Then we choose

$$\rho_0=\rho,\ \rho_{i+1}-\rho_i=(1-\lambda)\lambda^i\rho,\ i=0,1,\dots,\ \text{ where } \lambda^2\in(\frac{\widehat{c}}{1+\widehat{c}},1),$$

replace ρ by ρ_i and σ by ρ_{i+1}, and iterate to obtain

$$\begin{aligned}\int_{Q_\rho\cap D}&|\nabla u|^2\,\mathrm{d}z\\ &\le\left(\frac{\widehat{c}}{1+\widehat{c}}\right)^{k+1}\int_{Q_{\rho_{k+1}}\cap D}|\nabla u|^2\,\mathrm{d}z+\\ &\sum_{i=0}^{k}\left(\frac{\widehat{c}}{1+\widehat{c}}\right)^{i}\frac{c}{\widehat{c}+1}\Bigg[\frac{1}{(\rho_{i+1}-\rho_i)^2}\int_{Q_{\rho_{k+1}}\cap D}|u-\varphi|^2\,\mathrm{d}z\\ &\qquad+\int_{Q_{\rho_{k+1}}\cap D}\left(|\varphi'|^2+|\nabla\varphi|^2\right)\,\mathrm{d}z\Bigg].\end{aligned}$$

Letting $k\to\infty$, we obtain the result. □

We have not considered the regularity of the lateral boundary so far. Examples show that inward cusps are troublesome and that the boundary must satisfy some regularity conditions. Here we assume that the complement of a domain satisfies a uniform capacity density condition.

Next we recall how to calculate capacities in terms of quasicontinuous representatives. Let $1<p<\infty$. We call $u\in W^{1,p}(\Omega)$ *p-quasicontinuous* if for each $\varepsilon>0$ there exists an open set $V\subset\mathbb{R}^n$ such that

$$\operatorname{cap}_p(V,\mathbb{R}^n)\le\varepsilon$$

and

$$u|_{\Omega\setminus V}\text{ is continuous.}$$

The p-quasicontinuous functions are intimately related to the Sobolev space $W^{1,p}(\Omega)$. It is known, for example, that if $u\in W^{1,p}(\Omega)$, then u has a p-quasicontinuous representative.

Now, the variational p-capacity of a set $E \subset B_\rho(x) \subset \mathbb{R}^n$ can be written in the form

$$\operatorname{cap}_p(E, B_{2\rho}) = \inf_u \int_{B_{2\rho}} |\nabla u|^p \, \mathrm{d}x, \tag{3.5}$$

where $u \in W_0^{1,p}(B_{2\rho})$ is p-quasicontinuous and $u \geq 1$ in E except on a set of p-capacity zero.

For a ball we obtain that there exists a positive constant $c = c(n,p)$ such that

$$\operatorname{cap}_p(\overline{B}_\rho, B_{2\rho}) = c\rho^{n-p}.$$

For the basic properties of the capacity we refer to Chapter 2 of [HKM93].

Next we introduce a capacity density condition which we later impose on the complement of a domain. For the higher integrability results this condition is essentially sharp as pointed out in Remark 3.3 of [KK94] in the elliptic case.

Definition 3 Let $1 < p < \infty$. A set $E \subset \mathbb{R}^n$ is *uniformly p-thick* if there exist constants $\mu, \rho_0 > 0$ such that

$$\operatorname{cap}_p(E \cap \overline{B}_\rho(x), B_{2\rho}(x)) \geq \mu \operatorname{cap}_p(\overline{B}_\rho(x), B_{2\rho}(x))$$

for all $x \in E$ and for all $0 < \rho < \rho_0$.

If we replace the capacities with the Lebesgue measure, we obtain a measure density condition. A set E satisfying the measure density condition is uniformly p-thick for all $p > 1$. If $p > n$, then every nonempty set is uniformly p-thick.

The following lemma is sometimes useful when applying the capacity density condition. The result is based on capacity estimates Theorem 2.2 and Lemma 2.16 of [HKM93], but details are left for the reader.

Lemma 4 *Let Ω be a bounded open set, and suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly p-thick. Choose $y \in \Omega$ such that $B_{\frac{4}{3}\rho}(y) \setminus \Omega \neq \emptyset$. Then there exists a positive constant $\tilde{\mu} = \tilde{\mu}(\mu, \rho_0, n, p)$ such that*

$$\operatorname{cap}_p(\overline{B}_{2\rho}(y) \setminus \Omega, B_{4\rho}(y)) \geq \tilde{\mu} \operatorname{cap}_p(\overline{B}_{2\rho}(y), B_{4\rho}(y)).$$

A uniformly p-thick domain satisfies a deep self-improving property. This result is due to Lewis, see [Lew88]. See also page 52 of [Mik96] and [Anc86].

Theorem 5 *Let $1 < p \leq n$. If a set E is uniformly p-thick, then there exists q such that $1 < q < p$ for which E is uniformly q-thick.*

A uniformly q-thick set is also uniformly p-thick for all $p \geq q$. This is a simple consequence of Hölder's inequality.

Next we establish a well-known version of the Sobolev-Poincaré inequality. In this version the estimate depends on the capacity of a set in which the function equals zero. Later we use this estimate together with the boundary regularity condition. For a proof, see for example Lemma 3.1 of [KK94] or Lemma 8.11 of [Mik96].

Lemma 6 *Suppose that* $u \in W^{1,q}(B_{2\rho})$ *is* q*-quasicontinuous, where* $q \in [2^*, 2]$, $2^* = 2n/(n+2)$, $n \geq 2$. *Denote* $N_{B_\rho}(u) = \{ x \in \overline{B}_\rho : u(x) = 0\}$. *Then there exists a positive constant* $c = c(n)$ *such that*

$$\left(\fint_{B_{2\rho}} |u|^2 \, \mathrm{d}x \right)^{1/2} \leq \left(\frac{c}{\mathrm{cap}_q(N_{B_\rho}(u), B_{2\rho})} \int_{B_{2\rho}} |\nabla u|^q \, \mathrm{d}x \right)^{1/q}.$$

Next we prove parabolic Poincaré's inequality near the lateral boundary. The proof relies on the previous lemma and the pre-Caccioppoli type inequality.

