Helsinki University of Technology, Institute of Mathematics, Research Reports
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Espoo 2006 A483

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Lourenço Beirão da Veiga, Jarkko Niiranen, Rolf Stenberg: *A family of C^0 finite elements for Kirchhoff plates*; Helsinki University of Technology, Institute of Mathematics, Research Reports A483 (2006).

Abstract: *For the Kirchhoff plates a new finite element method, which is a modification of the one introduced in [25], is presented. This method has the advantages of allowing low order polynomials, and of holding convergence properties which does not deteriorate in the presence of the free boundary conditions. For the method optimal a-priori and a-posteriori error estimates are derived.*

AMS subject classifications: 65N30, 74S05, 74K20

Keywords: finite elements, Kirchhoff plate model, free boundary, a-priori error analysis, a-posteriori error analysis

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ISBN 951-22-7538-4
ISSN 0784-3143

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1 Introduction

A conforming finite element method for the Kirchhoff plate bending problem, i.e. the biharmonic problem, needs a discrete space for which it holds at least the global C^1-regularity. As a consequence, in order to retain minimal flexibility of the finite element space adopted, a fifth degree polynomial order is in general required. A classical way to avoid using high order polynomial spaces, is to write the Kirchhoff plate bending problem as the limit of the Reissner–Mindlin problem written in mixed form. On the other hand, in the presence of the free boundary conditions, this path leads to a method which is not completely consistent; in other words, the solution of the Kirchhoff problem is not exactly the solution of the mixed Reissner–Mindlin problem with thickness set equal to zero. We must observe that this point is in general ignored in the literature, where the more "classical" clamped case is typically considered.

Our aim in the present paper is to present for the Kirchhoff plate bending problem a family of "low order" finite elements for which it holds the optimal convergence rate even in the presence of free boundaries. This method is a modification of the stabilized method for the Reissner–Mindlin plates presented in [25]. The paper is organized as follows. In section 2 we describe the plate bending problem, while in section 3 we introduce the new family of finite elements. In section 4 an a-priori error analysis is derived. This analysis leads to optimal results, with respect to the solution regularity and to the polynomial degree used. In Section 5 an a-posteriori error analysis is accomplished. We devise a local error indicator which is shown to be both reliable and efficient.

We finally observe that the theoretical results here presented are in complete agreement with the numerical tests shown in [10].

2 The Kirchhoff plate bending problem

We consider the bending problem of an isotropic linearly elastic plate and assume that the undeformed plate midsurface is described by a given convex polygonal domain $\Omega \subset \mathbb{R}^2$. The plate is considered to be clamped on the part $\Gamma_{\rm c}$ of its boundary $\partial\Omega$, simply supported on the part $\Gamma_{\rm s} \subset \partial\Omega$ and free on $\Gamma_{\rm f} \subset \partial\Omega$. A transverse load $F = Gt^3 f$ is applied, where t is the thickness of the plate and G the shear modulus for the material.

In the sequel, we indicate with $\mathcal{V}$ the set of all the corner points in $\Gamma_{\rm f}$ corresponding to an angle of the boundary $\Gamma_{\rm f}$. Moreover, $\boldsymbol{n}$ and $\boldsymbol{s}$ represent the unit outward normal and the unit counterclockwise tangent to the boundary. Finally, for corner points $c \in \mathcal{V}$, we introduce the following notation. We indicate with $\boldsymbol{n}_1$ and $\boldsymbol{s}_1$ the unit vectors corresponding respectively to $\boldsymbol{n}$ and $\boldsymbol{s}$ on one of the two edges which form the boundary angle at c; we indicate with $\boldsymbol{n}_2$ and $\boldsymbol{s}_2$ the ones corresponding to the other edge. Note that which of the two edges correspond to the subscript 1 or 2 is not relevant.

Then, following the Kirchhoff plate bending model and assuming that the load is sufficiently regular, the deflection w of the plate can be found as the solution of the following well known biharmonic problem:

$$
\begin{aligned}
& D\Delta^2 w = Gf && \text{in } \Omega \\
& w = 0\,, \quad \tfrac{\partial w}{\partial \boldsymbol{n}} = 0 && \text{on } \Gamma_{\mathrm{c}} \\
& w = 0\,, \quad \boldsymbol{n}^T \boldsymbol{M} \boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{s}} \\
& \boldsymbol{n}^T \boldsymbol{M} \boldsymbol{n} = 0\,, \quad \tfrac{\partial}{\partial \boldsymbol{s}} \boldsymbol{s}^T \boldsymbol{M} \boldsymbol{n} + (\mathbf{div}\, \boldsymbol{M}) \cdot \boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{f}} \\
& \boldsymbol{s}_1^T \boldsymbol{M} \boldsymbol{n}_1(c) = \boldsymbol{s}_2^T \boldsymbol{M} \boldsymbol{n}_2(c) && \forall c \in \mathcal{V}\,,
\end{aligned}
\tag{2.1}
$$

where the scaled bending modulus and the bending moment are

$$
D = \frac{E}{12(1-\nu^2)}\,, \quad \boldsymbol{M} = \frac{G}{6}\big(\boldsymbol{\varepsilon}(\nabla w) + \frac{\nu}{1-\nu} \mathrm{div}\, \nabla w \boldsymbol{I}\big)\,, \tag{2.2}
$$

with E, ν the Young modulus and the Poisson ratio for the material. Note that it holds $G = \frac{E}{2(1+\nu)}$, respectively.

Due to the presence of the fourth order elliptic operator Δ^2 in (2.1), the natural space for the variational formulation of the problem (2.1) is the Sobolev space $H^2(\Omega)$. As a consequence, conforming finite element methods based on such a formulation need the C^1 regularity conditions. In order to keep minimal flexibility of the discrete space used, the C^1 regularity condition in turn requires a high order polynomial space, which may be preferable to avoid.

On the other hand, in the case of clamped and simply supported boundary conditions, the Kirchhoff problem can be treated similarly to a Reissner–Mindlin plate bending problem with the thickness t set to zero in the formulation. As a consequence, the following equivalent mixed variational formulation is obtained.

Next we introduce, respectively, the space for the deflection, rotation and for the "shear stress" Lagrange multiplier

$$
\begin{aligned}
W &= \big\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_{\mathrm{c}} \cup \Gamma_{\mathrm{s}} \big\}\,, && (2.3) \\
\boldsymbol{V} &= \big\{ \boldsymbol{\eta} \in [H^1(\Omega)]^2 \mid \boldsymbol{\eta} = 0 \text{ on } \Gamma_{\mathrm{c}}\,,\ \boldsymbol{\eta} \cdot \boldsymbol{s} = 0 \text{ on } \Gamma_{\mathrm{s}} \big\}\,, && (2.4) \\
\mathbf{H} &= \{ \boldsymbol{v} \in L^2(\Omega) \mid \mathrm{rot} \boldsymbol{v} \in L^2(\Omega)\,,\ \boldsymbol{v} \cdot \boldsymbol{s} = 0 \text{ in } \Gamma_{\mathrm{c}} \cup \Gamma_{\mathrm{s}} \}\,, && (2.5) \\
\boldsymbol{Q} &= \mathbf{H}'\,. && (2.6)
\end{aligned}
$$

Now the mixed variational formulation reads:

Find $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in W \times \boldsymbol{V} \times \boldsymbol{Q}$ such that

$$
\begin{aligned}
& a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \kappa \langle \boldsymbol{q}, \nabla v - \boldsymbol{\eta} \rangle = (f, v) && \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}\,, \\
& \langle \nabla w - \boldsymbol{\beta}, \boldsymbol{r} \rangle = 0 && \forall \boldsymbol{r} \in \boldsymbol{Q}\,,
\end{aligned}
\tag{2.7}
$$

where κ is the shear correction factor and a is the bilinear form

$$a(\boldsymbol{\phi}, \boldsymbol{\eta}) = \frac{1}{6}\big((\boldsymbol{\varepsilon}(\boldsymbol{\phi}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) + \frac{\nu}{1-\nu}(\operatorname{div}\boldsymbol{\phi}, \operatorname{div}\boldsymbol{\eta})\big) \quad \forall \boldsymbol{\phi} \in \boldsymbol{V},\ \boldsymbol{\eta} \in \boldsymbol{V}\,. \tag{2.8}$$

Note that the brackets $\langle \cdot, \cdot \rangle$ above indicate the duality product between functions of $\mathbf{H}$ and $\boldsymbol{Q}$.

The problem above could be equivalently rewritten setting W in H^2 and $\mathbf{H}$ in H^1. The advantage of formulation (2.7) is that, in the case of clamped or simply supported boundary conditions, a non conforming Kirchhoff element (i.e. which uses globally C^0 deflections) can be obtained using any locking free Reissner–Mindlin element [2, 4, 5, 7, 11, 12, 17, 18, 6, 13, 19]. The discrete deflection will converge to the continuous one at least in the H^1 norm.

However, this cannot be done in the case of the *free boundary conditions*. Introducing the rotation $\boldsymbol{\beta}$ and the shear stress $\boldsymbol{q}$, it is easy to check that the problem (2.1) is equivalent to

$$\begin{aligned}
&\boldsymbol{L\beta} + \boldsymbol{q} = \boldsymbol{0} && \text{in } \Omega\\
&\quad -\operatorname{div}\boldsymbol{q} = f && \text{in } \Omega\\
&\quad \nabla w - \boldsymbol{\beta} = \boldsymbol{0} && \text{in } \Omega\\
&w = 0\,,\ \boldsymbol{\beta} = 0 && \text{on } \Gamma_{\mathrm{c}}\\
&w = 0\,,\ \boldsymbol{\beta}\cdot\boldsymbol{s} = 0\,,\ \boldsymbol{n}^T\boldsymbol{M}\boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{s}}\\
&\boldsymbol{\beta}\cdot\boldsymbol{s} - \tfrac{\partial w}{\partial \boldsymbol{s}} = 0\,,\ \boldsymbol{n}^T\boldsymbol{M}\boldsymbol{n} = 0\,,\ \tfrac{\partial}{\partial \boldsymbol{s}}\boldsymbol{s}^T\boldsymbol{M}\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n} = 0 && \text{on } \Gamma_{\mathrm{f}}\\
&\boldsymbol{s}_1^T\boldsymbol{M}\boldsymbol{n}_1(c) = \boldsymbol{s}_2^T\boldsymbol{M}\boldsymbol{n}_2(c) && \forall c \in \mathcal{V}\,,
\end{aligned} \tag{2.9}$$

where now the scaled bending moment

$$\boldsymbol{M}(\boldsymbol{\phi}) = \frac{1}{6}\big(\boldsymbol{\varepsilon}(\boldsymbol{\phi}) + \frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{\phi}\,\boldsymbol{I}\big) \tag{2.10}$$

and the operator $\boldsymbol{L}$ is defined as

$$\boldsymbol{L\phi} = \mathbf{div}\,\boldsymbol{M}(\boldsymbol{\phi})\,. \tag{2.11}$$

On the other hand, the strong form related to the variational formulation (2.7) reads exactly identical to (2.9) apart the boundary condition on Γ_{f} which must be substituted with

$$\boldsymbol{n}^T\boldsymbol{M}\boldsymbol{n} = 0\,,\ \boldsymbol{s}^T\boldsymbol{M}\boldsymbol{n} = 0\,,\ \boldsymbol{q}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma_{\mathrm{f}} \tag{2.12}$$

and the vertex condition which is no longer valid.

