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Abstract: *The Schwartz kernel representation* $(A \mapsto K_A) : \mathcal{L}(\mathcal{H}) \to \mathcal{H}\widehat{\otimes}\mathcal{H}'$ *endows the space of continuous linear operators on a nuclear Hopf-Fréchet algebra* $\mathcal{H}$ *with a natural Hopf algebra structure. We study pseudodifferential symbolic calculus on a compact Lie group* G *related to the function Hopf algebra* $\mathcal{H} := \mathcal{D}(G)$.

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1 Introduction

Hopf algebras were introduced by Heinz Hopf in 1941, in the context of algebraic topology. Somewhat simplified, a Hopf algebra is an algebra for which the dual space is also an algebra so that the duality pairing is intertwined in a subtle symmetric way. Examples range from group algebras, their duals, and universal enveloping algebras to deformations of such structures.

The study of Hopf algebras can be considered as a kind of non-commutative geometry, stemming out from the 19th century observations of polynomial rings. Indeed, by the work of Hilbert and others, many commutative rings were realized as function algebras — recall the Gelfand theory of commutative Banach algebras culminating this line of thought in early 1940s: For a commutative unital C*-algebra $\mathcal{A}$, let $X := \mathrm{Hom}(\mathcal{A}, \mathbb{C})$. The Gelfand transform of $f \in \mathcal{A}$ is $\widehat{f} : X \to \mathbb{C}$ defined by $\widehat{f}(x) := x(f)$. If X is endowed with the weakest topology such that $\widehat{\mathcal{A}} := \{\widehat{f} \mid f \in \mathcal{A}\} \subset C(X)$ then it is a compact Hausdorff space and $C(X) = \widehat{\mathcal{A}} \cong \mathcal{A}$. That is, $\mathcal{A}$ is essentially an algebra of functions.

Hence, commutative algebra is closely related to geometry: instead of a space, we can study the function algebra on it. Non-commutative geometry is a concept referring to the study of not necessarily commutative algebras. We may associate C*-algebras to non-commutative topology (commutative case: $C(X)$), von Neumann algebras to non-commutative measure theory (commutative case: $L^\infty(X)$), and Lipschitz-algebras to non-commutative metric theory (commutative case: $\mathrm{Lip}(X)$) (see [7]).

What if $\mathrm{Hom}(\mathcal{A}, \mathbb{C})$ of a commutative C*-algebra $\mathcal{A}$ has a structure of a topological group? The group axioms give rise to operations on the algebra, and these new operations have natural symmetries reflected in the algebra; generalizing this to the non-commutative case, Hopf algebras arise. Nevertheless, Hopf algebras provide a satisfying duality theory for algebraic structures much more general than just groups. Purely algebraic Hopf theory is often spiced up with topology, and there are Hopf-von Neumann algebras, Hopf C*-algebras, Hopf-Fréchet algebras, etc.

In the sequel, we study Hopf algebras inspired by symbolic calculus of pseudodifferential operators on a compact Lie group G. There the Hopf algebra $\mathcal{H}$ is the nuclear Fréchet algebra $\mathcal{D}(G)$ of functions $f \in C^\infty(G)$; the pseudodifferential operators that map $\mathcal{H}$ to $\mathcal{H}$ form a subalgebra of $\mathcal{L}(\mathcal{H})$. Instead of studying an operator $A \in \mathcal{L}(\mathcal{H})$, we study its symbol $\sigma_A : G \to \mathcal{L}(\mathcal{H})$, in some sense a less complicated object. The symbol of a pseudodifferential operator composition is approximately the product of the symbols, $\sigma_{AB}(x) = \sigma_A(x)\sigma_B(x) + \ldots$, and the symbol of the adjoint operator is almost the adjoint of the original symbol, $\sigma_{A^*}(x) = \sigma_A(x)^* + \ldots$. Often just these first term approximations are studied, discarding the remainders. Thus a distorted composition $A \star B$ can be defined by $\sigma_{A\star B}(x) := \sigma_B(x)\sigma_A(x)$, and a distorted adjoint $A^\star$ by $\sigma_{A^\star}(x) := \sigma_A(x)^*$. But it turns out that $\mathcal{L}(\mathcal{H})$ has analogies of all the other Hopf operations as well.

