Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2003 A464

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SAMPLING AT EQUIANGULAR GRIDS ON THE 2-SPHERE AND ESTIMATES FOR SOBOLEV SPACE INTERPOLATION

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Ville Turunen: *Sampling at equiangular grids on the 2-sphere and estimates for Sobolev space interpolation*; Helsinki University of Technology Institute of Mathematics Research Reports A464 (2003).

Abstract: *We sample functions at $4n^2$ equiangularly spaced points on the unit sphere $\mathbb{S}^2$ of the 3-dimensional Euclidean space, and study the corresponding interpolation projections $Q_n : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ in the scale of L^2-type Sobolev spaces $H^s(\mathbb{S}^2)$. The main result is that if $-1 < s < t$ and $t > 7/2$ then*

$$\|f - Q_n f\|_{H^s(\mathbb{S}^2)} \leq c_{s,t,\varepsilon}\ n^{s-t}\ n^{4+\varepsilon}\ \|f\|_{H^t(\mathbb{S}^2)},$$

where $c_{s,t,\varepsilon} < \infty$ is a constant depending on s, t and $\varepsilon > 0$. Hence this result is useful only when $t > s + 4$. We also compare our methods to the well-known interpolation on the circle $\mathbb{S}^1$.

AMS subject classifications: 65T40, 33C55, 46E35, 42C10, 65T50.

Keywords: spherical harmonics, sampling, equiangular grid, interpolation, Sobolev space estimates.

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ISBN 951-22-6828-0
ISSN 0784-3143

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1 Introduction

Let $\mathbb{S}^2$ be the unit sphere of the Euclidean space $\mathbb{R}^3$. In this paper, we are concerned with approximation in the L^2-type Sobolev spaces $H^s(\mathbb{S}^2)$. A most natural set of sampling points on the sphere consists of the intersections of the equiangularly spaced longitudes and latitudes, and we shall restrict to this case. We study approximation in Sobolev spaces on the sphere $\mathbb{S}^2$ with comparison to the very basic case of interpolation on the circle $\mathbb{S}^1$. For $\mathbb{S}^1$, the old results are optimal and easy to obtain. However, non-commutative harmonic analysis poses hard problems, and for $\mathbb{S}^2$, our results are quite likely far from optimal, and yet require some work.

Our interest in this subject came originally from a need to solve integral and pseudodifferential equations by efficient algorithms. For example, a Dirichlet boundary value problem of a linear elliptic partial differential equation in a domain diffeomorphic to the unit disk of the plane $\mathbb{R}^2$ leads to a pseudodifferential equation on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. To solve such an equation numerically, one may use equispaced sampling points on $\mathbb{S}^1$, and compute with the now-classical FFT, the Fast Fourier Transform (see [8]).

Within last few years, the progress in Fast Fourier type algorithms on the sphere $\mathbb{S}^2$ (see [1], [2], [3]) has made it feasible to tackle applications, ranging from tomography to meteorology. E.g. boundary value problems in domains diffeomorphic to the unit ball of $\mathbb{R}^3$ can be considered ([6], [7]).

The structure of the paper is as follows: The second section reviews the well-known analysis on the circle. The third section is the core, dealing with the case of the sphere. The fourth section demonstrates the sins committed in estimates, applying the techniques of the sphere case to the circle.

2 Approximation on $\mathbb{S}^1$

To get some perspective to numerical harmonic analysis on the sphere $\mathbb{S}^2$, let us review some elementary facts from numerical Fourier analysis on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. We identify $\mathbb{S}^1$ naturally with the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the space of real numbers modulo integers): if $x \in [0,1[$ then $x + \mathbb{Z} \in \mathbb{T}$ can be identified with $(\cos(2\pi x), \sin(2\pi x))$. Of course, we may identify $\mathbb{T}$ also with the interval $[0,1[$, via $(x + \mathbb{Z} \mapsto x) : \mathbb{T} \to [0,1[$. The Fourier transform $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ of a measurable function $f : \mathbb{S}^1 \to \mathbb{C}$ is defined by

$$\widehat{f}(\xi) = \int_0^1 f(x)\ e^{-i2\pi x \cdot \xi}\ \mathrm{d}\xi.$$

The L^2-type Sobolev space of order $s \in \mathbb{R}$ on $\mathbb{S}^1$, denoted by $H^s(\mathbb{S}^1)$, is the completion of $C^\infty(\mathbb{S}^1)$ with respect to the norm

$$f \mapsto \|f\|_{H^s(\mathbb{S}^1)} := \left(\sum_{\xi \in \mathbb{Z}} \langle \xi \rangle^{2s}\ \left| \widehat{f}(\xi) \right|^2 \right)^{1/2},$$

where $\langle \xi \rangle := \max\{1, |\xi|\}$; the inner product is of course

$$(f, g) \mapsto \langle f, g \rangle_{H^s(\mathbb{S}^1)} = \sum_{\xi \in \mathbb{Z}} \langle \xi \rangle^{2s}\ \widehat{f}(\xi)\ \overline{\widehat{g}(\xi)}.$$