Lemma 7 (parabolic Poincaré) *Let* u *be a global quasiminizer with the boundary and initial conditions* (3.1). *Let* $Q_\rho = Q_\rho(x_0, t_0) \subset \mathbb{R}^{n+1}$, *suppose that* $\mathbb{R}^n \setminus \Omega$ *is uniformly* 2*-thick and that* $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$. *Suppose that* $\rho < M$ *for some* $M > 0$. *Then there exists a positive constant* $c = c(n, M, \mu, \rho_0, \alpha, \beta, K)$ *such that*

$$\begin{aligned} &\operatorname*{ess\,sup}_{t \in \Lambda_{2\rho} \cap (0,T)} \int_{B_{2\rho} \cap \Omega} |u - \varphi|^2 \, \mathrm{d}x \\ &\qquad \leq c \int_{Q_{4\rho} \cap D} |\nabla u|^2 \, \mathrm{d}z + c \int_{Q_{4\rho} \cap D} \left(|\varphi'|^2 + |\nabla \varphi|^2 \right) \mathrm{d}z. \end{aligned}$$

Proof: By Lemma 1, we conclude that

$$\begin{aligned} &\operatorname*{ess\,sup}_{t \in \Lambda_{2\rho} \cap (0,T)} \int_{B_{2\rho} \cap \Omega} |u - \varphi|^2 \, \mathrm{d}x \\ &\qquad \leq c \int_{Q_{4\rho} \cap D} |\nabla u|^2 \, \mathrm{d}z + \frac{c}{\rho^2} \int_{Q_{4\rho} \cap D} |u - \varphi|^2 \, \mathrm{d}z \\ &\qquad + c \int_{Q_{4\rho} \cap D} \left(|\varphi'|^2 + |\nabla \varphi|^2 \right) \mathrm{d}z. \end{aligned} \tag{3.6}$$

We extend $u(\bullet, t) - \varphi(\bullet, t) \in W_0^{1,2}(\Omega)$ by zero outside Ω. Then by Lemma 4 and the capacity of a ball, we obtain

$$\mathrm{cap}_2(N_{B_{2\rho}}(u - \varphi), B_{4\rho}(x_0)) \geq \tilde{\mu} \, \mathrm{cap}_2(\overline{B}_{2\rho}(x_0), B_{4\rho}(x_0)) = c\rho^{n-2}.$$

We estimate the second term on the right side of (3.6) by using Lemma 6 with $q = 2$ and the previous capacity estimate. We obtain

$$\begin{aligned} &\frac{c}{\rho^2} \int_{Q_{4\rho} \cap D} |u - \varphi|^2 \, \mathrm{d}z \\ &\leq \int_{\Lambda_{4\rho} \cap (0,T)} \frac{c\rho^n}{\rho^2 \, \mathrm{cap}_2(N_{B_{2\rho}}(u - \varphi), B_{4\rho})} \int_{B_{2\rho}} |\nabla (u - \varphi)|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\leq c \int_{Q_{4\rho} \cap D} |\nabla (u - \varphi)|^2 \, \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

and the result follows. $\square$

4 Reverse Hölder inequalities near the lateral boundary

In this section we prove that the gradient of a quasiminimizer is integrable to a higher power than assumed a priori. First we derive a reverse Hölder inequality and then apply the self-improving property.

Lemma 8 (Giaquinta-Modica type inequality) *Let u be a global quasiminizer with the boundary and initial conditions* (3.1). *Let $Q_\rho = Q_\rho(x_0, t_0)$, suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly 2-thick and that $B_{\frac{4}{3}\rho}(x_0) \setminus \Omega \neq \emptyset$. Suppose $\rho < M$ for some $M > 0$ and choose $\varepsilon > 0$. Then there exists a positive constant $c = c(n, M, \delta, \mu, \rho_0, \alpha, \beta, K, \varepsilon)$ and $q < 2$ such that*

$$\begin{aligned}
&\int_{Q_{2\rho}\cap D} |\nabla u|^2 \,\mathrm{d}z \\
&\qquad \le \frac{\varepsilon}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} |\nabla u|^2 \,\mathrm{d}z + \left(\frac{c}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} |\nabla u|^q \,\mathrm{d}z\right)^{2/q} \\
&\qquad + \frac{c}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} \left(|\varphi'|^2 + |\nabla\varphi|^2\right) \mathrm{d}z.
\end{aligned}$$

Proof: Again, we extend $u(\bullet, t) - \varphi(\bullet, t) \in W_0^{1,2}(\Omega)$ by zero outside Ω. Then we use Lemma 2 and divide the first term on the right into two parts

$$\begin{aligned}
&\frac{c}{\rho^2 |Q_{2\rho}|}\int_{Q_{2\rho}\cap D} |u-\varphi|^2 \,\mathrm{d}z \\
&\le \frac{c}{\rho^4}\int_{\Lambda_{2\rho}\cap(0,T)} \left(\fint_{B_{2\rho}} |u-\varphi|^2 \,\mathrm{d}x\right)^{1-q/2} \left(\fint_{B_{2\rho}} |u-\varphi|^2 \,\mathrm{d}x\right)^{q/2} \mathrm{d}t,
\end{aligned} \tag{4.1}$$

where $q \in [2n/(n+2), 2)$ is fixed later. Then Lemma 6 and Lemma 7 imply

$$\begin{aligned}
&\frac{1}{\rho^2 |Q_{2\rho}|}\int_{Q_{2\rho}\cap D} |u-\varphi|^2 \,\mathrm{d}z \\
&\le \frac{c}{\rho^2}\Bigg\{\frac{\rho^2}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} |\nabla(u-\varphi)|^2 \,\mathrm{d}z \\
&\qquad\qquad + \frac{\rho^2}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} \left(|\nabla\varphi|^2 + |\varphi'|^2\right) \mathrm{d}z\Bigg\}^{1-q/2} \\
&\cdot \frac{1}{\rho^2}\int_{\Lambda_{2\rho}\cap(0,T)} \frac{1}{\mathrm{cap}_q(N_{B_{2\rho}}(u-\varphi), B_{4\rho})}\int_{B_{4\rho}\cap D} |\nabla(u-\varphi)|^q \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.2}$$

Next we would like to use the uniform capacity density condition, but this is not possible straight away since $q < 2$, and we assumed that the complement of a domain is uniformly 2-thick. However, the density condition satisfies the self-improving property as stated in Theorem 5. This together with Lemma 4 implies

$$\operatorname{cap}_q(N_{B_{2\rho}}(u-\varphi), B_{4\rho}) \geq \tilde{\mu} \operatorname{cap}_q(\overline{B}_{2\rho}, B_{4\rho}) = c\rho^{n-q}$$

for large enough $q < 2$. We apply this and Young's inequality in (4.2) to obtain

$$\begin{aligned} &\frac{1}{\rho^2 |Q_{2\rho}|} \int_{Q_{2\rho}\cap D} |u-\varphi|^2 \, \mathrm{d}z \\ &\quad \leq \frac{\varepsilon}{|Q_{4\rho}|} \int_{Q_{4\rho}\cap D} |\nabla (u-\varphi)|^2 \, \mathrm{d}z + \frac{\varepsilon}{|Q_{4\rho}|} \int_{Q_{4\rho}\cap D} \left(|\nabla \varphi|^2 + |\varphi'|^2 \right) \mathrm{d}z \\ &\quad + \left(\frac{c}{|Q_{4\rho}|} \int_{Q_{4\rho}\cap D} |\nabla (u-\varphi)|^q \, \mathrm{d}z \right)^{2/q} . \end{aligned}$$

Lemma 8 follows now easily. □

Now we have all the tools to prove the higher integrability of the gradient of a quasiminimizer near the lateral boundary. The next theorem is one of our main results.