2 Schwartz Kernel Theorem.

Schwartz Kernel Theorem [5]. For nuclear Fréchet spaces and their duals the projective and injective tensor products coincide, and in the sequel $\widehat{\otimes}$ refers to these topological tensor products. Let $\mathcal{H}$ be a nuclear Fréchet space and $A \in \mathcal{L}(\mathcal{H})$. The Schwartz kernel $K_A \in \mathcal{H}\widehat{\otimes}\mathcal{H}'$ of A is defined by

$$\langle A\phi, f\rangle =: \langle K_A, f \otimes \phi\rangle$$

for every $\phi \in \mathcal{H}$ and $f \in \mathcal{H}' := \mathcal{L}(\mathcal{H}, \mathbb{C})$, where the duality brackets are for $\mathcal{H} \times \mathcal{H}'$ and $(\mathcal{H}\widehat{\otimes}\mathcal{H}') \times (\mathcal{H}'\widehat{\otimes}\mathcal{H})$, respectively. Then the mapping

$$(A \mapsto K(A) = K_A) : \mathcal{L}(\mathcal{H}) \to \mathcal{H}\widehat{\otimes}\mathcal{H}'$$

is a continuous linear isomorphism, for which

$$K_{BAC} = (B \otimes C')K_A,$$

where $C' \in \mathcal{L}(\mathcal{H}')$ is the adjoint of C, defined by $\langle \phi, C'f\rangle := \langle C\phi, f\rangle$.

3 Hopf algebras

Basic treatises of Hopf algebras are [2] and [1]. We shall use the following convention: all the vector spaces encountered in this paper are over the complex field $\mathbb{C}$, and $V \otimes W$ denotes the tensor product of vector spaces. We shall constantly identify the vector spaces $\mathbb{C} \otimes V$ and $V \otimes \mathbb{C}$ with V by respective mappings $\mu \otimes v \mapsto \mu v$ and $v \otimes \mu \mapsto \mu v$. The identity mapping in a vector space is denoted by I. The linear interchanging operator $\tau : V \otimes W \to W \otimes V$ is defined by $\tau(v \otimes w) := w \otimes v$.

Algebra. An *algebra* $\mathcal{A} = (\mathcal{A}, m, \eta)$ consists of a vector space $\mathcal{A}$ with linear mappings $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ (the product or multiplication) and $\eta : \mathbb{C} \to \mathcal{A}$ (the unit), satisfying

$$m(m \otimes I) = m(I \otimes m)$$

(associativity of the product) and

$$m(\eta \otimes I) = I = m(I \otimes \eta)$$

(the unit of the algebra; notice the identifications $\mathbb{C} \otimes V = V = V \otimes \mathbb{C}$). We shall use the following abbreviations: $m(f \otimes g) = fg$ and $\eta(1) = \mathbb{I}$. Then the algebra axioms are written as $(fg)h = f(gh)$ $(= fgh)$ and $\mathbb{I}f = f = f\mathbb{I}$. The algebra is *commutative* if $m = m\tau$. If $\mathcal{A}, \mathcal{B}$ are algebras then there is a natural *tensor product algebra* $\mathcal{A} \otimes \mathcal{B}$ with the unit

$$\eta_{\mathcal{A}\otimes\mathcal{B}} := \eta_{\mathcal{A}} \otimes \eta_{\mathcal{B}}$$

and with the product defined by

$$m_{\mathcal{A}\otimes\mathcal{B}} := (m_{\mathcal{A}} \otimes m_{\mathcal{B}})(I \otimes \tau \otimes I),$$

i.e. $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2) \otimes (b_1b_2)$.

Co-algebra. A *co-algebra* $\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon)$ consists of a vector space $\mathcal{C}$ with linear mappings $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ (the co-product or co-multiplication) and $\varepsilon : \mathcal{C} \to \mathbb{C}$ (the co-unit) satisfying

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$$

(co-associativity of the co-product) and

$$(\varepsilon \otimes I)\Delta = I = (I \otimes \varepsilon)\Delta$$

(the co-unit of the co-product). Notice that the co-algebra axioms are obtained by inverting the arrows of the commutative diagrams given by the algebra axioms; this is a dual concept. Co-algebra is called *co-commutative* if $\Delta = \tau\Delta$. The *tensor product co-algebra* $\mathcal{C} \otimes \mathcal{D}$ of co-algebras $\mathcal{C}, \mathcal{D}$ is endowed with operations

$$\varepsilon_{\mathcal{C}\otimes\mathcal{D}} := \varepsilon_{\mathcal{C}} \otimes \varepsilon_{\mathcal{D}}$$

and

$$\Delta_{\mathcal{C}\otimes\mathcal{D}} := (I \otimes \tau \otimes I)(\Delta_{\mathcal{C}} \otimes \Delta_{\mathcal{D}}).$$

Now if $\mathcal{A}$ is an algebra and $\mathcal{C}$ is a co-algebra, we can define the *convolution* $A * B$ of operators $A, B \in \mathcal{L}(\mathcal{C}, \mathcal{A})$ by $A * B := m(A \otimes B)\Delta$.