For $n \in \mathbb{N}$, let us define the orthogonal projection $P_n : H^s(\mathbb{S}^1) \to H^s(\mathbb{S}^1)$ by

$$P_n f := \sum_{\xi = -n+1}^{n} \widehat{f}(\xi)\ e_\xi,$$

where $e_\xi(x) = e^{i2\pi x \cdot \xi}$. If $s \leq t$ then it is easy to show that

$$\|f - P_n f\|_{H^s(\mathbb{S}^1)} \leq n^{s-t}\ \|f\|_{H^t(\mathbb{S}^1)}.$$

In practice, however, we are able to sample the values of a function f only at finitely many points of $\mathbb{S}^1$, and so we cannot even calculate $P_n f$; hence let us define the interpolation projection $Q_n : C(\mathbb{S}^1) \to C(\mathbb{S}^1)$ by

$$Q_n = P_n Q_n, \quad (Q_n f)\left(\frac{j}{2n}\right) = f\left(\frac{j}{2n}\right), \quad (0 \leq j < 2n);$$

recall that we can identify $\mathbb{S}^1$ with $[0,1[$. This indeed determines Q_n: if $1/2 < t$ and $f \in H^t(\mathbb{S}^1)$ then

$$Q_n f = \sum_{\xi = -n+1}^{n} e_\xi \sum_{\eta \in \mathbb{Z}} \widehat{f}(\xi + 2n\eta),$$

and if moreover $0 \leq s \leq t$ then

$$\|f - Q_n f\|_{H^s(\mathbb{S}^1)} \leq C_t\ n^{s-t}\ \|f\|_{H^t(\mathbb{S}^1)},$$

where $C_t < \infty$ depends only on t, and $\lim_{t \to \frac{1}{2}+} C_t = \infty$; this result is optimal (for a proof, see e.g. the lecture notes [8]). The Fourier coefficients $\widehat{Q_n f}$ can be computed efficiently by FFT, in time $\mathcal{O}(n \log n)$.

3 Approximation on $\mathbb{S}^2$

For the unit sphere

$$\mathbb{S}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ \|x\|_{\mathbb{R}^3} = (x_1^2 + x_2^2 + x_3^2)^{1/2} = 1\}$$

we use the spherical coordinates $x = x(\theta, \phi)$, where

$$x_1 = \cos(\phi)\sin(\theta), \quad x_2 = \sin(\phi)\sin(\theta), \quad x_3 = \cos(\theta),$$

$0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. The inner product for $L^2(\mathbb{S}^2)$ is

$$(f, g) \mapsto \langle f, g\rangle_{L^2(\mathbb{S}^2)} := \int_0^{2\pi} \int_0^{\pi} f(x) \ \overline{g(x)} \ \sin(\theta) \ \mathrm{d}\theta \ \mathrm{d}\phi.$$

Applying the Gramm–Schmidt process to the sequence $(z \mapsto z^l)_{l=0}^{\infty}$ we obtain the Legendre polynomials P_l forming an orthogonal basis for $L^2([-1, 1])$:

$$P_l^0(z) = P_l(z) := \frac{1}{2^l l!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^l (z^2 - 1)^l.$$

For $l \in \mathbb{N}$ and $m \in \{0, \ldots, l\}$, the associated Legendre function P_l^m is defined by

$$P_l^m(z) := (1 - z^2)^{m/2} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m P_l^0(z),$$

and furthermore

$$P_l^{-m}(z) := (-1)^m \ \frac{(l-m)!}{(l+m)!} \ P_l^m(z).$$

Then an orthonormal basis for $L^2(\mathbb{S}^2)$ consists of the spherical harmonic functions $Y_l^m \in C^\infty(\mathbb{S}^2)$ $(l \in \mathbb{N}, \ |m| \leq l)$,

$$Y_l^m(x(\theta, \phi)) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \ P_l^m(\cos(\theta)) \ e^{im\phi}.$$

For $l \in \mathbb{N}$ and $m \in \mathbb{Z}$, $|m| \leq l$, the Fourier coefficient $\widehat{f}(l, m)$ of a function $f \in C^\infty(\mathbb{S}^2)$ is

$$\widehat{f}(l, m) := \langle f, Y_l^m\rangle_{L^2(\mathbb{S}^2)} = \int_0^{2\pi} \int_0^{\pi} f(x) \ \overline{Y_l^m(x)} \ \sin(\theta) \ \mathrm{d}\theta \ \mathrm{d}\phi.$$

The Fourier inverse transform is now

$$f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \widehat{f}(l, m) \ Y_l^m.$$

It is often convenient to define $\widehat{f}(l, m) := 0$ whenever $l < 0$ or $|m| > l$. The spherical harmonic Fourier series can be obtained also from the Fourier series on groups SO(3) or U(2); this process is canonical, owing to the representation theory of the groups.

The Laplace operator Δ on $\mathbb{S}^2$ is the angular part of the Laplace operator $\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ on $\mathbb{R}^3$:

$$\Delta f = \frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta} \left(\sin(\theta) \frac{\partial}{\partial\theta} f \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial\phi^2} f.$$

Also Δ can be obtained using the representation theory of the symmetry group of $\mathbb{S}^2$. Notice that Y_l^m is an eigenfunction of the Laplacian, such that $\Delta Y_l^m = -l(l+1)\, Y_l^m$. Let the test function space $\mathcal{D}(\mathbb{S}^2)$ be $C^\infty(\mathbb{S}^2)$ endowed with the standard Fréchet space structure. The Sobolev space $H^s(\mathbb{S}^2)$ is the completion of the test function space in the norm

$$\|f\|_{H^s(\mathbb{S}^2)} := \left(\sum_{l=0}^{\infty} \langle l \rangle^{2s} \sum_{m=-l}^{l} \left| \widehat{f}(l,m) \right|^2 \right)^{1/2},$$

where $\langle l \rangle = \max\{1, l\}$. The inner product for $H^s(\mathbb{S}^2)$ is obvious. Let us define the orthogonal projection $P_N : H^s(\mathbb{S}^2) \to H^s(\mathbb{S}^2)$ by

$$P_n f := \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \widehat{f}(l,m)\; Y_l^m.$$