Therefore, if $\Gamma_{\mathrm{f}} \neq \emptyset$, the two formulations are *not* equivalent. This point is in general under-estimated or simply ignored in the literature, where typically only the "classical" clamped plate case is addressed. A direct correction to this nonequivalence could be done by substituting the space $(W, \boldsymbol{V}, \boldsymbol{Q})$ in (2.7) with the space

$$\{(v, \boldsymbol{\eta}, \boldsymbol{r}) \in W \times \boldsymbol{V} \times \boldsymbol{Q} \mid \boldsymbol{\eta} \cdot \boldsymbol{s} - \frac{\partial v}{\partial \boldsymbol{s}} = 0 \text{ on } \Gamma_{\mathrm{f}}\}\,. \tag{2.13}$$

Then, as noted in [15, 16, 9, 8], one would obtain a problem which is completely equivalent to (2.9). On the other hand, such a choice is not directly viable at the discrete level because a condition of (2.13) type generates an additional boundary locking effect. In the sequel we will present a family of low order finite elements for Kirchhoff plates which avoids this difficulty; in particular, its rate of convergence to the Kirchhoff problem solution does not deteriorate in the presence of the free boundary conditions.

Remark 2.1. As is well known, in the presence of the free boundary conditions the solution of the Reissner–Mindlin plate bending problem is strongly non-regular even if the load and domain boundaries are smooth (see [3]). This may lead to a very slow convergence of Reissner–Mindlin finite element methods. Therefore, it is exactly in the case of the free boundary conditions that the Kirchhoff plate model is particularly valuable.

3 Finite element formulation

In this section we introduce the numerical method, which is an extension of the method presented in [25]. In order to neglect plate rigid movements and the related technicalities, in the sequel we will assume that the one-dimensional measure

$$\mathrm{meas}(\Gamma_{\mathrm{c}} \cup \Gamma_{\mathrm{s}}) > 0\,. \tag{3.1}$$

3.1 The new finite element method

Let a regular family of triangular meshes on Ω be given. Given an integer $k \geq 1$, we then define the discrete spaces

$$W_h = \{v \in W \;\mid\; v_{|K} \in P_{k+1}(K) \;\; \forall K \in \mathcal{C}_h\}\,, \tag{3.2}$$

$$\boldsymbol{V}_h = \{\boldsymbol{\eta} \in \boldsymbol{V} \;\mid\; \boldsymbol{\eta}_{|K} \in [P_k(K)]^2 \;\; \forall K \in \mathcal{C}_h\}\,, \tag{3.3}$$

where $\mathcal{C}_h$ represents the set of all triangles K of the mesh and $P_k(K)$ is the space of polynomials of degree k on K. In the sequel, we will indicate with h_K the diameter of each element K, while h will indicate the maximum size of all the elements in the mesh. Also, we will indicate with e a general edge of the triangulation and with h_e the length of e.

Before introducing the method, we state the following result which follows trivially from classical scaling arguments and the coercivity of the form a.

Lemma 3.1. *Given any triangulation $\mathcal{C}_h$, let $\mathcal{T}_h$ indicate the set of all the triangle edges, and $\Gamma_{\mathrm{f},h}$ the set of the triangle edges in Γ_{f}. Then, there exist positive constants C_I and C'_I such that, for all meshes $\mathcal{C}_h$,*

$$C_I \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\boldsymbol{\phi}\|_{0,K}^2 \leq a(\boldsymbol{\phi}, \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_h \,, \tag{3.4}$$

$$C'_I \sum_{e \in \Gamma_{\mathrm{f},h}} h_e \, \|M_{ns}(\boldsymbol{\phi})\|_{0,e}^2 \leq a(\boldsymbol{\phi}, \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_h \,, \tag{3.5}$$

where the operator $M_{ns}(\boldsymbol{\eta}) = \boldsymbol{s}^T \boldsymbol{M}(\boldsymbol{\eta}) \, \boldsymbol{n}$ with $\boldsymbol{n}, \boldsymbol{s}$ unit outward normal and unit counterclockwise tangent to the edge e, and with $\boldsymbol{M}$ defined in (2.10)*.*

Let two real numbers γ and α be assigned, $\gamma > 2/C'_I$ and $0 < \alpha < C_I/4$. Then, the discrete problem reads:

Method 3.1. *Find $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$, such that*

$$\mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (f, v) \qquad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h \,, \tag{3.6}$$

where the form $\mathcal{A}_h$ is

$$\mathcal{A}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = \mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) + \mathcal{D}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) \,, \tag{3.7}$$

with

$$\begin{aligned} \mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\phi}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{L}\boldsymbol{\phi}, \boldsymbol{L}\boldsymbol{\eta})_K \\ &+ \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} (\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\phi}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \mathcal{D}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = \sum_{e \in \Gamma_{\mathrm{f},h}} \Big(&(M_{ns}(\boldsymbol{\phi}), [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s})_e \\ &+ ([\nabla z - \boldsymbol{\phi}] \cdot \boldsymbol{s}, M_{ns}(\boldsymbol{\eta}))_e + \frac{\gamma}{h_e} ([\nabla z - \boldsymbol{\phi}] \cdot \boldsymbol{s}, [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s})_e \Big) \end{aligned} \tag{3.9}$$

for all $(z, \boldsymbol{\phi}) \in W_h \times \boldsymbol{V}_h$, $(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$.

The bilinear form $\mathcal{B}_h$ constitutes essentially the original method of [25] with the thickness t set equal to zero, while the added form $\mathcal{D}_h$ is introduced to avoid the convergence deterioration in the presence of free boundaries.

Note that, instead of a global constant α, for stability reasons in practical implementation it may be preferable to use local constants α_K defined by

$$\frac{1}{\alpha_K} = \rho^{-1} \max_{\boldsymbol{\phi} \in P_k(K), a_K(\boldsymbol{\phi}, \boldsymbol{\phi}) \neq 0} \frac{h_K^2 \|\boldsymbol{L}\boldsymbol{\phi}\|_{0,K}^2}{a_K(\boldsymbol{\phi}, \boldsymbol{\phi})} \,, \tag{3.10}$$

where a_K represents the form a restricted to K and where $0 < \rho < \frac{1}{4}$ is a fixed value. Similarly, instead of using a global constant γ, local constants γ_e can be derived calculating the elementwise value of C'_I in Lemma 3.1.

We also introduce the discrete shear stress

$$\boldsymbol{q}_{h|K} = \frac{1}{\alpha h_K^2}(\nabla w_h - \boldsymbol{\beta}_h - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta}_h)_{|K} \quad \forall K \in \mathcal{C}_h \,. \tag{3.11}$$

Noting that, due to the first and third equation of (2.9), it holds

$$\boldsymbol{q}_{|K} = \frac{1}{\alpha h_K^2}(\nabla w - \boldsymbol{\beta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta})_{|K} \quad \forall K \in \mathcal{C}_h \,, \tag{3.12}$$

and it follows that the definition (3.11) is consistent with the exact "shear stress".

3.2 Boundary inconsistency of the original method

If the original method of [25] without the additional form $\mathcal{D}_h$ is employed, in the presence of a free boundary an inconsistency term arises. In other words,

$$\mathcal{B}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (f, v) + \sum_{e \in \Gamma_{\mathrm{f},h}} (M_{ns}(\boldsymbol{\beta}), [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s})_e \tag{3.13}$$

$\forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$, and therefore the additional inconsistency term

$$- \sum_{e \in \Gamma_{\mathrm{f},h}} (M_{ns}(\boldsymbol{\beta}), [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s})_e \tag{3.14}$$

arises, hindering severely the convergence of the method.

Regardless of the solution regularity and the polynomial degree k, the term in (3.14) can only be bounded with the order $O(h^{1/2})$. As well known (see for example [24]), the inconsistency error is a lower bound for the error of finite element methods. As a consequence, the original Kirchhoff method (i.e. without the additional correction $\mathcal{D}_h$) is not expected to converge with a rate better than $h^{1/2}$ if $\Gamma_{\mathrm{f}} \neq \emptyset$. This observation is also confirmed by the numerical tests shown in [10]. See also [14] for other numerical tests regarding this issue.

Note also that this boundary inconsistency term is connected not only to the formulation in [25] but is common to any other Kirchhoff method which follows a "Reissner–Mindlin limit" approach.

4 A-priori error estimates

For $(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$, we introduce the following mesh dependent norms:

$$|(v, \boldsymbol{\eta})|_h^2 = \sum_{K \in \mathcal{C}_h} h_K^{-2} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 \,, \tag{4.1}$$

$$\|v\|_{2,h}^2 = \|v\|_1^2 + \sum_{K \in \mathcal{C}_h} |v|_{2,K}^2 + \sum_{e \in \mathcal{I}_h} h_K^{-1} \| [\![\frac{\partial v}{\partial \boldsymbol{n}}]\!] \|_{0,e}^2 \,, \tag{4.2}$$

$$|||(v, \boldsymbol{\eta})|||_h = \|\boldsymbol{\eta}\|_1 + \|v\|_{2,h} + |(v, \boldsymbol{\eta})|_h \,, \tag{4.3}$$

and for $\boldsymbol{r} \in L^2(\Omega)$

$$\|\boldsymbol{r}\|_{-1,h} = \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{r}\|_{0,K}^2 \Big)^{1/2} . \tag{4.4}$$

Given the space

$$\boldsymbol{V}_* = \big\{ \boldsymbol{\eta} \in [H^1(\Omega)]^2 \mid \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{c}},\ \boldsymbol{\eta} \cdot \boldsymbol{s} = 0 \text{ on } \Gamma_{\mathrm{f}} \cup \Gamma_{\mathrm{s}} \big\} \tag{4.5}$$

we also introduce the norm

$$\|\boldsymbol{r}\|_{-1,*} = \sup_{\boldsymbol{\eta} \in \boldsymbol{V}_*} \frac{\langle \boldsymbol{r}, \boldsymbol{\eta} \rangle}{\|\boldsymbol{\eta}\|_1} . \tag{4.6}$$

Note that the norm $\| \cdot \|_{-1,*}$ bounds from above the classical norm $\| \cdot \|_{-1}$.