Bi-algebra. A *bi-algebra* $\mathcal{B} = (\mathcal{B}, m, \eta, \Delta, \varepsilon)$ is an algebra $(\mathcal{B}, m, \eta)$ and a co-algebra $(\mathcal{B}, \Delta, \varepsilon)$ such that

$$\Delta(fg) = \Delta(f)\Delta(g)$$

(the co-product is multiplicative, or the product is co-multiplicative),

$$\varepsilon(fg) = \varepsilon(f)\varepsilon(g)$$

(ε is a multiplicative linear functional on $\mathcal{B}$) and

$$\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$$

(and thereby $\varepsilon(\mathbb{I}) = 1$ follows). To state this in another way, Δ, ϵ are algebra morphisms and m, η are co-algebra morphisms. Now $\eta\varepsilon \in \mathcal{L}(\mathcal{B})$ is the neutral element with respect to the convolution product, i.e. $A*(\eta\varepsilon) = A = (\eta\varepsilon)*A$, and associativity of the convolution follows directly from both co-associativity of Δ and associativity of m.

Hopf algebra. A *Hopf algebra*

$$\mathcal{H} = (\mathcal{H}, m, \eta, \Delta, \varepsilon, S)$$

is a bi-algebra $(\mathcal{H}, m, \eta, \Delta, \varepsilon)$ with so called *antipode* $S \in \mathcal{L}(\mathcal{H})$ such that

$$I * S = \eta\varepsilon = S * I.$$

That is, S is the convolutive inverse of I. An *involutive Hopf algebra* is a Hopf algebra $\mathcal{H}$ with involution $j : \mathcal{H} \to \mathcal{H}$; i.e. j is conjugate-linear, $j^2 = I$, $j(fg) = j(g)j(f)$ and $(j \otimes j)\Delta = \Delta j$.

Group algebra example. Let G be a compact Lie group, $e \in G$ its neutral element. Let $\mathcal{D}(G)$ be the space $C^\infty(G)$ with the usual Fréchet space structure. We identify $\mathcal{D}(G)\otimes\mathcal{D}(G)$ with a subspace of $\mathcal{D}(G\times G)$. Then the vector space $\mathcal{D}(G)$ is endowed with co-algebra operations $(\Delta\phi)(x,y) := \phi(xy)$ and $\varepsilon(\phi) := \phi(e)$. Hence the co-algebra axioms here correspond to the monoid axioms of the underlying space. Trivially, the usual multiplication and the unit $\mathbb{I} \in \mathcal{D}(G)$ provide an algebra structure for $\mathcal{D}(G)$. Distribution $f \in \mathcal{D}'(G)$ is a linear functional on $\mathcal{D}(G)$, acting by $\langle \phi, f\rangle := \int_G \phi(x)\, f(x)\, \mathrm{d}\mu_G(x)$, where μ_G is the Haar measure of G. Then the convolution $f * g \in \mathcal{D}'(G) = \mathcal{L}(\mathcal{D}(G), \mathbb{C})$ of $f, g \in \mathcal{D}'(G)$ defined as above coincides with the usual convolution:

$$\begin{aligned}\langle \phi, f * g\rangle &:= \langle \Delta\phi, f \otimes g\rangle \\ &= \int_{G\times G} \phi(xy)\ f(x)\ g(y)\ \mathrm{d}\mu_{G\times G}(x,y) \\ &= \int_G \phi(x) \int_G f(xy^{-1})\ g(y)\ \mathrm{d}\mu_G(y)\ \mathrm{d}\mu_G(x).\end{aligned}$$

We notice that $\mathcal{D}(G)$ is in fact a bi-algebra with its canonical mappings, and $\eta\varepsilon(\phi) = \phi(e)\mathbb{I}$; the identity element with respect to the convolution is given essentially by the Dirac delta δ_e at the neutral element $e \in G$. The group bi-algebra $\mathcal{D}(G)$ is a natural involutive Hopf algebra with the antipode defined by $(S\phi)(x) := \phi(x^{-1})$ and the involution given by $(j\phi)(x) := \overline{\phi(x)}$. Thus here the antipode axiom is related to the existence and uniqueness of inverse elements in the underlying monoid. Notice also that $\mathcal{D}(G)\widehat{\otimes}\mathcal{D}(G)$ can be identified with $\mathcal{D}(G \times G)$.

Hopf algebra in a nutshell: In short, if we denote $m(A \otimes B)\Delta = A * B$ and $\mathbb{I}_* = \eta\varepsilon$, the axioms for a Hopf algebra $\mathcal{H}$ are:

$$m(m \otimes I) = m(I \otimes m), \quad \eta(1) = \mathbb{I},$$

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta, \quad I * \mathbb{I}_* = I = \mathbb{I}_* * I,$$

$$\Delta m = m_{\mathcal{H}\otimes\mathcal{H}}(\Delta \otimes \Delta) = (m \otimes m)\Delta_{\mathcal{H}\otimes\mathcal{H}}, \quad \varepsilon m = m_{\mathbb{C}}(\varepsilon \otimes \varepsilon),$$

$$\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}, \quad \varepsilon(\mathbb{I}) = 1,$$

$$I * S = \mathbb{I}_* = S * I,$$

all the mappings $m, \eta, \Delta, \varepsilon, S$ being linear. The axioms for an involution j in an involutive Hopf algebra are then

$$jm = \tau(m \otimes m)j, \quad \Delta j = (j \otimes j)\Delta,$$

j being conjugate-linear.