Lemma 1. *If $s \leq t$ then*

$$\|f - P_n f\|_{H^s(\mathbb{S}^2)} \leq n^{s-t}\; \|f\|_{H^t(\mathbb{S}^2)}.$$

Proof. This is a matter of a simple calculation:

$$\begin{aligned}
\|f - P_n f\|_{H^s(\mathbb{S}^2)} &= \left(\sum_{l=n}^{\infty} l^{2(s-t)}\; l^{2t} \sum_{m=-l}^{l} |\widehat{f}(l,m)|^2 \right)^{1/2} \\
&\overset{s\leq t}{\leq} n^{s-t} \left(\sum_{l=n}^{\infty} l^{2t} \sum_{m=-l}^{l} |\widehat{f}(l,m)|^2 \right)^{1/2} \\
&\leq n^{s-t} \left(\sum_{l=0}^{\infty} \langle l \rangle^{2t} \sum_{m=-l}^{l} |\widehat{f}(l,m)|^2 \right)^{1/2} \\
&= n^{s-t}\; \|f\|_{H^t(\mathbb{S}^2)}
\end{aligned}$$

□

Now we are facing the difficulty that a function f can be sampled only at finitely many points. As in the case of numerical Fourier analysis on the torus, we need interpolation projections Q_n, and study their approximation properties in the scale of Sobolev spaces. Moreover, it would be important to have some efficient computation method for the approximated Fourier coefficients $\widehat{Q_n f}(l,m)$.

Actually, there is an FFT-type algorithm for computing the Fourier coefficients of a function $f = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \widehat{f}(l,m)\, Y_l^m$, where $n \in \mathbb{N}$. Notice that here f has at most $N := n^2$ non-zero Fourier coefficients; a straightforward calculation for all these coefficients would cost $\mathcal{O}(N^2)$ operations, but with the FFT-type approach only $\mathcal{O}(N(\log N)^2)$ operations and memory is needed. The inverse Fourier transform has similar properties. The key to an FFT-type algorithm on $\mathbb{S}^2$ is that

$$Y_l^m(x) = c_{lm}\, P_l^m(\cos(\theta))\, e^{im\phi},$$

where the classical FFT on $\mathbb{S}^1$ can be applied to the exponential factor, and that there is a recurrence formula

$$(l-m+1)\, P_{l+1}^m(z) - (2l+1)\, z\, P_l^m(z) + (l+m)\, P_{l-1}^m(z) = 0$$

allowing one to construct a "Fast Legendre Transform". It can be said that all this is possible due to the rich symmetries, and there has been research on FFT-type algorithms on many compact groups [4].

Let $b \in \mathbb{N}$ and $n = 2^b$. The sampling points on $\mathbb{S}^2$ for degree n are $x_{jk}^{(n)} = x(\theta_j^{(n)}, \phi_k^{(n)})$,

$$\theta_j^{(n)} := \pi\, \frac{j}{2n}, \quad \phi_k^{(n)} := 2\pi\, \frac{k}{2n},$$

where $0 \le j,k < 2n$. Notice that $x_{0k}^{(n)} \in \mathbb{S}^2$ is the north pole for every k, and furthermore we will have sampling weight equal to 0 at the north pole; yet for convenience we may think of $4n^2$ points, which of course holds asymptotically as $n \to \infty$. Proposition 1 collects some results from [1]:

Proposition 1. *Let $l_0 \in \mathbb{N}$ and $m_0 \in \mathbb{Z}$, $|m_0| \le l_0$. If $f = \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \widehat{f}(l,m)\, Y_l^m$ then*

$$\begin{aligned} \widehat{f}(l_0,m_0) &= \frac{1}{4n^2} \sum_{j=0}^{2n-1} \sum_{k=0}^{2n-1} f\left(x_{jk}^{(n)}\right) \overline{Y_{l_0}^{m_0}\left(x_{jk}^{(n)}\right)} \sin\left(\theta_j^{(n)}\right) \times \\ &\quad \times \frac{4}{\pi} \sum_{l=0}^{n-1} \frac{1}{2l+1} \sin\left((2l+1)\theta_j^{(n)}\right). \end{aligned}$$

Notice that here we used (less than) $4n^2$ sampled values of f to compute n^2 Fourier coefficients $\widehat{f}(l,m)$. The sampling measure implicit here is

$$\mu_n := \frac{1}{4n^2} \sum_{j=0}^{2n-1} \sum_{k=0}^{2n-1} \delta_{x_{jk}^{(n)}} \sin\left(\theta_j^{(n)}\right) \frac{4}{\pi} \sum_{l=0}^{n-1} \frac{1}{2l+1} \sin\left((2l+1)\theta_j^{(n)}\right).$$

Let us define $Q_n : C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ by

$$Q_n f := P_n(f\ \mu_n).$$

Trivially $P_n Q_n = Q_n$. By Proposition 1, we have $Q_n P_n = P_n$. Hence $Q_n^2 = Q_n(P_n Q_n) = (Q_n P_n) Q_n = P_n Q_n = Q_n$, i.e. Q_n is a projection. We want to estimate $\|f - Q_n f\|_{H^s(\mathbb{S}^2)}$. First, let us present some auxiliary results.