In [23] the following lemma is proved:

Lemma 4.1. *There exists a positive constant C such that*

$$\|v\|_{2,h} \leq C\big(\|\boldsymbol{\eta}\|_1 + \|v\|_1 + |(v, \boldsymbol{\eta})|_h \big) \qquad \forall (v, \boldsymbol{\eta}) \in (W_h \times \boldsymbol{V}_h) . \tag{4.7}$$

Using the Poincaré inequality and the previous lemma, the following equivalence follows easily:

Lemma 4.2. *There is a positive constant C such that*

$$C |||(v, \boldsymbol{\eta})|||_h \leq \|\boldsymbol{\eta}\|_1 + |(v, \boldsymbol{\eta})|_h \leq |||(v, \boldsymbol{\eta})|||_h \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h . \tag{4.8}$$

The convergence of the method to the solution of the problem (2.9) is stated in Theorem 4.3 below. We need some preliminary results; the following theorem states the stability of the discrete formulation (3.6):

Theorem 4.1. *Let $0 < \alpha < C_I/4$ and $\gamma > 2/C_I'$. Then there exists a positive constant C such that*

$$\mathcal{A}_h(v, \boldsymbol{\eta}; v, \boldsymbol{\eta}) \geq C |||(v, \boldsymbol{\eta})|||_h^2 \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h . \tag{4.9}$$

Proof. Using the inverse estimate of Lemma 3.1 we easily get

$$\begin{aligned} &\mathcal{B}_h(v, \boldsymbol{\eta}; v, \boldsymbol{\eta}) \\ &= a(\boldsymbol{\eta}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 \|\boldsymbol{L\eta}\|_{0,K}^2 + \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} \|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L\eta}\|_{0,K}^2 \\ &\geq \big(1 - \frac{\alpha}{C_I}\big)\, a(\boldsymbol{\eta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} \|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L\eta}\|_{0,K}^2 . \end{aligned} \tag{4.10}$$

First using locally the arithmetic-geometric mean inequality with constant γ/h_e, then the second inverse inequality of Lemma 3.1, we get

$$\begin{aligned}
&\mathcal{D}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta})\\
&= \sum_{e\in\Gamma_{\mathrm{f},h}} 2(M_{ns}(\boldsymbol{\eta}),[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s})_e + \frac{\gamma}{h_e}\|[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\|_{0,e}^2)\\
&\geq \sum_{e\in\Gamma_{\mathrm{f},h}} \Big(-\frac{\gamma}{h_e}\|[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\|_{0,e}^2 - \gamma^{-1}h_e\,\|M_{ns}(\boldsymbol{\eta})\|_{0,e}^2 + \frac{\gamma}{h_e}\|[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\|_{0,e}^2\Big)\\
&= -\sum_{e\in\Gamma_{\mathrm{f},h}} \gamma^{-1}h_e\,\|M_{ns}(\boldsymbol{\eta})\|_{0,e}^2\\
&\geq -\frac{\gamma^{-1}}{C_I'}\,a(\boldsymbol{\eta},\boldsymbol{\eta})\\
&\geq -\frac{1}{2}\,a(\boldsymbol{\eta},\boldsymbol{\eta})\,. \qquad (4.11)
\end{aligned}$$

Joining (4.10) with (4.11) and using Korn's inequality we then obtain

$$\begin{aligned}
&\mathcal{B}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta}) + \mathcal{D}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta})\\
&\geq \big(\frac{1}{2}-\frac{\alpha}{C_I}\big)\,a(\boldsymbol{\eta},\boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\\
&\geq C\Big(\|\boldsymbol{\eta}\|_1^2 + \sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big)\,. \qquad (4.12)
\end{aligned}$$

From the triangle inequality, again the inverse estimate of Lemma 3.1 and the boundedness of the bilinear form a it follows

$$\begin{aligned}
&\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}\|_{0,K}^2\\
&\leq 2\Big(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + \sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big)\\
&\leq 2\Big(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + \sum_{K\in\mathcal{C}_h}\alpha h_K^2\|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big)\\
&\leq C\Big(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + a(\boldsymbol{\eta},\boldsymbol{\eta})\Big)\\
&\leq C\Big(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + \|\boldsymbol{\eta}\|_1^2\Big)\,, \qquad (4.13)
\end{aligned}$$

which combined with (4.12) gives

$$\mathcal{A}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta}) \geq C\big(\|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h\big)\,. \qquad (4.14)$$

The result then follows from Lemma 4.2. □

The following result states the consistency of the method. For simplicity, in the rest of this section we assume that w, the solution of the problem (2.1), or equivalently (2.9), is in $H^3(\Omega)$; this is a very reasonable assumption, as discussed at the end of this section. Note also that, with some additional technical work involving the appropriate Sobolev spaces and their duals, such assumption could be probably avoided.

Theorem 4.2. *The solution* $(w, \boldsymbol{\beta})$ *of the problem (2.9) satisfies*

$$\mathcal{A}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (f, v) \qquad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h \,. \tag{4.15}$$

Proof. The definition of the bilinear forms in Method 3.1, recalling (2.9) and the expression (3.12), we obtain

$$\begin{aligned}
\mathcal{B}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\beta}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{L}\boldsymbol{\beta}, \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} (\nabla w - \boldsymbol{\beta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{q}, \boldsymbol{L}\boldsymbol{\eta})_K + \sum_{K \in \mathcal{C}_h} (\boldsymbol{q}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) \,.
\end{aligned} \tag{4.16}$$

First by the definition, then integrating by parts on each triangle, finally using the regularity of the functions involved and the boundary conditions of (2.9) on Γ_{c}, Γ_{s}, we get

$$\begin{aligned}
a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) &= \sum_{K \in \mathcal{C}_h} \Big((\boldsymbol{M}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))_K + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta})_K \Big) \\
&= - \sum_{K \in \mathcal{C}_h} (\boldsymbol{L}\boldsymbol{\beta} + \boldsymbol{q}, \boldsymbol{\eta})_K + \sum_{e \in \Gamma_{\mathrm{f},h}} (\boldsymbol{M}(\boldsymbol{\beta}) \cdot \boldsymbol{n}, \boldsymbol{\eta})_e - \sum_{K \in \mathcal{C}_h} (\operatorname{div} \boldsymbol{q}, v)_K \\
&\quad + \sum_{e \in \Gamma_{\mathrm{f}}} (\boldsymbol{q} \cdot \boldsymbol{n}, v)_e \,.
\end{aligned} \tag{4.17}$$

Recalling the first two equations in (2.9), the identity above becomes

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) = (f, v) + \sum_{e \in \Gamma_{\mathrm{f},h}} \Big((\boldsymbol{M}(\boldsymbol{\beta}) \cdot \boldsymbol{n}, \boldsymbol{\eta})_e + (\boldsymbol{q} \cdot \boldsymbol{n}, v)_e \Big), \tag{4.18}$$

while, using the boundary conditions of (2.9) on Γ_{f}, an integration by parts along the boundary and the last condition in (2.9), finally leads to

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) = (f, v) - \sum_{e \in \Gamma_{\mathrm{f},h}} (M_{ns}(\boldsymbol{\beta}), [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s})_e \,. \tag{4.19}$$

Again, due to (2.9) and the definition of the bilinear forms in Method 3.1, we now get

$$
\begin{aligned}
&\mathcal{D}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta})\\
&=\sum_{e\in\Gamma_{\mathrm{f},h}}\Big((M_{ns}(\boldsymbol{\beta}),[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s})_e+([\nabla w-\boldsymbol{\beta}]\cdot\boldsymbol{s},M_{ns}(\boldsymbol{\eta}))_e\\
&\quad+\frac{\gamma}{h_e}([\nabla w-\boldsymbol{\beta}]\cdot\boldsymbol{s},[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s})_e\Big)\\
&=\sum_{e\in\Gamma_{\mathrm{f},h}}(M_{ns}(\boldsymbol{\beta}),[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s})_e\,.
\end{aligned}\tag{4.20}
$$

The result directly follows from (4.16), (4.19) and (4.20). □

We can now derive the error estimates for the method.

Theorem 4.3. *Let* $0<\alpha<C_I/4$ *and* $\gamma>2/C_I'$. *Let* $(w,\boldsymbol{\beta})$ *be the solution of the problem (2.9) and* $(w_h,\boldsymbol{\beta}_h)$ *the solution obtained with Method 3.1. Then it holds*

$$
|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h\le Ch^s\|w\|_{s+2}\tag{4.21}
$$

for all $1\le s\le k$.

Proof. Step 1. Let $(w_I,\boldsymbol{\beta}_I)\in W_h\times\boldsymbol{V}_h$ be the usual Lagrange interpolants to w and $\boldsymbol{\beta}$, respectively. Using first the stability result of Theorem 4.1 and then the consistency result of Theorem 4.2 one has the existence of a pair

$$
(v,\boldsymbol{\eta})\in W_h\times\boldsymbol{V}_h\,,\qquad |||(v,\boldsymbol{\eta})|||_h\le C\,,\tag{4.22}
$$

such that

$$
\begin{aligned}
|||(w_h-w_I,\boldsymbol{\beta}_h-\boldsymbol{\beta}_I)|||_h &\le \mathcal{A}_h(w_h-w_I,\boldsymbol{\beta}_h-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})\\
&= \mathcal{A}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})\,,
\end{aligned}\tag{4.23}
$$

where $\mathcal{A}_h=\mathcal{B}_h+\mathcal{D}_h$.

Step 2. For the $\mathcal{B}_h$-part we have

$$
\begin{aligned}
&\mathcal{B}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})=a(\boldsymbol{\beta}-\boldsymbol{\beta}_I,\boldsymbol{\eta})\\
&-\sum_{K\in\mathcal{C}_h}\alpha h_K^2(\boldsymbol{L}(\boldsymbol{\beta}-\boldsymbol{\beta}_I),\boldsymbol{L}\boldsymbol{\eta})_K\\
&+\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}(\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)-\alpha h_K^2\boldsymbol{L}(\boldsymbol{\beta}-\boldsymbol{\beta}_I),\\
&\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta})_K\,.
\end{aligned}\tag{4.24}
$$

Due to the first inverse inequality of Lemma 3.1 we get

$$
\Big(\sum_{K\in\mathcal{C}_h}h_K^2\|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2}\le C|||(v,\boldsymbol{\eta})|||_h\tag{4.25}
$$

and

$$\Big(\sum_{K\in\mathcal{C}_h}\frac{1}{\alpha h_K^2}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2}\le C|||(v,\boldsymbol{\eta})|||_h\,. \tag{4.26}$$

Using these bounds in (4.24) and recalling (4.22) we obtain

$$\begin{aligned}&\mathcal{B}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})\\&\le C\Big(|||(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I)|||_h+\Big(\sum_{K\in\mathcal{C}_h}h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_I|_{2,K}^2\Big)^{1/2}\Big)\,.\end{aligned} \tag{4.27}$$

Substituting the definition of the norm (4.3) in (4.27), using the triangle inequality, and finally applying the classical interpolation estimates it easily follows

$$\mathcal{B}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})\le Ch^s\big(\|w\|_{s+2}+\|\boldsymbol{\beta}\|_{s+1}\big)\,. \tag{4.28}$$

Step 3. For the $\mathcal{D}_h$-part in (4.23) we have, by the definition,

$$\begin{aligned}\mathcal{D}_h(w-w_I,\boldsymbol{\beta}-\boldsymbol{\beta}_I;v,\boldsymbol{\eta})=&\sum_{e\in\Gamma_{\mathrm{f},h}}\Big((M_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_I),[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s})_e\\&+([\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)]\cdot\boldsymbol{s},M_{ns}(\boldsymbol{\eta}))_e\\&+\frac{\gamma}{h_e}([\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)]\cdot\boldsymbol{s},[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s})_e\Big)\\=:&\,T_1+T_2+T_3\,.\end{aligned} \tag{4.29}$$