Theorem 9 *Let $u \in L^2(0,T;W^{1,2}(\Omega))$ be a global quasiminimizer, and suppose that $\varphi \in W^{1,2+\delta}(0,T;W^{1,2+\delta}(\Omega))$ is a boundary function such that*

$$u(\bullet,t) - \varphi(\bullet,t) \in W_0^{1,2}(\Omega) \text{ and } \frac{1}{h}\int_0^h \int_\Omega |u-\varphi|^2 \, \mathrm{d}x \, \mathrm{d}t \to 0 \text{ as } h \to 0.$$

Suppose that $\mathbb{R}^n \setminus \Omega$ is uniformly 2-thick, let $Q_\rho \subset \mathbb{R}^{n+1}$, and suppose that $\rho < M$ for some $M > 0$. Then there exist positive constants $\varepsilon_0 = \varepsilon_0(n, M, \delta, \mu, \rho_0, \alpha, \beta, K)$, $c = c(n, M, \delta, \mu, \rho_0, \alpha, \beta, K)$ such that for all $0 \leq \varepsilon < \varepsilon_0$, we have

$$\begin{aligned} &\left(\frac{1}{|Q_\rho|} \int_{Q_\rho \cap D} |\nabla u|^{2+\varepsilon} \, \mathrm{d}z \right)^{1/(2+\varepsilon)} \\ &\qquad \leq \left(\frac{c}{|Q_{4\rho}|} \int_{Q_{4\rho}\cap D} |\nabla u|^2 \, \mathrm{d}z \right)^{1/2} \\ &\qquad + \left(\frac{c}{|Q_{4\rho}|} \int_{Q_{4\rho}\cap D} |\nabla \varphi|^{2+\varepsilon} + |\varphi'|^{2+\varepsilon} \, \mathrm{d}z \right)^{1/(2+\varepsilon)} , \end{aligned}$$

where $D = \Omega \times (0,T)$.

Proof: We use the well-known Giaquinta-Modica lemma, see [GM79] or for example page 122 of [Gia83] or page 187 of [CW98]. See also [Geh73]. The Giaquinta-Modica lemma is formulated in the elliptic setting, but it extends to the parabolic case as pointed out in [GS82]. Later we prove a modification of this lemma, so for the proof we refer to Theorem 15.

We define

$$g(x,t) = \begin{cases} |\nabla u(x,t)|^q, & (x,t) \in \Omega \times (0,T), \\ 0, & \text{otherwise} , \end{cases}$$

$$f(x,t) = \begin{cases} |\nabla \varphi(x,t)|^q + |\varphi'(x,t)|^q, & (x,t) \in \Omega \times (0,T), \\ 0, & \text{otherwise} . \end{cases}$$

and $p = 2/q$. If $\Omega \setminus B_{\frac{4}{3}\rho} \neq \emptyset$, Lemma 8 holds and if $\Omega \setminus B_{\frac{4}{3}\rho} = \emptyset$, a modification of the local result, see [Wie87], holds. The conditions of the Giaquinta-Modica lemma are satisfied. □

5 Estimates near the initial boundary

In this section we study the higher integrability near the initial boundary $t = 0$. Here the regularity of the lateral boundary does not play a role, and weaker assumptions are used.

We start by deriving Caccioppoli type inequalities and parabolic Poincaré's inequality. These estimates are applied in the next section where we prove a reverse Hölder inequality near the initial boundary, and then show that it satisfies the self-improving property.

Let us denote $2^* = 2n/(n+2)$. We say that u is a quasiminimizer for an initial value problem if $u \in L^2(0,T; W^{1,2}_{\mathrm{loc}}(\Omega))$ satisfies (2.1) and the given initial condition

$$\frac{1}{h}\int_0^h \int_C |u(x,t) - \varphi(x)|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ as } h \to 0, \tag{5.1}$$

for all compact $C \subset \Omega$ and for a given $\varphi \in W^{1,2^*}(\Omega)$. In the proof we apply the weighted mean

$$u^\eta_\sigma(t) = \int_{B_\sigma} \eta^2(x,t) u(x,t)\,\mathrm{d}x \Big/ \int_{B_\sigma} \eta^2(x,t)\,\mathrm{d}x$$

instead of a standard mean $u_\sigma(t)$. The weighted mean is applied in the local case for example in [GS82] or [Cho93]. The weighted mean should approximate the standard mean, and therefore the weight η is defined to be a cut-off function such that $\eta \in C_0^\infty(Q_\sigma)$, $0 \le \eta \le 1$, $\eta = 1$ in Q_ρ, where $0 < \rho < \sigma < \infty$, and

$$\sup_{x \in B_\sigma} \eta(x,t) \le \tilde{c} \fint_{B_\sigma} \eta(x,t)\,\mathrm{d}x,\ \ t \in \Lambda_\sigma, \tag{5.2}$$

where $\Lambda_\sigma = \Lambda_\sigma(t_0) = (t_0 - \frac{1}{2}\sigma^2,\, t_0 + \frac{1}{2}\sigma^2)$.

The following lemma gives a detailed description of approximation properties of the weighted mean. The first inequality in the lemma is obtained easily by adding and subtracting $u_\sigma^\eta(t)$. The latter inequality is obtained by adding and subtracting $u_\sigma(t)$ and using Hölder's inequality together with (5.2). We omit the details.

Lemma 10 *Let $u(\bullet, t) \in L^2(\Omega)$ and $\eta,\ u_\sigma^\eta(t),\ u_\sigma(t)$ be as above. Then there exists a positive constant $c = c(p, \tilde{c})$ such that*

$$\int_{B_\sigma} |u - u_\sigma(t)|^2 \,\mathrm{d}x \le c \int_{B_\sigma} |u - u_\sigma^\eta(t)|^2 \,\mathrm{d}x \le c^2 \int_{B_\sigma} |u - u_\sigma(t)|^2 \,\mathrm{d}x.$$

Here $\tilde{c}$ is the constant in (5.2).