Lemma 2.

$$\widehat{\mu_n}(l,m) = \begin{cases} 1, & \text{when } l = 0 = m, \\ 0, & \text{when } 0 < l < 2n \quad \text{or} \quad m \notin 2n\mathbb{Z}. \end{cases}$$

Here $2n\mathbb{Z} = \{2nk :\ k \in \mathbb{Z}\}$. *Moreover,*

$$|\widehat{\mu_n}(l,m)| \leq C\ \langle l \rangle^{1/2},$$

where $C < 2.043$ *is a constant.*

Proof. The equations for the Fourier coefficients were proven in [1]. Let us then establish the estimate. Let

$$\operatorname{sgn}(\theta) = \begin{cases} 1, & 0 < \theta < \pi, \\ -1, & -\pi < \theta < 0. \end{cases}$$

Notice that in the $L^2([-\pi,\pi])$-sense,

$$\operatorname{sgn}(\theta) = \frac{4}{\pi} \sum_{l=0}^{\infty} \frac{1}{2l+1} \sin((2l+1)\theta).$$

Due to the Gibbs phenomenon, the partial sums of this Fourier series do not converge in L^∞, but yet we may say that

$$\left| \frac{4}{\pi} \sum_{l=0}^{n} \frac{1}{2l+1} \sin((2l+1)\theta) \right| < c$$

for every $n \in \mathbb{N}$, where $c < 1.179$ is a constant. Thereby

$$\begin{aligned} & |\widehat{\mu_n}(l_0, m_0)| \\ = & \left| \int_{\mathbb{S}^2} \overline{Y_{l_0}^{m_0}}\ \mathrm{d}\mu_n \right| \\ = & \left| \frac{1}{4n^2} \sum_{j=0}^{2n-1} \sum_{k=0}^{2n-1} \overline{Y_{l_0}^{m_0}\left(x_{jk}^{(n)}\right)}\ \sin\left(\theta_j^{(n)}\right)\ \frac{4}{\pi} \sum_{l=0}^{n-1} \frac{1}{2l+1}\ \sin\left((2l+1)\theta_j^{(n)}\right) \right| \\ \leq & \ c \sup_{0 \leq \theta \leq \pi} \sup_{0 \leq \phi < 2\pi} \left| Y_{l_0}^{m_0}(x(\theta,\phi)) \right| \\ \leq & \ c \sqrt{2l_0 + 1}\ \left\| Y_{l_0}^{m_0} \right\|_{L^2(\mathbb{S}^2)} \\ = & \ c \sqrt{2l_0 + 1}. \end{aligned}$$

In the last inequality we used Lemma 8 from [5]. Hence the desired estimate is obtained □

For a function or a distribution g on $\mathbb{S}^2$, denote $g_l^m := \widehat{g}(l,m)\ Y_l^m$.

Lemma 3. *Let $f \in \mathcal{D}(\mathbb{S}^2)$ and $g \in \mathcal{D}'(\mathbb{S}^2)$. Then $fg \in \mathcal{D}'(\mathbb{S}^2)$ satisfies*

$$\widehat{fg}(l,m) = \sum_{l_2=0}^{\infty} \sum_{l_1=|l_2-l|}^{l_2+l} \sum_{\substack{m_1,m_2: \\ |m_j|\le l_j \\ m_1+m_2=m}} \widehat{f_{l_1}^{m_1} g_{l_2}^{m_2}}(l,m).$$

Proof. There is a product formula

$$Y_{l_1}^{m_1}\, Y_{l_2}^{m_2} = \sum_{L=|l_1-l_2|}^{l_1+l_2} c_{m_1 m_2}^{l_1 l_2 L}\, Y_L^{m_1+m_2},$$

where

$$c_{m_1 m_2}^{l_1 l_2 L} = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2L+1)}}\, C_{0,0,0}^{l_1,l_2,L}\, C_{m_1,m_2,m_1+m_2}^{l_1,l_2,L},$$

$C_{m_1,m_2,m_3}^{l_1,l_2,l_3}$ being a Wigner symbol, see [9]. We use the convention $c_{m_1 m_2}^{l_1 l_2 L} = 0$ whenever $L < |l_1 - l_2|$ or $L > l_1 + l_2$ or $|m_1 + m_2| > L$. Thus

$$\widehat{f_{l_1}^{m_1} g_{l_2}^{m_2}}(l,m) = \delta_{m,m_1+m_2}\, c_{m_1 m_2}^{l_1 l_2 l}\, \widehat{f}(l_1,m_1)\, \widehat{g}(l_2,m_2),$$

so that

$$\begin{aligned}
\widehat{fg}(l,m) &= \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{l_2=0}^{\infty} \sum_{m_2=-l_2}^{l_2} \widehat{f_{l_1}^{m_1} g_{l_2}^{m_2}}(l,m) \\
&= \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sum_{m_2=-l_2}^{l_2} \delta_{m,m_1+m_2}\, c_{m_1 m_2}^{l_1 l_2 l}\, \widehat{f_{l_1}^{m_1} g_{l_2}^{m_2}}(l,m) \\
&= \sum_{l_2=0}^{\infty} \sum_{l_1=|l_2-l|}^{l_2+l} \sum_{\substack{m_1,m_2: \\ |m_j|\le l_j \\ m_1+m_2=m}} c_{m_1 m_2}^{l_1 l_2 l}\, \widehat{f}(l_1,m_1)\, \widehat{g}(l_2,m_2) \\
&= \sum_{l_2=0}^{\infty} \sum_{l_1=|l_2-l|}^{l_2+l} \sum_{\substack{m_1,m_2: \\ |m_j|\le l_j \\ m_1+m_2=m}} \widehat{f_{l_1}^{m_1} g_{l_2}^{m_2}}(l,m)
\end{aligned}$$

□

Lemma 4. *Let f be a function, and let μ_n be the sampling measure presented above. Then*

$$\left|\widehat{f_{l_1}^{m_1} \mu_{nl_2}^{m_2}}(l,m)\right| \le C_r\, \langle l\rangle^{-r}\, \langle l_1\rangle^{r}\, \langle l_2\rangle^{r+1/2}\, \left|\widehat{f}(l_1,m_1)\right|.$$

for every $r > 1$, where C_r is a constant depending on r, $\lim\limits_{r\to 1+} C_r = \infty$.