Note that the Agmon inequality (see [1])

$$\|v\|_{0,e}\le C\big(h_K^{-1/2}\|v\|_{0,K}+h_K^{1/2}\|v\|_{1,K}\big)\quad\forall v\in P_k(K)\,, \tag{4.30}$$

combined with the classical inverse estimate

$$|\nabla\boldsymbol{\phi}|_{0,K}\le Ch_K^{-1}\|\boldsymbol{\phi}\|_{0,K}\quad\forall\boldsymbol{\phi}\in[P_k(K)]^2\,, \tag{4.31}$$

gives

$$\|[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\|_{0,e}^2\le\|\nabla v-\boldsymbol{\eta}\|_{0,e}^2\le h_{K_e}^{-1}\|\nabla v-\boldsymbol{\eta}\|_{0,K_e}^2 \tag{4.32}$$

for all $e\in\Gamma_{\mathrm{f},h}$, where K_e is the only triangle pertaining to the boundary edge e. Following the same steps we also get

$$\|M_{ns}(\boldsymbol{\eta})\|_{0,e}^2\le h_{K_e}^{-1}\|\boldsymbol{\eta}\|_{1,K_e}^2\qquad\forall e\in\Gamma_{\mathrm{f},h}\,. \tag{4.33}$$

The l^2-Cauchy–Schwartz inequality, the bound (4.32) and the norm definition (4.3) now give

$$\begin{aligned}T_1\;&\le\;\Big(\sum_{e\in\Gamma_{\mathrm{f},h}}h_{K_e}\|M_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\Gamma_{\mathrm{f},h}}h_{K_e}^{-1}\|[\nabla v-\boldsymbol{\eta}]\cdot\boldsymbol{s}\|_{0,e}^2\Big)^{1/2}\\&\le\;\Big(\sum_{e\in\Gamma_{\mathrm{f},h}}h_{K_e}\|M_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|_{0,e}^2\Big)^{1/2}|||(v,\boldsymbol{\eta})|||_h\,.\end{aligned} \tag{4.34}$$

Recalling the bound (4.22), using the Agmon inequality and finally applying the classical polynomial interpolation properties, it follows

$$\begin{aligned} T_1 &\leq C\Big(\sum_{e\in\Gamma_{\mathrm{f},h}} \|\boldsymbol{M}(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|^2_{0,K_e} + h^2_{K_e}\|\boldsymbol{M}(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|^2_{1,K_e}\Big)^{1/2} \\ &\leq Ch^s\|\boldsymbol{\beta}\|_{s+1}\,. \end{aligned} \tag{4.35}$$

The l^2-Cauchy–Schwartz inequality, the bound (4.33) and the norm definition (4.3) now give

$$\begin{aligned} T_2 &\leq \Big(\sum_{e\in\Gamma_{\mathrm{f},h}} h^{-1}_{K_e}\|\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|^2_{0,e}\Big)^{1/2}\Big(\sum_{e\in\Gamma_{\mathrm{f},h}} h_{K_e}\|M_{ns}(\boldsymbol{\eta})\|^2_{0,e}\Big)^{1/2} \\ &\leq \Big(\sum_{e\in\Gamma_{\mathrm{f},h}} h^{-1}_{K_e}\|\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|^2_{0,e}\Big)^{1/2} |||(v,\boldsymbol{\eta})|||_h\,. \end{aligned} \tag{4.36}$$

Again recalling the bound (4.22), using the Agmon inequality, applying the triangle inequality and the classical polynomial interpolation properties, it follows

$$\begin{aligned} T_2 &\leq \Big(\sum_{e\in\Gamma_{\mathrm{f},h}} h^{-2}_{K_e}\|\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|^2_{0,K_e} \\ &\quad + \|\nabla(w-w_I)-(\boldsymbol{\beta}-\boldsymbol{\beta}_I)\|^2_{1,K_e}\Big)^{1/2} \\ &\leq Ch^s\big(\|\boldsymbol{\beta}\|_{s+1}+\|w\|_{s+2}\big)\,. \end{aligned} \tag{4.37}$$

The bound for T_3 follows combining the same techniques used for T_1 and T_2; we get

$$T_3 \leq Ch^s\big(\|\boldsymbol{\beta}\|_{s+1}+\|w\|_{s+2}\big)\,. \tag{4.38}$$

Now, joining all the bounds (4.23), (4.28), (4.29), (4.35), (4.37) and (4.38) we obtain

$$|||(w_h-w_I,\boldsymbol{\beta}_h-\boldsymbol{\beta}_I)|||_h \leq Ch^s\big(\|\boldsymbol{\beta}\|_{s+1}+\|w\|_{s+2}\big)\,. \tag{4.39}$$

The triangle inequality and the classical polynomial interpolation estimates (recalling that $\boldsymbol{\beta}=\nabla w$) then yield

$$\begin{aligned} |||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h &\leq Ch^s\big(\|\boldsymbol{\beta}\|_{s+1}+\|w\|_{s+2}\big) \\ &\leq Ch^s\|w\|_{s+2}\,. \end{aligned} \tag{4.40}$$

Note that the result holds for real values of the regularity parameter s because the interpolation results used above are valid for real values of s. □

We also have the following result for the shear stress Lagrange multiplier:

Lemma 4.3. *It holds*

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*} \leq Ch^k\|w\|_{k+2}\,. \tag{4.41}$$

Proof. The proof is essentially an application of the "Pitkäranta–Verfürth trick" (see [22, 26]). From the definitions (3.12),(3.11), the triangle inequality, the classical interpolation and the bound (4.40) it easily follows

$$\begin{aligned}
&\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,h} \\
&\leq |||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h + \Big(\sum_{K\in\mathcal{C}} h_K^2\|\boldsymbol{L}\boldsymbol{\beta}-\boldsymbol{L}\boldsymbol{\beta}_h\|_{0,K}^2\Big)^{1/2} \\
&\leq Ch^k\|w\|_{k+2}\,. \qquad (4.42)
\end{aligned}$$

By the definition of the norm $\|\cdot\|_{-1,*}$ there exists a function $\boldsymbol{\eta}\in\boldsymbol{V}_*$ such that

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*}\leq(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta})\,,\qquad \|\boldsymbol{\eta}\|_1\leq C\,. \qquad (4.43)$$

Using a Clément type interpolant we can find a piecewise linear function $\boldsymbol{\eta}_I\in\boldsymbol{V}_*$ such that, recalling also (4.43), it holds

$$h_K^{s-1}\|\boldsymbol{\eta}-\boldsymbol{\eta}_I\|_{s,K}\leq C\|\boldsymbol{\eta}\|_{1,K}\leq C'\,,\quad s=0,1 \qquad (4.44)$$

for all $K\in\mathcal{C}_h$. Using the Cauchy–Schwartz inequality, the bound (4.44) with $s=0$ and the definition (4.4) it follows

$$\begin{aligned}
(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta}) &= (\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta}-\boldsymbol{\eta}_I)+(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta}_I) \\
&\leq C\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,h}+(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta}_I)\,. \qquad (4.45)
\end{aligned}$$

Note that $\boldsymbol{\eta}_I$ is both in $\boldsymbol{V}_h$ and $\boldsymbol{V}_*$; moreover $\boldsymbol{L}\boldsymbol{\eta}_I=0$ on each single element K of $\mathcal{C}_h$. As a consequence, using (3.6),(3.11),(3.12) and Theorem 4.2, it follows

$$\begin{aligned}
&(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta}_I) \\
&= -a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta}_I)-\sum_{e\in\Gamma_{\mathrm{f},h}}([\nabla w_h-\boldsymbol{\beta}_h)]\cdot\boldsymbol{s},M_{ns}(\boldsymbol{\eta}_I))_e \\
&=: T_1+T_2\,. \qquad (4.46)
\end{aligned}$$

Due to the continuity of the bilinear form and using bound (4.44) with $s=1$ it immediately follows

$$\begin{aligned}
T_1 &\leq C\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_1 \\
&\leq C|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h\,. \qquad (4.47)
\end{aligned}$$

Using first the Cauchy–Schwartz inequality, then the Agmon inequality, finally the bound (4.44) with $s=1$, Lemma 3.1 and the definition (4.3), we get

$$\begin{aligned}
T_2 &\leq \Big(\sum_{e\in\Gamma_{\mathrm{f},h}} h_e^{-1}\|\nabla w_h-\boldsymbol{\beta}_h)\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\Gamma_{\mathrm{f},h}} h_e\|M_{ns}(\boldsymbol{\eta}_I)\|_{0,e}^2\Big)^{1/2} \\
&\leq \Big(\sum_{K\in\mathcal{C}_h} h_K^{-2}\|\nabla w_h-\boldsymbol{\beta}_h)\|_{0,K}^2\Big)^{1/2}\|\boldsymbol{\eta}_I\|_1 \\
&\leq C|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h\,, \qquad (4.48)
\end{aligned}$$

where in the last inequality we implicitly used the relation $\nabla w - \boldsymbol{\beta} = 0$. Combining (4.43), (4.45) with (4.46), (4.47) and (4.48) it follows

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*} \leq C\big(\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,h} + |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h\big) . \tag{4.49}$$

Joining (4.49), (4.42) and using Theorem 4.3 the proposition immediately follows. □

Combining Theorem 4.3 and Lemma 4.3 with the regularity of the solution w finally grants the convergence of the method.

The regularity of the solution of the Kirchhoff plate problems for convex polygonal domains, with all the three main types of boundary conditions, is very case dependent. We refer for example to the work [21] where a rather complete study is accomplished. Note that, if $f \in H^{-1}(\Omega)$, in most cases of interest the regularity condition $w \in H^3(\Omega)$ is indeed achieved.

Note that with classical duality arguments and technical calculations it is also possible to derive the error bound

$$\|w - w_h\|_0 \leq Ch^{k+1}\|w\|_{k+1} , \tag{4.50}$$

if the regularity estimate

$$\|w\|_3 \leq C\|f\|_0 \tag{4.51}$$

holds, and

$$\|w - w_h\|_1 \leq Ch^{k+1}\|w\|_{k+1} , \tag{4.52}$$

with the regularity estimate

$$\|w\|_3 \leq C\|f\|_{-1} . \tag{4.53}$$

Moreover, if $k \geq 2$ and the regularity estimate

$$\|w\|_4 \leq C\|f\|_0 \tag{4.54}$$

is satisfied, then the improved estimate holds:

$$\|w - w_h\|_0 \leq Ch^{k+2}\|w\|_{k+2} . \tag{4.55}$$

5 A-posteriori error estimates

In this section we prove the reliability and the efficiency for an a-posteriori error estimator for our method. To this end, we introduce

$$\tilde{\eta}_K^2 := h_K^4\|f_h + \operatorname{div} \boldsymbol{q}_h\|_{0,K}^2 + h_K^{-2}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2 , \tag{5.1}$$

$$\eta_e^2 := h_e^3\|[\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!]\|_{0,e}^2 + h_e\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e}^2 , \tag{5.2}$$

$$\eta_{s,e}^2 := h_e\|M_{nn}(\boldsymbol{\beta}_h)\|_{0,e}^2 , \tag{5.3}$$

$$\eta_{f,e}^2 := h_e\|M_{nn}(\boldsymbol{\beta}_h)\|_{0,e}^2 + h_e^3\|\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h \cdot \boldsymbol{n}\|_{0,e}^2 , \tag{5.4}$$

where f_h is some approximation of the load f, h_e denotes the length of the edge e and $[\![\cdot]\!]$ represents the jump operator (which is assumed to be equal to the function value on boundary edges). Here the normal unit vector $\boldsymbol{n}$ is fixed for each edge e.