From now on we assume that the cut-off function η also satisfies

$$\left|\frac{\partial \eta}{\partial t}\right| + |\nabla \eta|^2 \le \frac{c}{(\sigma - \rho)^2}.$$

Lemma 11 *Let u be a quasiminimizer to an initial value problem with the initial condition* (5.1). *Let $0 < \rho < \sigma < \infty$, and let $Q_\rho \subset Q_\sigma = Q_\sigma(x_0, t_0) \subset \mathbb{R}^{n+1}$ be concentric cylinders such that $\operatorname{dist}\{B_\sigma(x_0), \partial\Omega\} > a > 0$ and $0 \in \Lambda_\rho(t_0)$. Then there exists a positive constant $c = c(n, \alpha, \beta, \tilde{c}, K, a)$ such that*

$$\begin{aligned} &\int_{Q_\rho \cap D} |\nabla u|^2 \,\mathrm{d}z + \operatorname*{ess\,sup}_{t \in \Lambda_\rho \cap (0,T)} \int_{B_\rho} |u - u_\sigma^\eta(t)|^2 \,\mathrm{d}x \\ &\le c \int_{(Q_\sigma \setminus Q_\rho) \cap D} |\nabla u|^2 \,\mathrm{d}z + \frac{c}{(\sigma - \rho)^2} \int_{Q_\sigma \cap D} |u - u_\sigma^\eta(t)|^2 \,\mathrm{d}z \\ &+ c \left(\int_{B_\sigma} |\nabla \varphi|^{2^*} \,\mathrm{d}x \right)^{2/2^*}. \end{aligned}$$

Here $\tilde{c}$ is the constant in (5.2) *and $2^* = 2n/(n+2)$.*

Proof: We may assume that $Q_\rho \cap D \neq \emptyset$ since otherwise the claim is trivial. We choose a test function

$$\phi_\varepsilon(x,t) = \eta^2(x,t)(u_\varepsilon(x,t) - u_{\sigma,\varepsilon}^\eta(t))\chi_{0,t_1}^{h,\varepsilon}(t),\ t_1 \in \Lambda_\rho \cap (0,T),$$

where $u_{\sigma,\varepsilon}^\eta(t)$ is the weighted average of $u_\varepsilon(x,t)$ and otherwise the notation is the same as in Lemma 1. Now, let us consider the first term of (2.3). We insert the test function, add and subtract $u_{\sigma,\varepsilon}^\eta(t)\phi_\varepsilon'$ and have

$$-\int_{\mathbb{R}^{n+1}} u_\varepsilon \phi_\varepsilon' \,\mathrm{d}z = -\int_{\mathbb{R}^{n+1}} (u_\varepsilon - u_{\sigma,\varepsilon}^\eta(t))\phi_\varepsilon' \,\mathrm{d}z - \int_{\mathbb{R}^{n+1}} u_{\sigma,\varepsilon}^\eta(t)\phi_\varepsilon' \,\mathrm{d}z.$$

Integrating by parts and using the definition of $u^\eta_{\sigma,\varepsilon}(t)$, we notice that the last term vanishes

$$
\begin{aligned}
&-\int_{\mathbb{R}^{n+1}} u^\eta_{\sigma,\varepsilon}(t)\phi'_\varepsilon \,\mathrm{d}z \\
&= \int_{-\infty}^{\infty} \chi^{h,\varepsilon}_{0,t_1}(t) \left[\int_{B_\sigma} u_\varepsilon \eta^2 \,\mathrm{d}x - \frac{\int_{B_\sigma} \eta^2 \,\mathrm{d}x \int_{B_\sigma} \eta^2 u_\varepsilon \,\mathrm{d}x}{\int_{B_\sigma} \eta^2 \,\mathrm{d}x} \right] (u^\eta_{2\rho,\varepsilon}(t))' \,\mathrm{d}t = 0.
\end{aligned}
$$

Then we integrate the rest by parts, take limits, apply the initial condition and conclude that

$$
\begin{aligned}
-\int_{\mathbb{R}^{n+1}} u_\varepsilon \phi'_\varepsilon \,\mathrm{d}z \to &-\int_{\Omega\times(0,t_1)} |u - u^\eta_\sigma(t)|^2 \eta\eta' \,\mathrm{d}z \\
&+ \frac{1}{2}\int_{B_\sigma} |u(x,t_1) - u^\eta_\sigma(t_1)|^2 \eta^2(x,t_1) \,\mathrm{d}x \qquad (5.3)\\
&- \frac{1}{2}\int_{B_\sigma} |\varphi - \varphi^\eta_\sigma|^2 \eta^2(x,0) \,\mathrm{d}x,
\end{aligned}
$$

as first $\varepsilon \to 0$ and then $h \to 0$. Next we apply Lemma 10 together with Poincaré's inequality and conclude that

$$
\int_{B_\sigma} |\varphi - \varphi^\eta_\sigma|^2 \,\mathrm{d}x \le c \left(\int_{B_\sigma} |\nabla\varphi| \,\mathrm{d}x \right)^{2/2^*}.
$$

The rest of the proof is almost similar to the proof of Lemma 1 from (3.3) onwards, and we omit the details. □

Next we derive Caccioppoli's inequality by using the hole filling iteration.

Lemma 12 (Caccioppoli) *Let u be a quasiminimizer to an initial value problem with the initial condition* (5.1). *Let $0 < \rho < \infty$, and let $Q_\rho = Q_\rho(x_0,t_0) \subset \mathbb{R}^{n+1}$ such that $\operatorname{dist}\{B_{2\rho}(x_0), \partial\Omega\} > a > 0$ and $0 \in \Lambda_\rho(t_0)$. Then there exists a positive constant $c = c(n,\alpha,\beta,\tilde{c},K,a)$ such that*

$$
\begin{aligned}
\int_{Q_\rho\cap D} |\nabla u|^2 \,\mathrm{d}z \le &\frac{c}{\rho^2} \sup_{\widehat{\rho}\in[\rho,2\rho]} \int_{Q_{\widehat{\rho}}\cap D} |u - u_{\widehat{\rho}}(t)|^2 \,\mathrm{d}z \\
&+ c\left(\int_{B_{2\rho}} |\nabla\varphi|^{2^*} \,\mathrm{d}x \right)^{2/2^*}.
\end{aligned}
$$