Consequences of the Hopf axioms. It quite easily follows that S is anti-multiplicative and anti-co-multiplicative,

$$Sm = m(S \otimes S)\tau, \quad \Delta S = \tau(S \otimes S)\Delta,$$

and that

$$S\mathbb{I} = \mathbb{I}, \quad S\mathbb{I}_* = \mathbb{I}_* = \mathbb{I}_* S.$$

Furthermore, in an involutive Hopf algebra, the antipode has the inverse

$$S^{-1} = jSj.$$

Duality of Hopf algebras. Let $\mathcal{H}$ be a nuclear Hopf-Fréchet algebra, i.e. a nuclear Fréchet space and a Hopf algebra, with the algebraic tensor products replaced by the topological tensor products in the Hopf definitions. Then the dual space $\mathcal{H}' = \mathcal{L}(\mathcal{H}, \mathbb{C})$ has a natural dual Hopf algebra structure. Indeed, we define the Hopf structure $\mathcal{H}' = (\mathcal{H}', m, \eta, \Delta, \varepsilon, S)$ by dualities $\mathcal{H} \times \mathcal{H}' \to \mathbb{C}$, $(\mathcal{H}\widehat{\otimes}\mathcal{H}) \times (\mathcal{H}\widehat{\otimes}\mathcal{H})' \to \mathbb{C}$ and $\mathbb{C} \times \mathbb{C}' \to \mathbb{C}$, where $(\mathcal{H}\widehat{\otimes}\mathcal{H})' \cong \mathcal{H}'\widehat{\otimes}\mathcal{H}'$ and $\mathbb{C}' \cong \mathbb{C}$:

$$\langle \phi, m(f \otimes g) \rangle := \langle \Delta\phi, f \otimes g \rangle,$$

$$\langle \phi, \eta(1) \rangle := \langle \varepsilon(\phi), 1 \rangle = \varepsilon(\phi),$$

$$\langle \phi \otimes \psi, \Delta f \rangle := \langle m(\phi \otimes \psi), f \rangle,$$

$$\langle 1, \varepsilon(f) \rangle := \langle \eta(1), f \rangle,$$

$$\langle \phi, Sf \rangle := \langle S\phi, f \rangle.$$

If $\mathcal{H}$ is an involutive Hopf algebra, we can endow the dual with an antipode by

$$\langle \phi, j(f) \rangle := j_{\mathbb{C}} \langle j(S\phi), f \rangle,$$

where $j_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is the complex conjugation $z \mapsto \overline{z}$.

Group algebra dual. Let G be a compact Lie group and $f, g \in \mathcal{D}'(G)$. Then it is easy to verify that $m(f \otimes g) = f * g \in \mathcal{D}'(G)$, and that $\eta(1) = \delta_e \in \mathcal{D}'(G)$ is the Dirac delta at $e \in G$. Moreover, $\varepsilon f = \int_G f(x) \, \mathrm{d}\mu_G(x)$ and $Sf(x) = f(x^{-1})$ informally. The involution for distributions is given by $(j(f))(x) = \overline{f(x^{-1})}$. Notice that $\mathcal{D}'(G)\widehat{\otimes}\mathcal{D}'(G)$ can be identified with $\mathcal{D}'(G \times G)$.

Hopf structures via linear isomorphisms. Let $\mathcal{H}$ be a Hopf algebra, $\mathcal{B}$ a vector space and $\iota : \mathcal{B} \to \mathcal{H}$ a linear bijection. Then this isomorphism naturally endows $\mathcal{B}$ with a Hopf structure:

$$m_{\mathcal{B}} := \iota^{-1} m_{\mathcal{H}} (\iota \otimes \iota),$$

$$\eta_{\mathcal{B}} := \iota^{-1} \eta_{\mathcal{H}},$$

$$\Delta_{\mathcal{B}} := (\iota^{-1} \otimes \iota^{-1}) \Delta_{\mathcal{H}} \iota,$$

$$\varepsilon_{\mathcal{B}} := \varepsilon_{\mathcal{H}}\iota,$$

$$S_{\mathcal{B}} := \iota^{-1} S_{\mathcal{H}}\iota.$$

An involution, if it exists, is defined by

$$j_{\mathcal{B}} := \iota^{-1} j_{\mathcal{H}}\iota.$$

In the sequel, we equip $\mathcal{L}(\mathcal{H})$ with Hopf structures $\mathcal{L}_*$ and $\mathcal{L}_\star$ via linear bijections

$$\mathcal{L}_\star \xrightarrow{\rho} \mathcal{L}_* \xrightarrow{K} \mathcal{H}\widehat{\otimes}\mathcal{H}',$$

where $\mathcal{L}_\star = \mathcal{L}_* = \mathcal{L}(\mathcal{H})$ as topological vector spaces, K is the Schwartz kernel isomorphism, and ρ is a natural convolution isomorphism $A \mapsto A * S$.