Proof. The last claim follows simply because $\mu_{nl_2}^{m_2} \equiv 0$ when $m_2 \notin 2n\mathbb{Z}$. Notice that $H^r(\mathbb{S}^2) \subset C(\mathbb{S}^2)$ if and only if $r > \dim(\mathbb{S}^2)/2 = 1$. For functions $g, h : \mathbb{S}^2 \to \mathbb{C}$ let us denote $M_g h := gh$. When $r > 1$, $H^r(\mathbb{S}^2)$ is a Banach algebra when equipped with the norm

$$g \mapsto \sup_{h \in H^r(\mathbb{S}^2):\ \|h\|_{H^r(\mathbb{S}^2)} \leq 1} \|gh\|_{H^r(\mathbb{S}^2)} = \|M_g\|_{\mathcal{L}(H^r(\mathbb{S}^2))}.$$

This Banach algebra norm is equivalent to the Sobolev norm:

$$\|g\|_{H^r(\mathbb{S}^2)} \leq \|M_g\|_{\mathcal{L}(H^r(\mathbb{S}^2))} \leq c_r\ \|g\|_{H^r(\mathbb{S}^2)}.$$

Here c_r is a constant, and $\lim_{r \to 1+} c_r = \infty$. Thereby we get

$$\begin{aligned}
\left|\widehat{f_{l_1}^{m_1}\mu_{nl_2}^{m_2}}(l,m)\right|^2 &= \langle l\rangle^{-2r}\ \|(f_{l_1}^{m_1}\mu_{nl_2}^{m_2})_l^m\|_{H^r(\mathbb{S}^2)}^2 \\
&\leq \langle l\rangle^{-2r}\ \|f_{l_1}^{m_1}\mu_{nl_2}^{m_2}\|_{H^r(\mathbb{S}^2)}^2 \\
&= \langle l\rangle^{-2r}\ \|M_{f_{l_1}^{m_1}}\mu_{nl_2}^{m_2}\|_{H^r(\mathbb{S}^2)}^2 \\
&\leq \langle l\rangle^{-2r}\ \|M_{f_{l_1}^{m_1}}\|_{\mathcal{L}(H^r(\mathbb{S}^2))}^2\ \|\mu_{nl_2}^{m_2}\|_{H^r(\mathbb{S}^2)}^2 \\
&\leq c_r^2\ \langle l\rangle^{-2r}\ \|f_{l_1}^{m_1}\|_{H^r(\mathbb{S}^2)}^2\ \|\mu_{nl_2}^{m_2}\|_{H^r(\mathbb{S}^2)}^2 \\
&= c_r^2\ \langle l\rangle^{-2r}\ \langle l_1\rangle^{2r}\ \langle l_2\rangle^{2r}\ \left|\widehat{f}(l_1,m_1)\right|^2\ |\widehat{\mu_n}(l_2,m_2)|^2.
\end{aligned}$$

By Lemma 2 above, $|\widehat{\mu_n}(l_2,m_2)|^2 \leq C^2\ \langle l_2\rangle$, and this concludes the proof □

Theorem 1. *If $-1 < s < t$ and $t > 7/2$ then*

$$\|f - Q_n f\|_{H^s(\mathbb{S}^2)} \leq c_{s,t,\varepsilon}\ n^{s-t}\ n^{4+\varepsilon}\ \|f\|_{H^t(\mathbb{S}^2)}.$$

Here $c_{s,t,\varepsilon} < \infty$ is a constant depending only on s, t and $\varepsilon > 0$.

Proof. First, $H^{k+1+\varepsilon}(\mathbb{S}^2) \subset C^k(\mathbb{S}^2) \subset H^k(\mathbb{S}^2)$ for every $k \in \mathbb{N}$ and $\varepsilon > 0$, and furthermore $H^{k+1}(\mathbb{S}^2) \not\subset C^k(\mathbb{S}^2)$. Therefore we assume that $f \in H^t(\mathbb{S}^2)$ for some $t > 1$. Trivially

$$\begin{aligned}
\|f - Q_n f\|_{H^s(\mathbb{S}^2)} &= \|(f - P_n f) + (P_n f - Q_n f)\|_{H^s(\mathbb{S}^2)} \\
&\leq \|f - P_n f\|_{H^s(\mathbb{S}^2)} + \|P_n f - Q_n f\|_{H^s(\mathbb{S}^2)}.
\end{aligned}$$

Lemma 1 gave the optimal estimate for $\|f - P_n f\|_{H^s(\mathbb{S}^2)}$, and the difficulties come when dealing with $\|P_n f - Q_n f\|_{H^s(\mathbb{S}^2)}$. Now

$$\begin{aligned}
Q_n f &= P_n(f\mu_n) \\
&= \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \widehat{f\mu_n}(l,m)\ Y_l^m.
\end{aligned}$$

When $0 \le l < n$, we have

$$\widehat{f\mu_n}(l,m) = \widehat{f}(l,m) + \sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2: \\ m_2 \in 2n\mathbb{Z} \\ |m_j| \le l_j \\ m_1+m_2=m}} \widehat{f_{l_1}^{m_1} \mu_{nl_2}^{m_2}}(l,m)$$

by Lemmata 2 and 3. Thereby

$$\begin{aligned} & (Q_n - P_n)f \\ = & \sum_{l=0}^{n-1} \sum_{m=-l}^{l} Y_l^m \sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2: \\ m_2 \in 2n\mathbb{Z} \\ |m_j| \le l_j \\ m_1+m_2=m}} \widehat{f_{l_1}^{m_1} \mu_{nl_2}^{m_2}}(l,m). \end{aligned}$$