Here $\tilde{c}$ is the constant in (5.2) *and $2^* = 2n/(n+2)$.*

Proof: We start with Lemma 11, denote the constant of the first term on the right by $\widehat{c}$, add $\widehat{c}\int_{Q_\rho\cap D} |\nabla u|^2 \,\mathrm{d}z$ on both sides, divide by $\widehat{c}+1$, apply Lemma 10 and obtain

$$\begin{aligned}
&\int_{Q_\rho\cap D} |\nabla u|^2 \,\mathrm{d}z \\
&\quad \le \frac{\widehat{c}}{\widehat{c}+1}\int_{Q_\sigma\cap D} |\nabla u|^2 \,\mathrm{d}z + \frac{c}{(\widehat{c}+1)(\sigma-\rho)^2}\int_{Q_\sigma\cap D} |u-u_\sigma(t)|^2 \,\mathrm{d}z \\
&\quad + \frac{c}{\widehat{c}+1}\left(\int_{B_\sigma} |\nabla\varphi|^{2^*}\,\mathrm{d}x\right)^{2/2^*}.
\end{aligned}$$

Then we choose ρ_i similarly as in Lemma 2 replace ρ by ρ_i and σ by ρ_{i+1} and iterate to obtain the result. □

The next estimate is a parabolic Poincaré type inequality.

Lemma 13 (parabolic Poincaré) *Let u be a quasiminimizer to an initial value problem with the initial condition* (5.1). *Let $0<\rho<\infty$, and let $Q_\rho = Q_\rho(x_0,t_0)\subset \mathbb{R}^{n+1}$ such that $\operatorname{dist}\{B_{2\rho}(x_0),\,\partial\Omega\} > a > 0$ and $0\in\Lambda_\rho(t_0)$. Then there exists a positive constant $c = c(n,\alpha,\beta,\tilde{c},K,a)$ such that*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in\Lambda_\rho\cap(0,T)} \fint_{B_\rho} |u-u^\eta_{2\rho}(t)|^2 \,\mathrm{d}x \\
&\le c\rho^2\left(\frac{1}{|Q_{2\rho}|}\int_{Q_{2\rho}\cap D} |\nabla u|^2 \,\mathrm{d}z + \left(\fint_{B_{2\rho}} |\nabla\varphi|^{2^*}\,\mathrm{d}x\right)^{2/2^*}\right).
\end{aligned}$$

Here $\tilde{c}$ is the constant in (5.2) *and $2^* = 2n/(n+2)$.*

Proof: By Lemma 11, we have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in\Lambda_\rho\cap(0,T)} \int_{B_\rho} |u-u^\eta_{\sigma}(t)|^2 \,\mathrm{d}x \\
&\qquad \le c\int_{Q_{2\rho}\cap D} |\nabla u|^2 \,\mathrm{d}z + \frac{c}{\rho^2}\int_{Q_{2\rho}\cap D} |u-u^\eta_{2\rho}(t)|^2 \,\mathrm{d}z \\
&\qquad + c\left(\int_{B_{2\rho}} |\nabla\varphi|^{2^*}\,\mathrm{d}x\right)^{2/2^*}.
\end{aligned}$$

Then Lemma 10 and Poincaré's inequality imply

$$\frac{c}{\rho^2}\int_{Q_{2\rho}\cap D} |u-u^\eta_{2\rho}(t)|^2 \,\mathrm{d}z \le c\int_{Q_{2\rho}\cap D} |\nabla u|^2\,\mathrm{d}z.$$

The result follows by combining these estimates. □

Now we prove a reverse Hölder inequality for the gradient of a quasiminimizer.

Lemma 14 (Giaquinta-Modica type inequality) *Let u be a quasiminimizer to an initial value problem with the initial condition (5.1). Let $0<\rho<\infty$ and let $Q_\rho = Q_\rho(x_0,t_0)\subset\mathbb{R}^{n+1}$ such that $\operatorname{dist}\{B_{2\rho}(x_0),\,\partial\Omega\}>a>0$ and $0\in\Lambda_\rho(t_0)$. Choose $\varepsilon>0$. Then there exists a positive constant $c=c(n,\alpha,\beta,\tilde{c},K,\varepsilon,a)$ such that*

$$\begin{aligned}\frac{1}{|Q_\rho|}\int_{Q_\rho\cap D}|\nabla u|^2\,\mathrm{d}z \\ &\le \frac{\varepsilon}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D}|\nabla u|^2\,\mathrm{d}z + \left(\frac{c}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D}|\nabla u|^{2^*}\,\mathrm{d}z\right)^{2/2^*} \\ &+ c\left(\fint_{B_{4\rho}}|\nabla\varphi|^{2^*}\,\mathrm{d}x\right)^{2/2^*},\end{aligned}$$

where $2^*=2n/(n+2)$ and $\tilde{c}$ is the constant in (5.2).

Proof: We start with Lemma 12 and choose $\rho_0\in[\rho,2\rho]$ such that

$$\int_{Q_{\rho_0}\cap D}|u-u_{\rho_0}(t)|^2\,\mathrm{d}z = \sup_{\widehat{\rho}\in[\rho,2\rho]}\int_{Q_{\widehat{\rho}}\cap D}|u-u_{\widehat{\rho}}(t)|^2\,\mathrm{d}z, \tag{5.4}$$

and a cut-off function $\eta\in C_0^\infty(Q_{2\rho_0})$, $0\le\eta\le1$, $\eta=1$ in Q_{ρ_0}, satisfying (5.2). By Lemma 10 (lemma is valid also for $u^\eta_{2\rho_0}(t)$), we have

$$\int_{Q_{\rho_0}\cap D}|u-u_{\rho_0}(t)|^2\,\mathrm{d}z \le c\int_{Q_{\rho_0}\cap D}|u-u^\eta_{2\rho_0}(t)|^2\,\mathrm{d}z, \tag{5.5}$$

and thus

$$\begin{aligned}\int_{Q_\rho\cap D}|\nabla u|^2\,\mathrm{d}z &\le \frac{c}{\rho^2}\int_{Q_{\rho_0}\cap D}|u-u^\eta_{2\rho_0}(t)|^2\,\mathrm{d}z \\ &\quad + c\left(\int_{B_{2\rho}}|\nabla\varphi(x)|^{2^*}\,\mathrm{d}x\right)^{2/2^*}.\end{aligned}$$