4 Hopf structure via Schwartz kernels

The fundamental Hopf structure for $\mathcal{L}(\mathcal{H})$. Let $\mathcal{H}$ be a nuclear Hopf-Fréchet algebra; the most natural way to endow $\mathcal{L}(\mathcal{H})$ with a Hopf algebra structure is from $\mathcal{H}\widehat{\otimes}\mathcal{H}'$ via the Schwartz kernel isomorphism $A \mapsto K(A) = K_A$. For instance,

$$m_*(A \otimes B) := K^{-1}(m(K_A \otimes K_B)).$$

Let us denote this Hopf algebra by $\mathcal{L}_* = (\mathcal{L}(\mathcal{H}), m_*, \eta_*, \Delta_*, \varepsilon_*, S_*)$, and write $\mathbb{I}_* := \eta_*(1)$.

Theorem 1. *The operations in $\mathcal{L}_*$ can be written in terms of the basic Hopf operations $m, \eta, \Delta, \varepsilon, S$ of $\mathcal{H}$ as follows:*

$$m_*(A \otimes B) = m(A \otimes B)\Delta = A * B, \tag{1}$$

$$\mathbb{I}_* = \eta_*(1) = \eta\varepsilon, \tag{2}$$

$$\Delta_*(A) = \Delta A m, \tag{3}$$

$$\varepsilon_*(A) = \varepsilon(A(\mathbb{I}_{\mathcal{H}})), \tag{4}$$

$$S_*(A) = SAS. \tag{5}$$

If $\mathcal{H}$ is an involutive Hopf algebra then $\mathcal{L}_*$ has a Hopf structure with involution j_*, where

$$j_*(A) = jAjS. \tag{6}$$

Proof. In the following, $A, B \in \mathcal{L}(\mathcal{H})$ have respective Schwartz kernels $K_A, K_B \in \mathcal{H}\widehat{\otimes}\mathcal{H}'$. Let $f, g \in \mathcal{H}'$ and $\phi, \psi \in \mathcal{H}$. Then

$$\begin{aligned}
\langle m_*(A \otimes B)\phi, f\rangle &:= \langle m(K_A \otimes K_B), f \otimes \phi\rangle \\
&= \langle K_A \otimes K_B, \Delta(f \otimes \phi)\rangle \\
&= \langle (A \otimes B)\Delta\phi, \Delta f\rangle \\
&= \langle m(A \otimes B)\Delta\phi, f\rangle \\
&= \langle (A * B)\phi, f\rangle,
\end{aligned}$$

$$\begin{aligned}
\langle \mathbb{I}_*\phi, f\rangle &:= \langle \eta(1 \otimes 1), f \otimes \phi\rangle \\
&= \langle 1 \otimes 1, \varepsilon(f \otimes \phi)\rangle \\
&= \langle \varepsilon\phi, \varepsilon f\rangle \\
&= \langle \eta\varepsilon\phi, f\rangle,
\end{aligned}$$

$$\begin{aligned}
\langle \Delta_*(A)(\phi \otimes \psi), f \otimes g\rangle &:= \langle \Delta(K_A), (f \otimes \phi) \otimes (g \otimes \psi)\rangle \\
&= \langle K_A, m((f \otimes \phi) \otimes (g \otimes \psi))\rangle \\
&= \langle Am(\phi \otimes \psi), m(f \otimes g)\rangle \\
&= \langle \Delta Am(\phi \otimes \psi), f \otimes g\rangle,
\end{aligned}$$

$$\begin{aligned}
\varepsilon_*(A) &:= \varepsilon(K_A) \\
&= \langle K_A, \eta(1 \otimes 1)\rangle \\
&= \langle A\eta 1, \eta 1\rangle \\
&= \langle \varepsilon A\eta 1, 1\rangle \\
&= \varepsilon A\mathbb{I},
\end{aligned}$$

$$\begin{aligned}
\langle S_*(A)\phi, f\rangle &:= \langle S(K_A), f \otimes \phi\rangle \\
&= \langle K_A, S(f \otimes \phi)\rangle \\
&= \langle AS\phi, Sf\rangle \\
&= \langle SAS\phi, f\rangle.
\end{aligned}$$