Given any element $K \in \mathcal{C}_h$, let the local error indicator be

$$\eta_K := \Big(\tilde{\eta}_K^2 + \frac{1}{2}\sum_{e\in\Gamma_{\mathrm{i},h}\cap\partial K} \eta_e^2 + \sum_{e\in\Gamma_{\mathrm{s},h}\cap\partial K} \eta_{s,e}^2 + \sum_{e\in\Gamma_{\mathrm{f},h}\cap\partial K} \eta_{f,e}^2\Big)^{1/2}, \tag{5.5}$$

where $\Gamma_{\mathrm{i},h}$ represents the set of all the internal edges, while $\Gamma_{\mathrm{c},h}, \Gamma_{\mathrm{s},h}$ and $\Gamma_{\mathrm{f},h}$ represent the sets of all the boundary edges in $\Gamma_{\mathrm{c}}, \Gamma_{\mathrm{s}}$ and Γ_{f}, respectively.

Finally, the global error indicator is defined as

$$\eta := \Big(\sum_{K\in\mathcal{C}_h} \eta_K^2\Big)^{1/2}. \tag{5.6}$$

5.1 Upper bounds

In order to derive the reliability of the method we need the following saturation assumption.

Assumption 5.1. *Given a mesh $\mathcal{C}_h$, let $\mathcal{C}_{h/2}$ be the mesh obtained by splitting each triangle $K \in \mathcal{C}_h$ into four triangles connecting the edge midpoints. Let $(w_{h/2}, \boldsymbol{\beta}_{h/2}, \boldsymbol{q}_{h/2})$ be the discrete solution corresponding to the mesh $\mathcal{C}_{h/2}$. We assume that there exists a constant ρ, $0 < \rho < 1$, such that*

$$\begin{aligned} &|||(w - w_{h/2}, \boldsymbol{\beta} - \boldsymbol{\beta}_{h/2})|||_{h/2} + \|\boldsymbol{q} - \boldsymbol{q}_{h/2}\|_{-1,*} \\ &\le \rho\big(|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*}\big), \end{aligned} \tag{5.7}$$

where by $|||\cdot|||_{h/2}$ we indicate the $|||\cdot|||_h$ norm with respect to the new mesh $\mathcal{C}_{h/2}$.

In the sequel we will also need the following lemma:

Lemma 5.1. *Let, for $v \in W_{h/2}$, the local seminorm be*

$$|v|_{2,h/2,K} = \Big(\sum_{K'\in\mathcal{C}_{h/2}\cap K} |v|_{2,K'}^2\Big)^{1/2}. \tag{5.8}$$

Then, there is a positive constant C such that for all $v \in W_{h/2}$ there exists $v_I \in W_h$ such that

$$\|v - v_I\|_{0,K} \le C h_K^2 |v|_{2,h/2,K} \qquad \forall K \in \mathcal{C}_h. \tag{5.9}$$

Moreover, v_I interpolates v at all the vertices of the triangulation $\mathcal{C}_{h/2}$.

Proof. We choose v_I as the only function in $H^1(\Omega)$ such that

$$
\begin{aligned}
&v_I|_K \in P_2(K)\ \forall K \in \mathcal{C}_h\,,\\
&v_I(N) = v(N)\ \forall N \in \mathcal{V}_{h/2}\,,
\end{aligned}
\tag{5.10}
$$

where $\mathcal{V}_{h/2}$ represents the set of all the vertices of $\mathcal{C}_{h/2}$. Note that it is trivial to check that $v_I \in W_h$ for all $k \geq 1$. Observing that

$$
|v|_{2,h/2,K} + \sum_{N \in \mathcal{V}_{h/2} \cap K} |v(N)|\,, \qquad v \in W_{h/2}\,, K \in \mathcal{C}_h\,,
\tag{5.11}
$$

is indeed a norm on the finite dimensional space of the functions $v \in W_{h/2}$ restricted to K, the result follows applying the classical scaling argument. □

For simplicity, in the sequel we will treat the case $\Gamma_s = \emptyset$, the general case following with identical arguments as the ones that follow. We have the following preliminary result:

Theorem 5.1. *It holds*

$$
|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2} \leq C\Big(\sum_{K \in \mathcal{C}_h} \eta_K^2 + h_K^4 \|f - f_h\|_{0,K}^2\Big)^{1/2}.
\tag{5.12}
$$

Proof. Step 1. Due to the stability of the discrete formulation, proved in Theorem 4.1, there exists a couple $(v, \boldsymbol{\eta}) \in W_{h/2} \times \boldsymbol{V}_{h/2}$ such that

$$
|||(v, \boldsymbol{\eta})|||_{h/2} \leq C
\tag{5.13}
$$

and

$$
|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2} \leq \mathcal{A}_{h/2}(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h; v, \boldsymbol{\eta})\,.
\tag{5.14}
$$

We also have

$$
\mathcal{A}_{h/2}(w_{h/2}, \boldsymbol{\beta}_{h/2}; v, \boldsymbol{\eta}) = (f, v)\,.
\tag{5.15}
$$

Simple calculations and the definition (3.11) give

$$
\begin{aligned}
\mathcal{B}_{h/2}(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) &= a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_{h/2}} \alpha h_K^2 (\boldsymbol{L}\boldsymbol{\beta}_h, \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + \sum_{K \in \mathcal{C}_{h/2}} \frac{1}{\alpha h_K^2} (\nabla w_h - \boldsymbol{\beta}_h - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_{h/2}} (\nabla w_h - \boldsymbol{\beta}_h, \boldsymbol{L}\boldsymbol{\eta})_K + \sum_{K \in \mathcal{C}_{h/2}} (\boldsymbol{q}_h, \nabla v - \boldsymbol{\eta})_K \\
&\quad + R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \\
&= \mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) + R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta})\,,
\end{aligned}
\tag{5.16}
$$

where $\boldsymbol{q}_h$ is defined as in (3.11), i.e. based on the coarser mesh, and

$$
\begin{aligned}
R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = & \sum_{K \in \mathcal{C}_{h/2}} \frac{1}{\alpha h_K^2} (\nabla w_h - \boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta})_K \\
- \sum_{K \in \mathcal{C}_h} \frac{1}{\alpha h_K^2} & (\nabla w_h - \boldsymbol{\beta}_h, \nabla v - \boldsymbol{\eta})_K \,.
\end{aligned} \tag{5.17}
$$

Note that the last term on the right hand side is well defined since $\nabla v - \boldsymbol{\eta}$ is piecewise L^2.

Let now $\mathcal{T}_h$, $\mathcal{T}_{h/2}$ indicate respectively the set of all the edges of $\mathcal{C}_h$ and $\mathcal{C}_{h/2}$, while $\Gamma_{\mathrm{f},h}$, $\Gamma_{\mathrm{f},h/2}$ represent the subset of all the edges on Γ_{f}. Adding and subtracting the difference between the two forms it then follows

$$
\mathcal{D}_{h/2}(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = \mathcal{D}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) + R'(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \,, \tag{5.18}
$$

where

$$
\begin{aligned}
R'(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = & \sum_{e \in \Gamma_{\mathrm{f},h/2}} \frac{\gamma}{h_e} ([\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s})_e \\
- \sum_{e \in \Gamma_{\mathrm{f},h}} \frac{\gamma}{h_e} & ([\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla v - \boldsymbol{\eta}] \cdot \boldsymbol{s})_e \,,
\end{aligned} \tag{5.19}
$$

and where the first member on the right hand side is indeed well defined because of the piecewise regularity of $(v, \boldsymbol{\eta})$.

Step 2. Finally, let $v_I \in W_h$ be the interpolant of v described in Lemma 5.1, and $\boldsymbol{\eta}_I \in \boldsymbol{V}_h$ the classical piecewise linear node interpolant of $\boldsymbol{\eta}$ for the mesh $\mathcal{C}_h$. Joining (5.15), (5.16), (5.18) and using

$$
\begin{aligned}
\mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v_I, \boldsymbol{\eta}_I) + \mathcal{D}_h(w_h, \boldsymbol{\beta}_h; v_I, \boldsymbol{\eta}_I) &= \mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v_I, \boldsymbol{\eta}_I) \\
&= (f, v_I) \,,
\end{aligned} \tag{5.20}
$$

some simple algebra gives

$$
\begin{aligned}
& \mathcal{A}_{h/2}(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \\
& = (f, v - v_I) - \mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v - v_I, \boldsymbol{\eta} - \boldsymbol{\eta}_I) \\
& \quad - \mathcal{D}_h(w_h, \boldsymbol{\beta}_h; v - v_I, \boldsymbol{\eta} - \boldsymbol{\eta}_I) - R(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \\
& \quad - R'(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \,.
\end{aligned} \tag{5.21}
$$

From the definitions (3.8), (2.8), (3.11), (2.10) and (2.11), integrating by parts on each element $K \in \mathcal{C}_h$ and rearranging the terms we obtain

$$
\begin{aligned}
&(f, v - v_I) - \mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v - v_I, \boldsymbol{\eta} - \boldsymbol{\eta}_I) \\
&= (f, v - v_I) - a(\boldsymbol{\beta}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_I) \\
&\quad + \sum_{K \in \mathcal{C}_h} (\nabla w_h - \boldsymbol{\beta}_h, \boldsymbol{L}(\boldsymbol{\eta} - \boldsymbol{\eta}_I))_K - \sum_{K \in \mathcal{C}_h} (\boldsymbol{q}_h, \nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I))_K \\
&= \sum_{K \in \mathcal{C}_h} \Big((f + \operatorname{div} \boldsymbol{q}_h, v - v_I)_K + (\boldsymbol{q}_h + \boldsymbol{L}\boldsymbol{\beta}_h, \boldsymbol{\eta} - \boldsymbol{\eta}_I)_K \\
&\quad - (\nabla w_h - \boldsymbol{\beta}_h, \boldsymbol{L}(\boldsymbol{\eta} - \boldsymbol{\eta}_I))_K \Big) \\
&\quad + \sum_{e \in \Gamma_{\mathrm{i},h}} \Big(([\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!], \boldsymbol{\eta} - \boldsymbol{\eta}_I)_e + ([\![\boldsymbol{q}_h \cdot \boldsymbol{n}]\!], v - v_I)_e \Big) \\
&\quad + \sum_{e \in \Gamma_{\mathrm{f},h}} \Big((\boldsymbol{q}_h \cdot \boldsymbol{n}, v - v_I)_e + (\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}, \boldsymbol{\eta} - \boldsymbol{\eta}_I)_e \Big) . \qquad (5.22)
\end{aligned}
$$