Then we divide the first term on the right into two parts, estimate the first part by essential supremum and apply Lemma 10 to the latter. We obtain

$$\begin{aligned}\frac{1}{\rho^2|Q_{\rho_0}|}\int_{Q_{\rho_0}\cap D}|u-u^\eta_{2\rho_0}(t)|^2\,\mathrm{d}z \\ &\le \frac{c}{\rho^2}\operatorname*{ess\,sup}_{t\in\Lambda_{\rho_0}\cap(0,T)}\left(\fint_{B_{\rho_0}}|u-u^\eta_{2\rho_0}(t)|^2\,\mathrm{d}x\right)^{1-2^*/2} \\ &\quad \frac{1}{\rho_0^2}\int_{\Lambda_{\rho_0}\cap(0,T)}\left(\fint_{B_{2\rho_0}}|u-u_{2\rho_0}(t)|^2\,\mathrm{d}x\right)^{2^*/2}\,\mathrm{d}t.\end{aligned}$$

Then we apply Lemma 13 to the first part, Poincaré's inequality to the latter part, and have

$$\frac{1}{\rho^2\,|Q_{\rho_0}|}\int_{Q_{\rho_0}\cap D}|u-u^{\eta}_{2\rho_0}(t)|^2\,\mathrm{d}z$$

$$\leq c\left(\frac{1}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D}|\nabla u|^2\,\mathrm{d}z+\left(\fint_{B_{4\rho}}|\nabla\varphi|^{2^*}\,\mathrm{d}x\right)^{2/2^*}\right)^{1-2^*/2}$$

$$\cdot\frac{1}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D}|\nabla u|^{2^*}\,\mathrm{d}z$$

Finally, the result is obtained by using Young's inequality. □

6 Reverse Hölder inequalities near the initial boundary

The previous lemma makes sense if the gradient of the initial value function is integrable to the power $2n/(n+2)$ instead of 2. Next we show that the reverse Hölder inequality has the self-improving property also in this setting.

Theorem 15 *Let $D=\Omega\times(0,T)$, $p>1$, $q=pn/(n+2)$ and $\gamma>0$. Choose $\tilde{\varepsilon}>0$ and denote $\delta_{\Lambda_{4\rho}(\tilde{t}_0)}=1$ if $0\in\Lambda_{4\rho}(\tilde{t}_0)$ and $\delta_{\Lambda_{4\rho}(\tilde{t}_0)}=0$ otherwise. Suppose that $g\geq 0$, $g\in L^p(Q_{4\rho}(\tilde{x}_0,\tilde{t}_0)\cap D)$, $f\geq 0$, $f\in L^{q+\gamma}(Q_{4\rho}(\tilde{x}_0,\tilde{t}_0)\cap D)$ and suppose that there exists a positive constant $b=b(\tilde{\varepsilon})$ such that*

$$\frac{1}{|Q_\rho|}\int_{Q_\rho\cap D}g^p\,\mathrm{d}z\leq\frac{\tilde{\varepsilon}}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D}g^p\,\mathrm{d}z+b\left(\frac{1}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D}g^q\,\mathrm{d}z\right)^{p/q}+b\delta_{\Lambda_{4\rho}(\tilde{t}_0)}\left(\fint_{B_{4\rho}}f^q\,\mathrm{d}x\right)^{p/q},\tag{6.1}$$

for all bounded cylinders $Q_{4\rho}=Q_{4\rho}(\tilde{x}_0,\tilde{t}_0)\subset\mathbb{R}^{n+1}$ such that $\operatorname{dist}\{B_{4\rho}(\tilde{x}_0),\partial\Omega\}>a>0$. Then there exist positive constants $\varepsilon_0=\varepsilon_0(b,\gamma,n,p,a)$ and $c=c(b,\gamma,n,p,a)$ such that for all $0\leq\varepsilon<\varepsilon_0$, we have

$$\left(\frac{1}{|Q_R|}\int_{Q_R\cap D}g^{p+\varepsilon}\,\mathrm{d}z\right)^{1/(p+\varepsilon)}\leq c\left(\frac{1}{|Q_{4R}|}\int_{Q_{4R}\cap D}g^p\,\mathrm{d}z\right)^{1/p}+c\delta_{\Lambda_{4R}}\left(\fint_{B_{4R}}f^{q+\varepsilon}\,\mathrm{d}x\right)^{1/q+\varepsilon},$$

for all bounded cylinders $Q_{4R} = Q_{4R}(x_0, t_0) \subset \mathbb{R}^{n+1}$ *such that* $\operatorname{dist}\{B_{4R}(x_0), \partial\Omega\} > a > 0$.

Proof: The proof consists of several steps. First we divide the space-time cylinder into smaller Whitney-type cylinders. In each Whitney-type cylinder we are able to derive estimates with constants that are independent of the place. Then we divide the space-time cylinder into a good set and a bad set. In the good set the function g^p is bounded, and in the bad set we can estimate the average of the function. The Calderón-Zygmund decomposition is usually applied for this, but here we use a different strategy which seems to work better in the parabolic case also with more general growth conditions. Finally, we obtain the higher integrability by using Fubini's theorem.

We denote $Q_0 = Q_{4R}(z_0) = Q_{4R}(x_0, t_0)$ and divide Q_0 into the Whitney-type cylinders (see for example page 15 of [Ste93])

$$Q_i = Q_{r_i}(z_i), \; i = 1, 2, \dots,$$

where r_i is comparable to the parabolic distance of Q_i to ∂Q_0. The *parabolic distance* is defined to be

$$\operatorname{dist}_p \{E, F\} = \inf_{E,F} \left\{ |x - \overline{x}| + |t - \overline{t}|^{1/2} \right\},$$

where the infimum is taken taken over the sets E and F, that is, $(x, t) \in E$, $(\overline{x}, \overline{t}) \in F$. In addition, the cylinders Q_i are of bounded overlap (meaning that every z belongs at the most to a fixed finite number of cylinders), and

$$Q_{5r_i} \subset Q_0.$$

We choose

$$\lambda_0 = \left(\frac{1}{|Q_0|} \int_{Q_0 \cap D} g^p \, \mathrm{d}z \right)^{1/p} \quad \text{and } \lambda > \lambda_0.$$