We have

$$\begin{aligned} & \|(P_n - Q_n)f\|_{H^s(\mathbb{S}^2)}^2 \\ = & \sum_{l=0}^{n-1} \langle l \rangle^{2s} \sum_{m=-l}^{l} \left| \sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2: \\ m_2 \in 2n\mathbb{Z} \\ |m_j| \le l_j \\ m_1+m_2=m}} \widehat{f_{l_1}^{m_1} \mu_{nl_2}^{m_2}}(l,m) \right|^2 \\ \le & \sum_{l=0}^{n-1} \langle l \rangle^{2s} \sum_{m=-l}^{l} \left(\sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2: \\ m_2 \in 2n\mathbb{Z} \\ |m_j| \le l_j \\ m_1+m_2=m}} \left| \widehat{f_{l_1}^{m_1} \mu_{nl_2}^{m_2}}(l,m) \right| \right)^2, \end{aligned}$$

and by Lemma 4 we get

$$\begin{aligned} & \|(P_n - Q_n)f\|_{H^s(\mathbb{S}^2)}^2 \\ \le & C_r^2 \sum_{l=0}^{n-1} \langle l \rangle^{2s-2r} \sum_{m=-l}^{l} \\ & \left(\sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2: \\ m_2 \in 2n\mathbb{Z} \\ |m_j| \le l_j \\ m_1+m_2=m}} \langle l_1 \rangle^r \ \langle l_2 \rangle^{r+1/2} \ \left| \widehat{f}(l_1,m_1) \right| \right)^2. \end{aligned}$$

The Hölder inequality gives us the next estimate:

$$\begin{aligned}
&\|(P_n - Q_n)f\|^2_{H^s(\mathbb{S}^2)} \\
\leq\ & C_r^2 \sum_{l=0}^{n-1} \langle l\rangle^{2s-2r} \sum_{m=-l}^{l} \\
&\left(\sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2:\\ m_2\in 2n\mathbb{Z}\\ |m_j|\leq l_j\\ m_1+m_2=m}} \langle l_1\rangle^{2r-2t}\ \langle l_2\rangle^{2r+1} \right) \times \\
&\times \left(\sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2:\\ m_2\in 2n\mathbb{Z}\\ |m_j|\leq l_j\\ m_1+m_2=m}} \langle l_1\rangle^{2t} \left| \widehat{f_{l_1}^{m_1} \mu_{nl_2}^{m_2}}(l,m) \right|^2 \right)
\end{aligned}$$

for some t large enough. Moreover,

$$\begin{aligned}
& \sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2:\\ m_2\in 2n\mathbb{Z}\\ |m_j|\leq l_j\\ m_1+m_2=m}} \langle l_1\rangle^{2r-2t}\ \langle l_2\rangle^{2r+1} \\
\leq\ & \sum_{l_2=2n}^{\infty} l_2^{2r+1} \sum_{l_1=l_2-l}^{l_2+l} (1 + 2\min\{l_1, l_2/(2n)\})\ l_1^{2r-2t} \\
\leq\ & \sum_{l_2=2n}^{\infty} l_2^{2r+1}\ (1 + l_2/n) \sum_{l_1=l_2-l}^{l_2+l} l_1^{2r-2t} \\
\leq\ & (2l+1) \sum_{l_2=2n}^{\infty} l_2^{2r+1}\ (1 + l_2/n)\ (l_2 - n)^{2r-2t} \\
\overset{k:=l_2-n}{\leq}\ & (2l+1) \sum_{k=n}^{\infty} (k+n)^{2r+1}\ (2 + k/n)\ k^{2r-2t} \\
\leq\ & (2l+1) \sum_{k=n}^{\infty} 2^{2r+1}\ k^{2r+1}\ (2 + k/n)\ k^{2r-2t} \\
\leq\ & c_{t,r}\ \langle l\rangle\ n^{4r-2t+2},
\end{aligned}$$

where $c_{t,r}$ depends on t and r. In the previous calculation, we had to demand that $4r - 2t + 2 < -1$, i.e. $t > 2r + 3/2$. Since $r > 1$, we must have $t > 7/2$.

With this knowledge, we get

$$\begin{aligned}
&\|(P_n - Q_n)f\|^2_{H^s(\mathbb{S}^2)} \\
\leq\ & C_r^2\ c_{t,r}\ n^{4r-2t+2} \sum_{l=0}^{n-1} \langle l\rangle^{2s-2r+1} \sum_{m=-l}^{l} \\
& \left(\sum_{l_2=2n}^{\infty} \sum_{l_1=l_2-l}^{l_2+l} \sum_{\substack{m_1,m_2:\\ m_2\in 2n\mathbb{Z}\\ |m_j|\leq l_j\\ m_1+m_2=m}} \langle l_1\rangle^{2t} \left|\widehat{f}(l_1,m_1)\right|^2 \right) \\
=\ & C_r^2\ c_{t,r}\ n^{4r-2t+2} \sum_{l=0}^{n-1} \langle l\rangle^{2s-2r+1} \sum_{l_1=n+1}^{\infty} \langle l_1\rangle^{2t} \sum_{l_2:\max\{2n,l_1-l\}\leq l_2\leq l_1+l} \\
& \sum_{m=-l}^{l} \sum_{\substack{m_1,m_2:\\ m_2\in 2n\mathbb{Z}\\ |m_j|\leq l_j\\ m_1+m_2=m}} \left|\widehat{f}(l_1,m_1)\right|^2 \\
\leq\ & C_r^2\ c_{t,r}\ n^{4r-2t+2} \sum_{l=0}^{n-1} \langle l\rangle^{2s-2r+1} \sum_{l_1=n+1}^{\infty} \langle l_1\rangle^{2t}\ (2l+1)\ \times \\
& \times\ (2l+1) \sum_{m_1=-l_1}^{l_1} \left|\widehat{f}(l_1,m_1)\right|^2 \\
\leq\ & 9\ C_r^2\ c_{t,r}\ n^{4r-2t+2} \sum_{l=0}^{n-1} \langle l\rangle^{2s-2r+3} \sum_{l_1=n+1}^{\infty} \langle l_1\rangle^{2t} \sum_{m_1=-l_1}^{l_1} \left|\widehat{f}(l_1,m_1)\right|^2 \\
\leq\ & 9\ C_r^2\ c_{t,r}\ n^{4r-2t+2} \sum_{l=0}^{n-1} \langle l\rangle^{2s-2r+3}\ \|f\|^2_{H^t(\mathbb{S}^2)}.
\end{aligned}$$