If $\mathcal{H}$ is an involutive Hopf algebra with involution j then $\mathcal{H}'$ has involution i given by $\langle \phi, i(f)\rangle = \overline{\langle j(S\phi), f\rangle}$; thereby $\mathcal{L}_*$ has the Hopf structure with involution $A \mapsto jAjS$, because

$$\begin{aligned}
\langle (j \otimes i)K_A, f \otimes \phi\rangle &= \overline{\langle (S^{-1} \otimes I)K_A, (i \otimes jS)(f \otimes \phi)\rangle} \\
&= \overline{\langle S^{-1}AjS\phi, if\rangle} \\
&= \langle jSS^{-1}AjS\phi, f\rangle \\
&= \langle K_{jAjS}, f \otimes \phi\rangle;
\end{aligned}$$

notice that we used the fact $K_{BAC} = (B \otimes C')K_A$ □

Hopf operations for $\mathcal{L}(\mathcal{D}(G))$. Let G be a compact Lie group and $A, B \in \mathcal{L}(\mathcal{D}(G))$ with respective Schwartz kernels K_A, K_B. Then informally

$$K_{A*B}(x,y) = \int_G K_A(x, yz^{-1})\ K_B(x,z)\ \mathrm{d}\mu_G(z),$$

$$\varepsilon_*(A) = \int_G K_A(e,z)\ \mathrm{d}\mu_G(z),$$
$$K_{S_*(A)}(x,y) = K_A(x^{-1}, y^{-1}),$$
$$K_{j_*(A)}(x,y) = \overline{K_A(x, y^{-1})},$$

and so on.

5 Hopf homomorphism by convolution

In Theorem 1 we equipped $\mathcal{L}(\mathcal{H})$ with the involutive Hopf structure $\mathcal{L}_*$ from $\mathcal{H}\widehat{\otimes}\mathcal{H}'$ via the Schwartz kernel isomorphism. There the product is the operator convolution $(A,B) \mapsto A * B = m(A \otimes B)\Delta$, with the unit element $\mathbb{I}_* = \eta\varepsilon \in \mathcal{L}(\mathcal{H})$. We know that I and S are convolution inverses to each other, $I * S = \eta\varepsilon = S * I$. A simple way to endow $\mathcal{L}(\mathcal{H})$ with an involutive Hopf structure with the unit element I is via the linear bijection

$$\rho = (A \mapsto A * S) : \mathcal{L}(\mathcal{H}) \to \mathcal{L}_*.$$

Alternatively, this Hopf algebra is begotten by the isomorphism

$$L = (A \mapsto L_A = K_{A*S}) : \mathcal{L}(\mathcal{H}) \to \mathcal{H}\widehat{\otimes}\mathcal{H}'.$$

We denote this Hopf structure by

$$\mathcal{L}_\star = (\mathcal{L}(\mathcal{H}), m_\star, \eta_\star, \Delta_\star, \varepsilon_\star, S_\star, j_\star).$$

Since $S * I = \eta\varepsilon = I_*$ and $(A * S)(\mathbb{I}) = A\mathbb{I}$, we get:

Theorem 2. *The operations in $\mathcal{L}_\star$ can be written in terms of the basic Hopf operations $m, \eta, \Delta, \varepsilon, S$ of $\mathcal{H}$ as follows:*

$$m_\star(A \otimes B) = A * S * B, \tag{7}$$

$$\mathbb{I}_\star = \eta_\star(1) = I \tag{8}$$

$$\Delta_\star(A) = (\Delta(A * S)m) * (I \otimes I), \tag{9}$$

$$\varepsilon_\star(A) = \varepsilon(A(\mathbb{I}_\mathcal{H})), \tag{10}$$

$$S_\star(A) = (S(A * S)S) * I. \tag{11}$$

If $\mathcal{H}$ is an involutive Hopf algebra then $\mathcal{L}_\star$ has a Hopf structure with involution $j_\star$, where

$$j_\star(A) = (j(A * S)jS) * I. \tag{12}$$

6 Pseudodifferential operators

Pseudodifferential operators. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz test function space of rapidly decreasing smooth functions $\mathbb{R}^n \to \mathbb{C}$. An operator $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ is called a *pseudodifferential operator* of order $m \in \mathbb{R}$ on $\mathbb{R}^n$, denoted by $A \in \Psi^m(\mathbb{R}^n)$, if it is of the form

$$(Af)(x) = \int_{\mathbb{R}^n} a(x,\xi)\ \widehat{f}(\xi)\ e^{i2\pi x\cdot\xi}\ \mathrm{d}x,$$

where the *symbol* $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies the inequalities