For $(x, t) \in Q_0 \cap D$, we define

$$h(x, t) = \frac{1}{\widehat{c}\,|Q_0|^{1/p}} \min\{|Q_i|^{1/p} : (x, t) \in Q_i\} g(x, t),$$

where $\widehat{c} \geq 1$ is fixed later. Suppose that we have $(\widehat{x}, \widehat{t}) \in Q_i$ such that $h(\widehat{x}, \widehat{t}) > \lambda$, and define

$$\alpha = \frac{|Q_0|}{|Q_i|}.$$

Then for r, $r_i/20 \leq r \leq r_i$, we have

$$\frac{1}{|Q_r|} \int_{Q_r \cap D} g^p \, \mathrm{d}z \leq \frac{c\,|Q_0|}{|Q_i|} \frac{1}{|Q_0|} \int_{Q_0 \cap D} g^p \, \mathrm{d}z \leq \widehat{c}^p \alpha \lambda^p,$$

where $\widehat{c}$ is chosen to be large enough. By Lebesgue's theorem

$$\lim_{r\to 0}\frac{1}{|Q_r(\widehat{x},\widehat{t})|}\int_{Q_r(\widehat{x},\widehat{t})\cap D} g^p\,\mathrm{d}z = g^p(\widehat{x},\widehat{t}) > \widehat{c}^p\alpha\lambda^p$$

for almost all $(\widehat{x},\widehat{t})$. By these two estimates and continuity of the integral there exists $\rho,\ 0<\rho\le r_i/20$ and $c(n,p)\ge 1$ such that

$$c^{-1}\alpha\lambda^p \le \frac{1}{|Q_\rho|}\int_{Q_\rho\cap D} g^p\,\mathrm{d}z \le \frac{c}{|Q_{20\rho}|}\int_{Q_{20\rho}\cap D} g^p\,\mathrm{d}z \le c^2\alpha\lambda^p. \tag{6.2}$$

First, this chain of inequalities implies that we can absorb the first term on the right side of (6.1) into the left by choosing $\tilde{\varepsilon}>0$ small enough, and thus we have

$$\frac{1}{|Q_\rho|}\int_{Q_\rho\cap D} g^p\,\mathrm{d}z \le c\left(\frac{1}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} g^q\,\mathrm{d}z\right)^{p/q} + c\delta_{\Lambda_{4\rho}}\left(\fint_{B_{4\rho}} f^q\,\mathrm{d}x\right)^{p/q}.$$

Together with properties of the Whitney decomposition, (6.2) also implies that there exists $c\ge 1$ such that

$$c^{-1}\lambda^p \le \frac{1}{|Q_\rho|}\int_{Q_\rho\cap D} h^p\,\mathrm{d}z \le \frac{c}{|Q_{20\rho}|}\int_{Q_{20\rho}\cap D} h^p\,\mathrm{d}z \le c^2\lambda^p. \tag{6.3}$$

We have $\alpha^{-p/q} \le (|Q_i|\,/\,|Q_0|)^{p/q} \le 1$ and thus by the previous estimates, we obtain

$$\begin{aligned}\frac{1}{|Q_{20\rho}|}\int_{Q_{20\rho}\cap D} h^p\,\mathrm{d}z \le c&\left(\frac{1}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} h^q\,\mathrm{d}z\right)^{p/q}\\ &+ c\delta_{\Lambda_{4\rho}}\left(\fint_{B_{4\rho}} f^q\,\mathrm{d}x\right)^{p/q}.\end{aligned} \tag{6.4}$$

We define the level sets

$$\begin{aligned}G(\lambda) &= \{(x,t)\in Q_0\cap D\,:\,h(x,t)>\lambda\},\\ \tilde{G}(\lambda) &= \{x\in B_0\,:\,f(x)>\lambda\},\end{aligned}$$

where $B_0 = B_{4R}(x_0)$. Next we use (6.4) and the level sets to calculate

$$\begin{aligned}\frac{1}{|Q_{20\rho}|}\int_{Q_{20\rho}\cap D} h^p\,\mathrm{d}z \le &c\eta^p\lambda^p + \left(|Q_{4\rho}|^{-1}\int_{Q_{4\rho}\cap G(\eta\lambda)} h^q\,\mathrm{d}z\right)^{p/q}\\ &+ c\delta_{\Lambda_{4\rho}}\left(|B_{4\rho}|^{-1}\int_{B_{4\rho}\cap\tilde{G}(\eta\lambda)} f^q\,\mathrm{d}x\right)^{p/q}.\end{aligned} \tag{6.5}$$

By Hölder's inequality and (6.3), there exists $c \geq 1$ such that

$$\left(\frac{1}{|Q_{4\rho}|}\int_{Q_{4\rho}\cap D} h^q \,\mathrm{d}z\right)^{(p-q)/q} \leq c\lambda^{p-q}. \tag{6.6}$$

Then we choose $\eta > 0$ small enough and use (6.3) to absorb the first term on the right of (6.5) into the left. Next we apply (6.6) and arrive at

$$\begin{aligned}\frac{1}{|Q_{20\rho}|}\int_{Q_{20\rho}\cap D} h^p \,\mathrm{d}z \leq & c\,|Q_{4\rho}|^{-1}\,\lambda^{p-q}\int_{Q_{4\rho}\cap G(\eta\lambda)} h^q \,\mathrm{d}z \\ & + c\delta_{\Lambda_{4\rho}}\left(|B_{4\rho}|^{-1}\int_{B_{4\rho}\cap\tilde{G}(\eta\lambda)} f^q \,\mathrm{d}x\right)^{p/q}.\end{aligned} \tag{6.7}$$

By Vitali's covering theorem, we have a disjoint set of cylinders

$$\{Q_{4\rho_i}(\tilde{z}_i)\}_{i=1}^{\infty},\ \ \tilde{z}_i \in G(\lambda)$$

such that almost everywhere

$$G(\lambda) \subset \cup_{i=1}^{\infty} Q_{20\rho_i}(\tilde{z}_i) \subset Q_0,$$

and (6.7) holds in every cylinder. Multiplying (6.7) by $|Q_{4\rho}|$ remembering $q = pn/(2+n)$ to get rid of $|B_{4\rho}|^{-1}$ and summing over i, we obtain

$$\begin{aligned}\int_{G(\lambda)} h^p \,\mathrm{d}z &\leq \sum_{i=1}^{\infty}\int_{Q_{20\rho_i}\cap D} h^p \,\mathrm{d}z \\ &\leq c\lambda^{p-q}\int_{G(\eta\lambda)} h^q \,\mathrm{d}z + c\delta_{\Lambda_{4R}(t_0)}\left(\int_{\tilde{G}(\eta\lambda)} f^q \,\mathrm{d}x\right)^{p/q}.\end{aligned} \tag{6.8}$$