In order to treat the boundary pieces we need two observations. First, note that integration by parts along the boundary edges gives

$$
\sum_{e \in \Gamma_{\mathrm{f},h}} (M_{ns}(\boldsymbol{\beta}_h), \nabla(v - v_I) \cdot \boldsymbol{s})_e = - \sum_{e \in \Gamma_{\mathrm{f},h}} (\frac{\partial}{\partial \boldsymbol{s}} M_{ns}(\boldsymbol{\beta}_h), v - v_I) , \qquad (5.23)
$$

where there are no additional terms because $v_I(N) = v(N)$ for all the vertices N of $\mathcal{C}_h$. Second, note that the last term in (5.22) can be splitted as

$$
\begin{aligned}
&\sum_{e \in \Gamma_{\mathrm{f},h}} (\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}, \boldsymbol{\eta} - \boldsymbol{\eta}_I)_e \\
&= \sum_{e \in \Gamma_{\mathrm{f},h}} \Big((M_{nn}(\boldsymbol{\beta}_h), (\boldsymbol{\eta} - \boldsymbol{\eta}_I) \cdot \boldsymbol{n})_e + (M_{ns}(\boldsymbol{\beta}_h), (\boldsymbol{\eta} - \boldsymbol{\eta}_I) \cdot \boldsymbol{s})_e \Big) \qquad (5.24)
\end{aligned}
$$

As a consequence, applying (5.23) and (5.24) we obtain

$$
\begin{aligned}
&\sum_{e \in \Gamma_{\mathrm{f},h}} \Big((\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}, \boldsymbol{\eta} - \boldsymbol{\eta}_I)_e + (\boldsymbol{q}_h \cdot \boldsymbol{n}, v - v_I)_e \Big) + \mathcal{D}_h(w_h, \boldsymbol{\beta}_h; v - v_I, \boldsymbol{\eta} - \boldsymbol{\eta}_I) \\
&= \sum_{e \in \Gamma_{\mathrm{f},h}} \Big((M_{nn}(\boldsymbol{\beta}_h), (\boldsymbol{\eta} - \boldsymbol{\eta}_I) \cdot \boldsymbol{n})_e - (\frac{\partial}{\partial \boldsymbol{s}} M_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h \cdot \boldsymbol{n}, v - v_I)_e \\
&\quad + ([\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, M_{ns}(\boldsymbol{\eta} - \boldsymbol{\eta}_I))_e \\
&\quad + \frac{\gamma}{h_e} ([\nabla w_h - \boldsymbol{\beta}_h] \cdot \boldsymbol{s}, [\nabla(v - v_I) - (\boldsymbol{\eta} - \boldsymbol{\eta}_I)] \cdot \boldsymbol{s})_e \Big) . \qquad (5.25)
\end{aligned}
$$

Combining (5.21), (5.22) and (5.25) we have

$$
\begin{aligned}
&\mathcal{A}_{h/2}(w_{h/2}-w_h,\boldsymbol{\beta}_{h/2}-\boldsymbol{\beta}_h;v,\boldsymbol{\eta})\\
&=\sum_{K\in\mathcal{C}_h}\Big((f+\operatorname{div}\boldsymbol{q}_h,v-v_I)_K+(\boldsymbol{q}_h+\boldsymbol{L}\boldsymbol{\beta}_h,\boldsymbol{\eta}-\boldsymbol{\eta}_I)_K\\
&\quad-(\nabla w_h-\boldsymbol{\beta}_h,\boldsymbol{L}(\boldsymbol{\eta}-\boldsymbol{\eta}_I))_K\Big)\\
&\quad+\sum_{e\in\Gamma_{\mathrm{i},h}}\Big((\llbracket\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}\rrbracket,\boldsymbol{\eta}-\boldsymbol{\eta}_I)_e+(\llbracket\boldsymbol{q}_h\cdot\boldsymbol{n}\rrbracket,v-v_I)_e\Big)\\
&\quad+\sum_{e\in\Gamma_{\mathrm{f},h}}\Big((M_{nn}(\boldsymbol{\beta}_h),(\boldsymbol{\eta}-\boldsymbol{\eta}_I)\cdot\boldsymbol{n})_e\\
&\quad-(\frac{\partial}{\partial\boldsymbol{s}}M_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n},v-v_I)_e+([\nabla w_h-\boldsymbol{\beta}_h]\cdot\boldsymbol{s},M_{ns}(\boldsymbol{\eta}-\boldsymbol{\eta}_I))_e\\
&\quad+\frac{\gamma}{h_e}([\nabla w_h-\boldsymbol{\beta}_h]\cdot\boldsymbol{s},[\nabla(v-v_I)-(\boldsymbol{\eta}-\boldsymbol{\eta}_I)]\cdot\boldsymbol{s})_e\Big)\\
&\quad-R(w_h,\boldsymbol{\beta}_h;v,\boldsymbol{\eta})-R'(w_h,\boldsymbol{\beta}_h;v,\boldsymbol{\eta})\,.
\end{aligned}
\tag{5.26}
$$

Step 3. Finally, we have to bound all the terms above. We first treat the last two addenda (see (5.17) and (5.19) for the definitions). First recalling that $\mathcal{C}_{h/2}$ is a subdivision of $\mathcal{C}_h$, then from the Hölder inequality and finally from (4.3),(5.13) it follows

$$
\begin{aligned}
&|R(w_h,\boldsymbol{\beta}_h;v,\boldsymbol{\eta})|\le 2|\sum_{K\in\mathcal{C}_{h/2}}\frac{1}{\alpha h_K^2}(\nabla w_h-\boldsymbol{\beta}_h,\nabla v-\boldsymbol{\eta})_K|\\
&\le C\Big(\sum_{K\in\mathcal{C}_{h/2}}\frac{1}{h_K^2}\|\nabla w_h-\boldsymbol{\beta}_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_{h/2}}\frac{1}{h_K^2}\|\nabla v-\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2}\\
&\le C\Big(\sum_{K\in\mathcal{C}_{h/2}}\frac{1}{h_K^2}\|\nabla w_h-\boldsymbol{\beta}_h\|_{0,K}^2\Big)^{1/2}.
\end{aligned}
\tag{5.27}
$$

Using the Agmon inequality with arguments similar to those already adopted in (5.27) it can be checked that

$$
|R'(w_h,\boldsymbol{\beta}_h;v,\boldsymbol{\eta})|\le\Big(\sum_{K\in\mathcal{C}_{h/2}}\frac{1}{h_K^2}\|\nabla w_h-\boldsymbol{\beta}_h\|_{0,K}^2\Big)^{1/2}.
\tag{5.28}
$$

Combining (5.27) and (5.28) we get

$$
|-R(w_h,\boldsymbol{\beta}_h;v,\boldsymbol{\eta})-R'(w_h,\boldsymbol{\beta}_h;v,\boldsymbol{\eta})|\ \le C\Big(\sum_{K\in\mathcal{C}_h}\tilde{\eta}_K^2\Big)^{1/2}.
\tag{5.29}
$$

For the other terms in (5.26), we show in detail only a couple of examples, because the remaining cases easily follow applying the same arguments. We start by observing that from the definitions (5.8), (4.2), (4.3) and the bound (5.13) it follows immediately that

$$
\sum_{K\in\mathcal{C}_h}|v|_{2,h/2,K}^2\ \le\ C|||(v,\boldsymbol{\eta})|||_{h/2}\le C'\,.
\tag{5.30}
$$

We have

$$\sum_{K\in\mathcal{C}_h}(f+\operatorname{div}\boldsymbol{q}_h, v-v_I)_K$$
$$=\sum_{K\in\mathcal{C}_h}(f_h+\operatorname{div}\boldsymbol{q}_h, v-v_I)_K+(f-f_h, v-v_I)_K\,. \tag{5.31}$$

Recalling Lemma 5.1 and using (5.30) we get

$$\sum_{K\in\mathcal{C}_h}(f_h+\operatorname{div}\boldsymbol{q}_h, v-v_I)_K$$
$$\leq\Big(\sum_{K\in\mathcal{C}_h}h_K^4\|f_h+\operatorname{div}\boldsymbol{q}_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_h}h_K^{-4}\|v-v_I\|_{0,K}^2\Big)^{1/2}$$
$$\leq C\Big(\sum_{K\in\mathcal{C}_h}h_K^4\|f_h+\operatorname{div}\boldsymbol{q}_h\|_{0,K}\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_h}|v|_{2,h/2,K}^2\Big)^{1/2}$$
$$\leq C\Big(\sum_{K\in\mathcal{C}_h}\tilde{\eta}_K^2\Big)^{1/2}. \tag{5.32}$$

The same argument gives

$$\sum_{K\in\mathcal{C}_h}(f-f_h, v-v_I)_K\leq\Big(\sum_{K\in\mathcal{C}_h}h_K^4\|f-f_h\|_{0,K}^2\Big)^{1/2}, \tag{5.33}$$

which, joined with (5.31) and (5.32), bounds the first term in (5.26).

First using the Agmon and the inverse inequality, then again applying Lemma 5.1 and (5.30), we have

$$\sum_{e\in\Gamma_{\mathrm{f},h}}(\frac{\partial}{\partial\boldsymbol{s}}M_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}, v-v_I)_e$$
$$\leq\Big(\sum_{e\in\Gamma_{\mathrm{f},h}}h_e^3\|\frac{\partial}{\partial\boldsymbol{s}}M_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\Gamma_{\mathrm{f},h}}h_e^{-3}\|v-v_I\|_{0,e}^2\Big)^{1/2}$$
$$\leq C\Big(\sum_{e\in\Gamma_{\mathrm{f},h}}\eta_{f,e}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{C}_h}h_e^{-4}\|v-v_I\|_{0,K}^2\Big)^{1/2}$$
$$\leq C\Big(\sum_{e\in\Gamma_{\mathrm{f},h}}\eta_{f,e}^2\Big)^{1/2}. \tag{5.34}$$

The remaining terms are bounded using the same techniques.