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{Am\alpha\beta}\ \|\xi\|_{\mathbb{R}^n}^{m-|\alpha|}$$

for $\|\xi\|_{\mathbb{R}^n} > 1$, where $C_{Am\alpha\beta} < \infty$ is a constant depending on A, m, α, β. Let M be a compact smooth manifold without a boundary, and let $\mathcal{D}(M)$ be the test function space of C^∞-smooth functions on M. An operator $A \in \mathcal{L}(\mathcal{D}(M))$ is called a *pseudodifferential operator* of order $m \in \mathbb{R}$ on M, denoted by $A \in \Psi^m(M)$, if all of its localizations belong to $\Psi^m(\mathbb{R}^{\dim(M)})$; this definition makes sense, since $\Psi^m(\mathbb{R}^n)$ is invariant under smooth changes of local coordinates. A *principal symbol* of a pseudodifferential operator $A \in \Psi^m(M)$ is a function on the cotangent bundle of M defining A up to $\Psi^{m-1}(M)$. If pseudodifferential operators $A \in \Psi^{m_A}(M)$ and $B \in \Psi^{m_B}(M)$ have respective principal symbols a, b, then the composition AB has a principal symbol ab, and the adjoint A^* has a principal symbol $\overline{a}$; in some sense, pseudodifferential operator algebras behave like function algebras. For more about pseudodifferential calculus, see [4].

Compact Lie groups. The nuclear Fréchet space of interest for us is $\mathcal{D}(G)$, where G is a compact Lie group. Let μ_G be the normalized Haar measure of G. For $A \in \mathcal{L}(\mathcal{D}(G))$, let us define a mapping $s_A : G \to \mathcal{D}'(G)$ and a convolution operator $\sigma_A(x) \in \mathcal{L}(\mathcal{D}(G))$ by

$$\begin{aligned}
(A\phi)(x) &= \int_G K_A(x,y)\ \phi(y)\ \mathrm{d}\mu_G(y) \\
&=: \int_G s_A(x)(xy^{-1})\ \phi(y)\ \mathrm{d}\mu_G(y) \\
&= (s_A(x) * \phi)(x) \\
&=: (\sigma_A(x)\phi)(x).
\end{aligned}$$

Let us call the mapping $\sigma_A : G \to \mathcal{L}(\mathcal{D}(G))$ the *symbol of* A. It is noteworthy that $A \mapsto \sigma_A$ is a one-to-one mapping. For pseudodifferential operators on G, there is a symbolic calculus with asymptotic expansions analogous to the Euclidean case, see [3] and [6]. One of the consequences is that if pseudodifferential operators $A_1 \in \Psi^{m_1}(G)$ and $A_2 \in \Psi^{m_2}(G)$ have the respective symbols $\sigma_{A_1}, \sigma_{A_2}$, then the composition $A_1A_2 \in \Psi^{m_1+m_2}(G)$ has the symbol $x \mapsto \sigma_{A_1}(x)\sigma_{A_2}(x)$ modulo $\Psi^{m_1+m_2-1}(G)$, and the adjoint

$A_1^* \in \Psi^{m_1}(G)$ has the symbol $x \mapsto \sigma_{A_1}(x)^*$ modulo $\Psi^{m_1-1}(G)$. Again, the behavior of pseudodifferential operator algebras resembles function algebra case. Next we show how symbolic calculus is related to the Hopf algebra $\mathcal{D}(G)$.

Remark. The distribution $L_A \in \mathcal{D}(G)\widehat{\otimes}\mathcal{D}'(G)$ introduced in the previous section satisfies $L_A = (I \otimes S)s_A$, because

$$\begin{aligned} L_A(x,y) &= K_{A*S}(x,y) \\ &= \int_G K_A(x, yz^{-1})\ K_S(x,z)\ \mathrm{d}\mu_G(z) \\ &= K_A(x, yx) \\ &= s_A(x, y^{-1}). \end{aligned}$$

Conversely, $s_A = (I \otimes S)L_A$, since here $S^2 = I$.

Theorem 3. *Let $A, B \in \mathcal{L}(\mathcal{D}(G))$. Then*

$$\sigma_{A\star B}(x) = \sigma_B(x)\sigma_A(x)$$

for every $x \in G$. If $A \in \Psi^{m_1}(G)$ and $B \in \Psi^{m_2}(G)$ then

$$A \star B \in \Psi^{m_1+m_2}(G) \quad \text{and} \quad A \star B - AB \in \Psi^{m_1+m_2-1}(G).$$

Moreover, $\sigma_{I_\star}(x) \equiv I$.

Proof. Notice that $\mathcal{D}(G)$ is commutative and that $S : \mathcal{D}'(G) \to \mathcal{D}'(G)$ is antimultiplicative. Thus we get

$$\begin{aligned} s_{A\star B} &= (I \otimes S)L_{A\star B} \\ &= (I \otimes S)(L_A\ L_B) \\ &= (I \otimes S)(((I \otimes S)s_A)\ ((I \otimes S)s_B)) \\ &= ((I \otimes S)(I \otimes S)s_B)\ ((I \otimes S)(I \otimes S)s_A) \\ &= s_B\ s_A, \end{aligned}$$

and consequently $\sigma_{A\star B}(x) = \sigma_B(x)\sigma_A(x)$.