By integrating, using Fubini's theorem and (6.8), we have

$$\begin{aligned}&\int_{G(\lambda_0)} h^{p+\varepsilon} \,\mathrm{d}z \\ &\quad = \int_{G(\lambda_0)}\left(\int_{\lambda_0}^{h} (\lambda^{\varepsilon})' \,\mathrm{d}\lambda + (\lambda_0)^{\varepsilon}\right) h^p \,\mathrm{d}z \\ &\quad = \varepsilon\int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1}\int_{G(\lambda)} h^p \,\mathrm{d}z\,\mathrm{d}\lambda + (\lambda_0)^{\varepsilon}\int_{G(\lambda_0)} h^p \,\mathrm{d}z \\ &\quad \leq c\int_{\lambda_0}^{\infty} \varepsilon\lambda^{\varepsilon-1+p-q}\int_{G(\eta\lambda)} h^q \,\mathrm{d}z\,\mathrm{d}\lambda \\ &\quad + c\varepsilon\lambda^{\varepsilon-1}\delta_{\Lambda_{4R}(t_0)}\int_{\lambda_0}^{\infty}\left(\int_{\tilde{G}(\eta\lambda)} f^q \,\mathrm{d}x\right)^{p/q} \mathrm{d}\lambda + (\lambda_0)^{\varepsilon}\int_{G(\lambda_0)} h^p \,\mathrm{d}z.\end{aligned}$$

We estimate this integral in two parts. First, by Fubini's theorem, we see that

$$\begin{aligned}
&\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1+p-q} \int_{G(\eta\lambda)} h^q \,\mathrm{d}z \,\mathrm{d}\lambda \, + (\lambda_0)^{\varepsilon} \int_{G(\lambda_0)} h^p \,\mathrm{d}z \\
&= c\varepsilon \int_{G(\eta\lambda_0)} \left(\int_{\lambda_0}^{h/\eta} \lambda^{\varepsilon-1+p-q} \,\mathrm{d}\lambda \right) h^q \,\mathrm{d}z \, + (\lambda_0)^{\varepsilon} \int_{G(\lambda_0)} h^p \,\mathrm{d}z \\
&\leq \frac{c\varepsilon}{\varepsilon+p-q} \int_{G(\lambda_0)} h^{\varepsilon+p} \,\mathrm{d}z + c(\lambda_0)^{\varepsilon} \int_{G(\eta\lambda_0)} h^p \,\mathrm{d}z.
\end{aligned}$$

Then we divide the boundary term into two parts. By Fubini's theorem and Hölder's inequality, we have

$$\begin{aligned}
&\varepsilon \int_{\lambda_0}^{\infty} \lambda^{\varepsilon-1} \left(\int_{\tilde{G}(\eta\lambda)} f^q \,\mathrm{d}x \right)^{p/q} \mathrm{d}\lambda \\
&\qquad \leq \left(\int_{\tilde{G}(\eta\lambda_0)} f^q \,\mathrm{d}x \right)^{p/q-1} \int_{\tilde{G}(\eta\lambda_0)} \int_{\lambda_0}^{f/\eta} \varepsilon \lambda^{\varepsilon-1} f^q \,\mathrm{d}\lambda \,\mathrm{d}x \\
&\qquad \leq cR^{2\varepsilon/(q+\varepsilon)} \left(\int_{\tilde{G}(\eta\lambda_0)} f^{q+\varepsilon} \,\mathrm{d}x \right)^{(p+\varepsilon)/(q+\varepsilon)}.
\end{aligned}$$

We collect the estimates, choose $\varepsilon > 0$ small enough to absorb the term containing $h^{p+\varepsilon}$ into the left and conclude that

$$\begin{aligned}
\int_{G(\lambda_0)} h^{p+\varepsilon} \,\mathrm{d}z &\leq c(\lambda_0)^{\varepsilon} \int_{G(\eta\lambda_0)} h^p \,\mathrm{d}z \\
&\qquad + c\delta_{\Lambda_{4R}} R^{2\varepsilon/(q+\varepsilon)} \left(\int_{\tilde{G}(\eta\lambda_0)} f^{q+\varepsilon} \,\mathrm{d}x \right)^{(p+\varepsilon)/(q+\varepsilon)}.
\end{aligned}$$

Notice that if the term we would like to absorb is infinite, we can replace h by $\min\{h, k\}$, $k > \lambda_0$, for which (6.8) continues to hold, and finally let $k \to \infty$. We remember that $q = pn/(n+2)$ and easily obtain

$$\begin{aligned}
\frac{1}{|Q_R|} \int_{Q_R \cap D} h^{p+\varepsilon} \,\mathrm{d}z &\leq \frac{c(\lambda_0)^{\varepsilon}}{|Q_{4R}|} \int_{Q_{4R} \cap D} h^p \,\mathrm{d}z \\
&\qquad + c\delta_{\Lambda_{4R}} \left(\fint_{B_{4R}} f^{q+\varepsilon} \,\mathrm{d}x \right)^{(p+\varepsilon)/(q+\varepsilon)}.
\end{aligned}$$

Since we are far away from the boundary of Q_{4R} on the left side, the definition of $h(z)$ and λ_0 implies the result. □

The next theorem is the higher integrability for the gradient of a quasimimizer near the initial boundary.

Theorem 16 *Let u be a quasiminimizer to an initial value problem with the initial condition* (5.1)*. Let $0 < R < \infty$ and let $Q_R = Q_R(x_0, t_0) \subset \mathbb{R}^{n+1}$ such that* $\operatorname{dist}\{B_{4R}(x_0), \partial\Omega\} > a > 0$ *and $0 \in \Lambda_R(t_0)$. Then there exist positive constants $\varepsilon_0 = \varepsilon_0(n, \delta, \alpha, \beta, \tilde{c}, K, a)$ and $c = c(n, \delta, \alpha, \beta, \tilde{c}, K, a)$ such that for every $0 \le \varepsilon < \varepsilon_0$, we have*

$$
\begin{aligned}
&\left(\frac{1}{|Q_R|}\int_{Q_R\cap D} |\nabla u|^{2+\varepsilon}\,\mathrm{d}z\right)^{1/(2+\varepsilon)} \\
&\le c\left(\frac{1}{|Q_{4R}|}\int_{Q_{4R}\cap D} |\nabla u|^{2}\,\mathrm{d}z\right)^{1/2} + c\left(\fint_{B_{4R}} |\nabla \varphi|^{2^*+\varepsilon}\,\mathrm{d}x\right)^{1/(2^*+\varepsilon)},
\end{aligned}
$$

where $2^* = 2n/(2+n)$ and $\tilde{c}$ is the constant in (5.2)*.*

Proof: We choose

$$g = |\nabla u|\,,\ p = 2,\ q = 2n/(2+n),\ f = |\nabla \varphi(x)|$$

and use Theorem 15. If we are near the initial boundary Lemma 14 holds and if we are far away from the initial boundary, we can use the local result, see [Wie87], to satisfy the condition of Theorem 15. □

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