It is worth noting that, by the definition (3.11),

$$(\boldsymbol{q}_h+\boldsymbol{L}\boldsymbol{\beta}_h)_{|K}=\frac{1}{\alpha h_k^2}(\nabla w_h-\boldsymbol{\beta}_h)_{|K}\qquad\forall K\in\mathcal{C}_h\,, \tag{5.35}$$

which is the reason why there appears no terms of the kind $\|\boldsymbol{q}_h+L\boldsymbol{\beta}_h\|_{0,K}$ in the error estimator. Note also that the Agmon and the inverse inequality easily give

$$\sum_{e\in\Gamma_{\mathrm{f},h}}h_e^{-1}\|\nabla w_h-\boldsymbol{\beta}_h\|_{0,e}^2\leq C\sum_{K\in\mathcal{C}_h}h_K^{-2}\|\nabla w_h-\boldsymbol{\beta}_h\|_{0,K}^2\,, \tag{5.36}$$

which is the reason why there appears no boundary terms of the kind $\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,e}$. We finally get

$$\mathcal{A}_{h/2}(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) \leq C\big(\eta_K^2 + h_K^4 \|f - f_h\|_{0,K}^2\big)^{1/2}, \quad (5.37)$$

which combined with (5.14) proves the proposition. □

We also have the following lemma for the shear stress:

Lemma 5.2. *It holds*

$$\|\boldsymbol{q}_{h/2} - \boldsymbol{q}_h\|_{-1,*}^2 \leq C\Big(|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2}^2 + \sum_{K \in \mathcal{C}_h} \tilde{\eta}_K^2\Big). \quad (5.38)$$

Proof. We start observing that, referring to the definition (3.11) and its "$h/2$" counterpart, $\boldsymbol{q}_h$ and $\boldsymbol{q}_{h/2}$ are defined on different meshes and therefore with different h_K^2 coefficients. However, recalling that the size ratio between the two meshes is bounded, it is easy to check that an opportune splitting and the triangle inequality give

$$\begin{aligned}\|\boldsymbol{q}_{h/2} - \boldsymbol{q}_h\|_{-1,h}^2 \leq C\Big(&\sum_{K \in \mathcal{C}_{h/2}} \|\nabla(w_{h/2} - w_h) - (\boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)\|_{0,K}^2 \\ &+ \sum_{K \in \mathcal{C}_h} \|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2 + \sum_{K \in \mathcal{C}_{h/2}} h_K^2 \|\boldsymbol{L}\boldsymbol{\beta}_{h/2} - \boldsymbol{L}\boldsymbol{\beta}_h\|_{0,K}^2\Big). \quad (5.39)\end{aligned}$$

The first and the last term in (5.39) can be bounded in terms of the $|||\cdot|||_{h/2}$ norm, simply using the definition (4.3) and the inverse inequality

$$h_K^2 \|\boldsymbol{L}\boldsymbol{\beta}_{h/2} - \boldsymbol{L}\boldsymbol{\beta}_h\|_{0,K}^2 \leq C\|\boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h\|_{1,K}^2. \quad (5.40)$$

Therefore, recalling the definition (5.1), we get

$$\|\boldsymbol{q}_{h/2} - \boldsymbol{q}_h\|_{-1,h}^2 \leq C|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2}^2 + \sum_{K \in \mathcal{C}_h} \tilde{\eta}_K^2. \quad (5.41)$$

The transition from the $\|\boldsymbol{q}_{h/2} - \boldsymbol{q}_h\|_{-1,h}$ norm to the $\|\boldsymbol{q}_{h/2} - \boldsymbol{q}_h\|_{-1,*}$ norm is done using the "Pitkäranta-Verfürth trick" with steps almost identical to those used in Lemma 4.3, therefore omitted. □

Joining Theorem 5.1 and Lemma 5.2 gives the following a-posteriori upper bound for the method:

Theorem 5.2. *It holds*

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*} \leq C\Big(\sum_{K \in \mathcal{C}_h} \eta_K^2 + h_K^4 \|f - f_h\|_{0,K}^2\Big)^{1/2} \quad (5.42)$$

Proof. Theorem 5.1 combined with Lemma 5.2 trivially gives

$$\begin{aligned}&|||(w_{h/2} - w_h, \boldsymbol{\beta}_{h/2} - \boldsymbol{\beta}_h)|||_{h/2} + \|\boldsymbol{q}_{h/2} - \boldsymbol{q}_h\|_{-1,*} \\ &\leq C\Big(\sum_{K \in \mathcal{C}_h} \eta_K^2 + h_K^4 \|f - f_h\|_{0,K}^2\Big)^{1/2}, \quad (5.43)\end{aligned}$$

which, by recalling the saturation assumption, proves the theorem. □

5.2 Lower bounds

In this section we prove the efficiency of the error estimator. Given any edge e of the triangulation, we define ω_e as the set of all the triangles $K \in \mathcal{C}_h$ that have e as an edge. Given any $K \in \mathcal{C}_h$, we define ω_K as the set of all the triangles $K \in \mathcal{C}_h$ that share an edge with K. We then have the following lemma [20]:

Lemma 5.3. *Given any edge e of the triangulation $\mathcal{C}_h$, let $P_k(e)$ be the space of polynomials of degree at most k on e. There exists a linear operator*

$$\Pi_e \,:\, P_k(e) \longrightarrow H^2_0(\omega_e) \tag{5.44}$$

such that for all $p_k \in P_k(e)$ it holds

$$C_1 \|p_k\|^2_{0,e} \leq (p_k, \Pi_e(p_k))_e \leq \|p_k\|^2_{0,e}\,, \tag{5.45}$$

$$\|\Pi_e(p_k)\|_{0,\omega_e} \leq C_2 h_e^{1/2} \|p_k\|_{0,e}\,, \tag{5.46}$$

where the positive constants C_i above depend only on k and the minimum angle of the triangles in $\mathcal{C}_h$.

We have the following efficiency result:

Theorem 5.3. *It holds*

$$\eta_K \leq |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_{h,\omega_K} + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*,\omega_K} + h_K^2 \|f - f_h\|_{0,\omega_K}\,, \tag{5.47}$$

where $|||\cdot|||_{h,\omega_K}$, $\|\cdot\|_{-1,*,\omega_K}$, $\|\cdot\|_{0,\omega_K}$ represent respectively the norms $|||\cdot|||_h$, $\|\cdot\|_{-1,*}$, $\|\cdot\|_0$ restricted to the domain ω_K.

Proof. Step 1. The proof consists of bounding separately all the addenda in η_K. First we bound the terms of $\tilde{\eta}_K^2$ in (5.1). Given any $K \in \mathcal{C}_h$, let b_K indicate the standard third order polynomial bubble on K, scaled such that $\|b_K\|_{L^\infty(K)} = 1$. Given $K \in \mathcal{C}_h$, let now $\varphi_K \in H^2_0(K)$ be defined as

$$\varphi_K = (f_h + \operatorname{div} \boldsymbol{q}_h)\, b_K^2\,. \tag{5.48}$$

The standard scaling arguments then easily show that

$$\|f_h + \operatorname{div} \boldsymbol{q}_h\|^2_{0,K} \leq C(f_h + \operatorname{div} \boldsymbol{q}_h, \varphi_K)_K\,, \tag{5.49}$$

$$\|\varphi_K\|_{0,K} \leq C\|f_h + \operatorname{div} \boldsymbol{q}_h\|_{0,K}\,. \tag{5.50}$$

Recalling (2.9), integration by parts gives

$$\begin{aligned} & h_K^2 \|f_h + \operatorname{div} \boldsymbol{q}_h\|^2_{0,K} \leq C h_K^2 (f_h + \operatorname{div} \boldsymbol{q}_h, \varphi_K)_K \\ & = C h_K^2 \big((f + \operatorname{div} \boldsymbol{q}_h, \varphi_K)_K + (f_h - f, \varphi_K)_K \big) \\ & = C h_K^2 \big((-\operatorname{div} \boldsymbol{q} + \operatorname{div} \boldsymbol{q}_h, \varphi_K)_K + (f_h - f, \varphi_K)_K \big) \\ & \leq C h_K^2 \big((\boldsymbol{q}_h - \boldsymbol{q}, \nabla \varphi_K)_K + (f_h - f, \varphi_K)_K \big)\,. \end{aligned} \tag{5.51}$$

Note that in particular $\nabla\varphi_K \in \boldsymbol{V}_*$ (see the definition (4.5)). Therefore the duality inequality and the Cauchy–Schwartz inequality, followed by the inverse inequality and the bound (5.50) lead to the estimate

$$\begin{aligned}
&Ch_K^2\big((\boldsymbol{q}_h - \boldsymbol{q}, \nabla\varphi_K)_K + (f_h - f, \varphi_K)\big) \\
&\le C\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*,K}\, h_K^2\|\nabla\varphi_K\|_{1,K} + Ch_K^2\|f - f_h\|_{0,K}\|\varphi_K\|_{0,K} \\
&\le C\big(\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*,K} + h_K^2\|f - f_h\|_{0,K}\big)\|f_h + \operatorname{div}\boldsymbol{q}_h\|_{0,K}\,. \qquad (5.52)
\end{aligned}$$

Combining (5.51) with (5.52) finally gives

$$h_K^2\|f_h + \operatorname{div}\boldsymbol{q}_h\|_{0,K} \le C\big(\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,*,K} + h_K^2\|f - f_h\|_{0,K}\big)\,. \qquad (5.53)$$

The second term in (5.1) can be bounded directly using $(2.9)_3$

$$\begin{aligned}
h_K^{-1}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K} &= h_K^{-1}\|\nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{0,K}^2 \\
&\le |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_{h,K}\,. \qquad (5.54)
\end{aligned}$$

Step 2. Second we bound the terms of η_e^2 in (5.2). Given now e in the set $\Gamma_{\mathrm{i},h}$ of the internal edges of $\mathcal{C}_h$, let

$$\boldsymbol{\varphi}_e = \Pi_e([\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!])\,, \qquad (5.55)$$

where, with a little abuse of notation, the operator Π_e is intended as applied on each single component. Then, from (5.45) and with integration by parts, it follows

$$\begin{aligned}
&h_e^{1/2}\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e}^2 \\
&\le Ch_e^{1/2}([\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!], \boldsymbol{\varphi}_e)_e \\
&= Ch_e^{1/2}\big((\boldsymbol{L}\boldsymbol{\beta}_h, \boldsymbol{\varphi}_e)_{\omega_e} + (\boldsymbol{M}(\boldsymbol{\beta}_h), \nabla\boldsymbol{\varphi}_e)_{\omega_e}\big)\,, \qquad (5.56)
\end{aligned}$$

where we recall that ω_e was defined at the start of this section. Integration by parts and the first identity in (2.9) immediately yield

$$\begin{aligned}
(\boldsymbol{M}(\boldsymbol{\beta}), \nabla\boldsymbol{\varphi}_e)_{\omega_e} &= -(\boldsymbol{L}\boldsymbol{\beta}, \boldsymbol{\varphi}_e)_{\omega_e} \\
&= (\boldsymbol{q}, \boldsymbol{\varphi}_e)_{\omega_e}\,, \qquad (5.57)
\end{aligned}$$

which, used in (5.56), gives

$$\begin{aligned}
&h_e^{1/2}\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e}^2 \\
&\le Ch_e^{1/2}\big((\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}, \boldsymbol{\varphi}_e)_{\omega_e} + (\boldsymbol{M}(\boldsymbol{\beta}_h) - \boldsymbol{M}(\boldsymbol{\beta}), \nabla\boldsymbol{\varphi}_e)_{\omega_e}\big) \\
&= Ch_e^{1/2}\big((\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{\varphi}_e)_{\omega_e} + (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\varphi}_e)_{\omega_e} \\
&\quad + (\boldsymbol{M}(\boldsymbol{\beta}_h) - \boldsymbol{M}(\boldsymbol{\beta}), \nabla\boldsymbol{\varphi}_e)_{\omega_e}\big)\,. \qquad (5.58)
\end{aligned}$$