Let $A \in \Psi^{m_1}(G)$ and $B \in \Psi^{m_2}(G)$. As it is well-known, $AB \in \Psi^{m_1+m_2}(G)$ and $[A, B] = AB - BA \in \Psi^{m_1+m_2-1}(G)$. From the symbolic calculus of [3] and [6] it follows that the operator $A \star B$ with the symbol

$$x \mapsto \sigma_B(x)\sigma_A(x)$$

belongs to $\Psi^{m_1+m_2}(G)$, and moreover that $A \star B - BA \in \Psi^{m_1+m_2-1}(G)$, because the first term in the asymptotic expansion for $\sigma_{BA}(x)$ is $\sigma_B(x)\sigma_A(x)$. Hence also

$$A \star B - AB = A \star B - BA - [A, B]$$

belongs to $\Psi^{m_1+m_2-1}(G)$ □

Theorem 4. *Let $A \in \mathcal{L}(\mathcal{D}(G))$. Then*

$$\sigma_{j_\star(A)}(x) = \sigma_A(x)^*$$

for every $x \in G$, where B^* for $B \in \mathcal{L}(\mathcal{D}(G))$ is defined by

$$\langle \phi, \overline{B^* f} \rangle := \langle B\phi, \overline{f} \rangle.$$

If $A \in \Psi^m(G)$ then

$$j_\star(A) \in \Psi^m(G) \quad \text{and} \quad j_\star(A) - A^* \in \Psi^{m-1}(G).$$

Proof. Now

$$\begin{aligned} s_{j_\star(A)} &= (I \otimes S) L_{j_\star(A)} \\ &= (I \otimes S)(j \otimes j) L_A \\ &= (I \otimes S)(j \otimes j)(I \otimes S) s_A \\ &= (j \otimes SjS) s_A \\ &= (j \otimes j) s_A, \end{aligned}$$

and combining this with $\langle g * \phi, \overline{f} \rangle = \langle \phi, \overline{j(g) * f} \rangle$, we get $\sigma_{j_\star(A)}(x) = \sigma_A(x)^*$. If $A \in \Psi^m(G)$ then $A^* \in \Psi^m(G)$ and

$$x \mapsto \sigma_{A^*}(x) - \sigma_A(x)^*$$

is the symbol of an operator belonging to $\Psi^{m-1}(G)$, by [3] and [6] □

Theorem 5. *Let $A \in \mathcal{L}(\mathcal{D}(G))$. Then*

$$\sigma_{S_\star(A)}(x) = \sigma_A(x^{-1})'$$

for every $x \in G$. If $A \in \Psi^m(G)$ then $S_\star(A) \in \Psi^m(G)$.

Proof. Here

$$\begin{aligned} s_{S_\star(A)} &= (I \otimes S) L_{S_\star(A)} \\ &= (I \otimes S)(S \otimes S) L_A \\ &= (I \otimes S)(S \otimes S)(I \otimes S) s_A \\ &= (S \otimes S) s_A. \end{aligned}$$

Combining this fact with $\langle g * \phi, f \rangle = \langle \phi, (Sg) * f \rangle$, we get $\sigma_{S_\star(A)}(x) = \sigma_A(x^{-1})'$. Let $A \in \Psi^m(G)$. Then

$$x \mapsto \sigma_{A'}(x) - \sigma_A(x)'$$

is the symbol of an operator belonging to $\Psi^{m-1}(G)$, due to the analogous result for $A \mapsto A^*$ presented in [3] and [6]. If $B \in \Psi^m(G)$ and $\kappa : G \to G$ is C^∞-smooth then

$$x \mapsto \sigma_B(\kappa(x))$$

is the symbol of an operator belonging to $\Psi^m(G)$, due to the symbol operator inequalities in [6]. Finally, by choosing $\sigma_B(x) := \sigma_A(x)'$ and $\kappa := (x \mapsto x^{-1})$, we obtain $S_\star(A) \in \Psi^m(G)$ □

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References

[1] E. Abe: Hopf Algebras. Cambridge University Press. 1977.

[2] M. E. Sweedler: Hopf Algebras. W. A. Benjamin, Inc. 1969.

[3] M. E. Taylor: *Noncommutative microlocal analysis.* Mem. Amer. Math. Soc. 52 (1984), 313.

[4] M. E. Taylor: Partial Differential Equations II. Qualitative Studies of Linear Equations. Springer-Verlag. 1997.

[5] F. Treves: Topological Vector Spaces, Distributions and Kernels. Academic Press. 1967.

[6] V. Turunen: *Pseudodifferential calculus on compact Lie groups and homogeneous spaces.* Ph. D. Thesis. Helsinki Univ. Techn. 2001. `http://lib.hut.fi/Diss/2001/isbn9512256940/`

[7] N. Weaver: Lipschitz algebras. World Scientific. 1999.

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