We may consider $r > 1$ to depend on t. We must suppose that $2s-2r+3 > -1$, i.e. $s > r-2 > -1$. Then $\sum_{l=0}^{n-1} \langle l\rangle^{2s-2r+3} \leq c'_{s-r}\ n^{2s-2r+4}$, so that

$$\|(P_n - Q_n)f\|^2_{H^s(\mathbb{S}^2)} \leq C^2_{s,t,r}\ n^{2s-2t}\ n^{2r+6}\ \|f\|^2_{H^t(\mathbb{S}^2)}.$$

Here $2r+6 = 2(r+3) > 2\cdot 4$. This yields

$$\|(P_n - Q_n)f\|_{H^s(\mathbb{S}^2)} \leq C'_{s,t,\varepsilon}\ n^{s-t}\ n^{4+\varepsilon}\ \|f\|_{H^t(\mathbb{S}^2)},$$

where $C_{t\varepsilon}$ is a constant depending on t and $\varepsilon > 0$. Consequently, an estimate for $\|f - Q_n f\|_{H^s(\mathbb{S}^2)}$ is established □

4 Insufficiency of methods

It is also important to notice that the results in numerical Fourier analysis on $\mathbb{S}^2$ above are quite robust; more delicate treatises will be needed. To demonstrate this, let us apply our technique on $\mathbb{S}^2$ to get estimates for interpolation projections on $\mathbb{S}^1$. The notation in this section is parallel to the $\mathbb{S}^2$-case.

We identify $\mathbb{S}^1$ with $[0,1[\subset \mathbb{R}$. Recall that here

$$P_n f = \sum_{\xi=-n+1}^{n} \widehat{f}(\xi)\ e_\xi.$$

Lemma 1'. *If $s \leq t$ then*

$$\|f - P_n f\|_{H^s(\mathbb{S}^1)} \leq n^{s-t}\ \|f\|_{H^t(\mathbb{S}^1)}.$$

This result is optimal, but we will present non-optimal results, yet in analogy to the interpolation on the sphere $\mathbb{S}^2$ — we must remember that the main point of this section is to demonstrate how insufficient our methods for $\mathbb{S}^2$ may be.

The sampling points are $\{j/(2n) :\ 0 \leq j < 2n\}$, and each point carries the equal weight $1/(2n)$. The corresponding sampling measure is

$$\mu_n = \frac{1}{2n} \sum_{j=0}^{2n-1} \delta_{j/(2n)}.$$

Let $Q_n f := P_n(f\mu_n)$. Now we can calculate *all* the Fourier coefficients of μ_n:

$$\widehat{\mu_n}(\xi) = \begin{cases} 1, & \text{when } \xi \in 2n\mathbb{Z}, \\ 0, & \text{when } \xi \notin 2n\mathbb{Z}. \end{cases}$$

But we want to demonstrate what could possibly cause bad estimates for $\mathbb{S}^2$ in the previous section. Hence let us modestly state the following:

Lemma 2'.

$$\widehat{\mu_n}(\xi) = \begin{cases} 1, & \text{when } \xi = 0, \\ 0, & \text{when } \xi \notin 2n\mathbb{Z}, \end{cases}$$

Moreover,

$$|\widehat{\mu_n}(\xi)| \leq 1.$$

For a distribution $g \in \mathcal{D}'(\mathbb{S}^1)$, let $g_\xi = \widehat{g}(\xi)\ e_\xi$. We write the well-known formula $\widehat{fg}(\xi) = (\widehat{f} * \widehat{g})(\xi)$ in the following form:

Lemma 3'. *Let $f \in \mathcal{D}(\mathbb{S}^1)$ and $g \in \mathcal{D}'(\mathbb{S}^1)$. Then $fg \in \mathcal{D}'(\mathbb{S}^1)$ satisfies*

$$\widehat{fg}(\xi) = \sum_{\xi_2 \in \mathbb{Z}} \sum_{\xi_1 = \xi - \xi_2} \widehat{f_{\xi_1} g_{\xi_2}}(\xi).$$

The Fourier coefficients of $Q_n f$ can be written down precisely, but we settle for a weaker result:

Lemma 4'. *Let f be a function, and let μ_n be the sampling measure presented above. Then*

$$\left|\widehat{f\mu_n}(\xi)\right| \leq C_r \ \langle \xi \rangle^{-r} \ \langle \xi_1 \rangle^r \ \langle \xi_2 \rangle^r \ \left|\widehat{f}(\xi_1)\right|,$$

where $r > 1/2$.

Proof. The operator norm endows $H^r(\mathbb{S}^1)$ with a Banach algebra structure if and only if $r > \dim(\mathbb{S}^1)/2 = 1/2$. Now the estimation process goes as in the proof of Lemma 4, and we notice that $|\widehat{\mu_n}(\xi_2)| \leq 1$ here □

Theorem 1'. *If $-1/2 < s < t$ and $t > 3/2$ then*

$$\|f - Q_n f\|_{H^s(\mathbb{S}^1)} \leq c_{s,t,\varepsilon} \ n^{s-t} \ n \ \|f\|_{H^t(\mathbb{S}^1)}.$$

Here $c_{t,s,\varepsilon} < \infty$ is a constant depending only on s, t and $\varepsilon > 0$.