We now bound the three terms on the right hand side of (5.58). The identity (5.35), the Cauchy–Schwartz inequality, the definition (5.55) and the bound (5.46) give

$$\begin{aligned}
&h_e^{1/2}(\boldsymbol{L}\boldsymbol{\beta}_h + \boldsymbol{q}_h, \boldsymbol{\varphi}_e)_{\omega_e} \\
&\le C\Big(\sum_{K\in\mathcal{C}_h\cap\omega_e} h_K^{-2}\|\nabla w_h - \boldsymbol{\beta}_h\|_{0,K}^2\Big)^{1/2}\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e} \\
&\le C\,|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_{h,\omega_e}\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e}\,. \qquad (5.59)
\end{aligned}$$

Note that in particular $\boldsymbol{\varphi}_e \in \boldsymbol{V}_*$ (see the definition (4.5)). Therefore, first the duality inequality, then the definition (5.55) combined with the bound (5.46) and the inverse inequality give

$$
\begin{aligned}
h_e^{1/2}(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\varphi}_e)_{\omega_e} &\le h_e^{1/2}\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_e}\|\boldsymbol{\varphi}_e\|_{1,\omega_e} \\
&\le C\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_e}\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e}\,. \qquad (5.60)
\end{aligned}
$$

Now again the Cauchy–Schwartz inequality, then the inverse inequality and finally (5.55) combined with the bound (5.46) lead to the estimate

$$
\begin{aligned}
&h_e^{1/2}(\boldsymbol{M}(\boldsymbol{\beta}_h)-\boldsymbol{M}(\boldsymbol{\beta}),\nabla\boldsymbol{\varphi}_e)_{\omega_e} \le C\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{1,\omega_e}h_K^{-1/2}\|\boldsymbol{\varphi}_e\|_{0,\omega_e} \\
&\le C\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{1,\omega_e}\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e}\,. \qquad (5.61)
\end{aligned}
$$

Combining (5.59),(5.60) and (5.61) with (5.58) it follows

$$
h_e^{1/2}\|[\![\boldsymbol{M}(\boldsymbol{\beta}_h)\boldsymbol{n}]\!]\|_{0,e} \le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_e}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_e}\big), \qquad (5.62)
$$

while the remaining term in (5.2) is bounded with similar arguments:

$$
h_e^{3/2}\|[\![\boldsymbol{q}_h\cdot\boldsymbol{n}]\!]\|_{0,e} \le C\big(\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,K}+h_K^2\|f-f_h\|_{0,K}\big)\,. \qquad (5.63)
$$

Step 3. Third we bound the only term of $\eta_{s,e}^2$ in (5.3) which appears also in (5.4). Given now a triangulation edge e in $\Gamma_{\mathrm{f},h}\cup\Gamma_{\mathrm{s},h}$, let

$$
\varphi_e = \Pi_e(M_{nn}(\boldsymbol{\beta}_h))\,. \qquad (5.64)
$$

Due to (5.45) and $(2.9)_6$ integration by parts gives

$$
\begin{aligned}
&h_e^{1/2}\|M_{nn}(\boldsymbol{\beta}_h)\|_{0,e}^2 \le h_e^{1/2}(M_{nn}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_e)_e \\
&= h_e^{1/2}(\boldsymbol{M}_n(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_e\boldsymbol{n})_e \\
&= h_e^{1/2}\big((\boldsymbol{M}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\nabla(\varphi_e\boldsymbol{n}))_{\omega_e}+(\boldsymbol{L}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_e\boldsymbol{n})_{\omega_e}\big)\,, \qquad (5.65)
\end{aligned}
$$

where $\boldsymbol{n}$ is, as usual, the chosen normal unit vector to e. Using the Cauchy–Schwartz inequality, then the inverse inequality and finally the bound (5.46) we easily get

$$
\begin{aligned}
h_e^{1/2}(\boldsymbol{M}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\nabla(\varphi_e\boldsymbol{n}))_{\omega_e} &\le h_e^{1/2}\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{1,\omega_e}\|\nabla(\varphi_e\boldsymbol{n})\|_{0,\omega_e} \\
&\le C\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{1,\omega_e}\|M_{nn}(\boldsymbol{\beta}_h)\|_{0,e}\,. \qquad (5.66)
\end{aligned}
$$

Recalling $(2.9)_1$, for the second term in (5.65), we have

$$
\begin{aligned}
&h_e^{1/2}(\boldsymbol{L}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_e\boldsymbol{n})_{\omega_e} \\
&= h_e^{1/2}(\boldsymbol{L}\boldsymbol{\beta}_h+\boldsymbol{q}_h,\varphi_e\boldsymbol{n})_{\omega_e}+h_e^{1/2}(\boldsymbol{q}-\boldsymbol{q}_h,\varphi_e\boldsymbol{n})_{\omega_e}\,. \qquad (5.67)
\end{aligned}
$$

Observing that $\varphi_e\boldsymbol{n}\in\boldsymbol{V}_*$, the two terms on the right hand side of (5.67) can be bounded with the same arguments used respectively in (5.59) and (5.60). Omitting the details, we therefore get

$$
\begin{aligned}
&h_e^{1/2}(\boldsymbol{L}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_e\boldsymbol{n})_{\omega_e} \le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_e} \\
&\quad +\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_e}\big)\|M_{nn}(\boldsymbol{\beta}_h)\|_{0,e}\,. \qquad (5.68)
\end{aligned}
$$

From (5.65), (5.66) and (5.68) we get

$$h_e^{1/2}\|M_{nn}(\boldsymbol{\beta}_h)\|_{0,e} \le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_e} + \|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_e}\big)\,. \quad (5.69)$$

Step 4. Finally we bound the last term of $\eta_{f,e}^2$ in (5.4). Given again a triangulation edge e in $\Gamma_{\mathrm{f},h}$, let

$$\varphi_e = \Pi_e\big(\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\big)\,. \quad (5.70)$$

Using (5.45) and recalling $(2.9)_6$ we obtain

$$\begin{aligned} &h_e^{3/2}\|\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\|_{0,e}^2 \\ &\le h_e^{3/2}\big((\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_e)_e + ((\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},\varphi_e)_e\big)\,. \end{aligned} \quad (5.71)$$

For the first term, integration by parts on the edge and simple algebra give

$$\begin{aligned} &h_e^{3/2}(\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h-\boldsymbol{\beta}),\varphi_e)_e = h_e^{3/2}(M_{ns}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_e\cdot\boldsymbol{s})_e \\ &= h_e^{3/2}\big((\boldsymbol{M}(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\boldsymbol{n},\nabla\varphi_e)_e - (M_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_e\cdot\boldsymbol{n})_e\big)\,. \end{aligned} \quad (5.72)$$

Using again integration by parts, the first term in (5.72) can be written as

$$\begin{aligned} &h_e^{3/2}(\boldsymbol{M}(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\boldsymbol{n},\nabla\varphi_e)_e \\ &= h_e^{3/2}\big((\boldsymbol{L}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_e)_{\omega_e} + (\boldsymbol{M}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\nabla\varphi_e)_{\omega_e}\big)\,. \end{aligned} \quad (5.73)$$

The second term in (5.71), again due to integration by parts and also recalling $(2.9)_2$, is instead equivalent to

$$\begin{aligned} h_e^{3/2}((\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},\varphi_e)_e &= h_e^{3/2}\big((\boldsymbol{q}-\boldsymbol{q}_h,\nabla\varphi_e)_{\omega_e} \\ &-(f_h+\operatorname{div}\boldsymbol{q}_h,\varphi_e)_{\omega_e} - (f-f_h,\varphi_e)_{\omega_e}\big)\,. \end{aligned} \quad (5.74)$$

For the first term, due to $(2.9)_1$ and (3.11), we now have

$$\begin{aligned} &h_e^{3/2}(\boldsymbol{q}-\boldsymbol{q}_h,\nabla\varphi_e)_{\omega_e} \\ &= h_e^{3/2}\big((\boldsymbol{L}\boldsymbol{\beta}_h-\boldsymbol{L}\boldsymbol{\beta},\nabla\varphi_e)_{\omega_e} - \frac{1}{\alpha h_{\omega_e}^2}(\nabla w_h-\boldsymbol{\beta}_h,\nabla\varphi_e)_{\omega_e}\big)\,, \end{aligned} \quad (5.75)$$

where h_{ω_e} is the size of the triangle ω_e. Combining all the identities from (5.71) to (5.75) it follows that

$$\begin{aligned} &h_e^{3/2}\|\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h) - \boldsymbol{q}_h\cdot\boldsymbol{n}\|_{0,e}^2 \\ &\le h_e^{3/2}\big((\boldsymbol{M}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\nabla\varphi_e)_{\omega_e} - (M_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_e\cdot\boldsymbol{n})_e \\ &\quad -\frac{1}{\alpha h_{\omega_e}^2}(\nabla w_h-\boldsymbol{\beta}_h,\nabla\varphi_e)_{\omega_e} - (f_h+\operatorname{div}\boldsymbol{q}_h,\varphi_e)_{\omega_e} \\ &\quad -(f-f_h,\varphi_e)_{\omega_e}\big)\,. \end{aligned} \quad (5.76)$$

For the second term on the right hand side of (5.76), recalling $(2.9)_6$, using the Cauchy–Schwartz inequality and due to the bound (5.69) we have

$$h_e^{3/2}(M_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_e\cdot\boldsymbol{n})_e \le h_e^{1/2}\|M_{nn}(\boldsymbol{\beta}_h)\|_{0,e}h_e\|\nabla\varphi_e\|_{0,e}$$
$$\le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_e}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_e}\big)h_e\|\nabla\varphi_e\|_{0,e}\,,\quad(5.77)$$

which, using the inverse inequality and the bound (5.46) gives

$$h_e^{3/2}(M_{nn}(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\nabla\varphi_e\cdot\boldsymbol{n})_e \le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_e}$$
$$+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,*,\omega_e}\big)\|\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}\|_{0,e}\,. \quad (5.78)$$

The remaining terms on the right hand side of (5.76) can all be bounded using the Cauchy–Schwartz inequality, the inverse inequality and the bounds (5.53), (5.46) as already shown for the similar previous cases. Without showing all the details, we finally get

$$h_e^{3/2}\|\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}\|_{0,e}^2$$
$$\le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_e}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,h,\omega_e}$$
$$+h_K^2\|f-f_h\|_{0,K}\big)\;|||_{h,\omega_e}\|\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}\|_{0,e}\,, \quad (5.79)$$

or, trivially,

$$h_e^{3/2}\|\frac{\partial}{\partial s}M_{ns}(\boldsymbol{\beta}_h)-\boldsymbol{q}_h\cdot\boldsymbol{n}\|_{0,e}$$
$$\le C\big(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_{h,\omega_e}+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,h,\omega_e}$$
$$+h_K^2\|f-f_h\|_{0,K}\big)\,. \quad (5.80)$$

The proposition is now proved. □

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ISSN 0784-3143