Proof. First,

$$\|f - Q_n f\|_{H^s(\mathbb{S}^1)} \leq \|f - P_n f\|_{H^s(\mathbb{S}^1)} + \|P_n f - Q_n f\|_{H^s(\mathbb{S}^1)}.$$

Lemma 1' gave the optimal estimate for $\|f - P_n f\|_{H^s(\mathbb{S}^1)}$. We must estimate $\|P_n f - Q_n f\|_{H^s(\mathbb{S}^1)}$, but not too well: we follow the path taken in the proof of Theorem 1. If $-n < \xi \leq n$ then Lemma 2' yields

$$\widehat{f\mu_n}(\xi) = \widehat{f}(\xi) + \sum_{\xi_2:\ |\xi_2| \geq 2n} \sum_{\xi_1 = \xi - \xi_2} \widehat{f_{\xi_1} \mu_{n\xi_2}}(\xi).$$

Hence

$$(Q_n - P_n) f = \sum_{\xi = -n+1}^{n} e_\xi \sum_{\xi_2:\ |\xi_2| \geq 2n} \sum_{\xi_1 = \xi - \xi_2} \widehat{f_{\xi_1} \mu_{n\xi_2}}(\xi),$$

so that Lemma 4 gives

$$
\begin{aligned}
&\|(P_n - Q_n)f\|^2_{H^s(\mathbb{S}^1)} \\
= &\sum_{\xi=-n+1}^{n} \langle\xi\rangle^{2s} \left| \sum_{\xi_2:\ |\xi_2|\geq 2n} \ \sum_{\xi_1=\xi-\xi_2} \widehat{f_{\xi_1}\mu_{n\xi_2}}(\xi) \right|^2 \\
\leq &\sum_{\xi=-n+1}^{n} \langle\xi\rangle^{2s} \left(\sum_{\xi_2:\ |\xi_2|\geq 2n} \ \sum_{\xi_1=\xi-\xi_2} \left|\widehat{f_{\xi_1}\mu_{n\xi_2}}(\xi)\right| \right)^2 \\
\leq &\ C_r^2 \sum_{\xi=-n+1}^{n} \langle\xi\rangle^{2s-2r} \left(\sum_{\xi_2:\ |\xi_2|\geq 2n} \ \sum_{\xi_1=\xi-\xi_2} \langle\xi_1\rangle^r \ \langle\xi_2\rangle^r \ \left|\widehat{f_{\xi_1}\mu_{n\xi_2}}(\xi)\right| \right)^2 .
\end{aligned}
$$

The Hölder inequality yields

$$
\begin{aligned}
&\|(P_n - Q_n)f\|^2_{H^s(\mathbb{S}^1)} \\
\leq &\sum_{\xi=-n+1}^{n} \langle\xi\rangle^{2s} \left(\sum_{\xi_2:\ |\xi_2|\geq 2n} \ \sum_{\xi_1=\xi-\xi_2} \langle\xi_1\rangle^{2r-2t} \ \langle\xi_2\rangle^{2r} \right) \times \\
&\times \left(\sum_{\xi_2:\ |\xi_2|\geq 2n} \ \sum_{\xi_1=\xi-\xi_2} \langle\xi_1\rangle^{2t} \ \left|\widehat{f}(\xi_1)\right|^2 \right).
\end{aligned}
$$

Here

$$
\sum_{\xi_2:\ |\xi_2|\geq 2n} \ \sum_{\xi_1=\xi-\xi_2} \langle\xi_1\rangle^{2r-2t} \ \langle\xi_2\rangle^{2r} \leq C_{t,r} \ n^{4r-2t+1};
$$

here $4r - 2t < -1$, i.e. $t > 2r + 1/2 > 3/2$. Consequently,

$$
\begin{aligned}
&\|(P_n - Q_n)f\|^2_{H^s(\mathbb{S}^1)} \\
\leq &\ c_{t,r} \ n^{4r-2t+1} \sum_{\xi=-n+1}^{n} \langle\xi\rangle^{2s-2r} \sum_{\xi_2:\ |\xi_2|\geq 2n} \langle\xi_1\rangle^{2t} \ \left|\widehat{f}(\xi_1)\right|^2 \\
\leq &\ c'_{t,r} \ n^{2(s-t)} \ n^{2(r+1)} \ \|f\|^2_{H^t(\mathbb{S}^1)},
\end{aligned}
$$

where $2s - 2r > -1$, i.e. $s > r - 1 > -1/2$. Since $r > 1/2$, we have $r + 1 > 3/2$. Thus we eventually get

$$
\|f - Q_n f\|_{H^s(\mathbb{S}^1)} \leq c_{t,s,\varepsilon} \ n^{s-t} \ n^{3/2+\varepsilon} \ \|f\|_{H^t(\mathbb{S}^1)},
$$

where $-1/2 \leq s < t$, $t > 3/2$ and $\varepsilon > 0$. □

5 Conclusion

Theorem 1' shows that our method does not give the optimal result even in the extremely simple case of the circle $\mathbb{S}^1$: We lost $3/2 + \varepsilon$ Sobolev space degrees, and we had to suppose that $f \in H^t(\mathbb{S}^1)$ with $t > 3/2$, even though actually $t > 1/2$ can be assumed. This may suggest that our results for the 2-sphere $\mathbb{S}^2$ are not the best possible.

Acknowledgements. The work was funded by the Academy of Finland.

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ISBN 951-22-6828-0
ISSN 0784-3143