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Abstract: *We prove that a necessary condition for a Fourier-multiplier $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{B}(X;Y))$ to give a bounded operator $T_m : f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ from $H^1_{\mathrm{at}}(\mathbb{R}^n; X)$ to $L^1(\mathbb{R}^n; Y)$ is the R-boundedness of $\{m(y) |\ y \neq 0\}$. Here X, Y are arbitrary Banach spaces and H^1_{at} is the atomic Hardy space. As a consequence of this result, we show that the maximal L^p-regularity of the abstract Cauchy problem $u' = Au + f$ on a UMD-space is equivalent to maximal H^1_{at}-regularity.*

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1 Introduction

By now it is understood that R-boundedness of the pointwise values of an operator-valued function is intimately connected with the boundedness of the related Fourier-multiplier between L^p-spaces of vector-valued functions. To be more precise, let $m \in C(\mathbb{R} \setminus \{0\}; \mathcal{B}(X;Y))$ and $1 < p < \infty$, and consider the following three statements:

A. The collection $\{m(t) : X \to Y \,|\, t \neq 0\}$ is R-bounded.

B. The function m is C^1 away from the origin and $\{tm'(t) : X \to Y \,|\, t \neq 0\}$ is R-bounded.

C. The multiplier operator $T_m : f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ is bounded from $L^p(\mathbb{R};X)$ to $L^p(\mathbb{R};Y)$.

If X and Y are UMD spaces, the following implications hold:

$$A \cap B \implies C \implies A.$$

The first one is due to L. Weis [W] and the second (which is true for general Banach spaces) was shown by Ph. Clément and J. Prüss [CP]. Analogous results for Fourier-multipliers in the periodic case have been proved by W. Arendt and S. Bu [AB].

It was shown by the author in [H] that the implication $A \cap B \implies C$ remains valid also with the Lebesgue–Bôchner spaces $L^p(\mathbb{R};X)$, $L^p(\mathbb{R};Y)$ replaced by the atomic Hardy spaces $H^1_{\mathrm{at}}(\mathbb{R};X)$ and $H^1_{\mathrm{at}}(\mathbb{R};Y)$. The purpose of this paper is to establish the partial converse $C \implies A$ in this setting (again, with arbitrary Banach spaces X, Y), i.e., to show that R-boundedness is a necessary condition for the boundedness of Fourier-multipliers also on H^1.

The paper is organised as follows: The main theorem stated in the title is precisely formulated and proved in §2, whereas the implications of this result to maximal regularity are considered in §3. In §4 we give a simpler proof of the main result for the one-dimensional domain and in §5 we briefly consider some related ideas in the periodic situation, including an interesting necessary condition for bounded multipliers from $L^\infty(\mathbb{T};X)$ to $\mathrm{BMO}(\mathbb{T};Y)$.

Notations and conventions. We denote by ε_j, $j = 1, 2, \ldots$, a sequence of independent random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$ which satisfy $\mathbb{P}(\varepsilon_j = 1) = \mathbb{P}(\varepsilon_j = -1) = 1/2$. $\mathbb{E}$ denotes the expectation related to the probability measure $\mathbb{P}$. The random variables ε_j are referred to as the *Rademacher functions.*

We recall [CP, W] that $\mathcal{T} \subset \mathcal{B}(X;Y)$ is called *R-bounded*, the "R" being short for Rademacher, randomized or Riesz, if for some $p \in (0, \infty)$ and $C < \infty$ and for all $N \in \mathbb{Z}_+$, $x_j \in X$, $T_j \in \mathcal{T}$ the inequality

$$\left(\mathbb{E} \left| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right|_Y^p \right)^{\frac{1}{p}} \leq C \left(\mathbb{E} \left| \sum_{j=1}^{N} \varepsilon_j x_j \right|_X^p \right)^{\frac{1}{p}}$$

holds. It follows from the Khintchine–Kahane inequality that for each fixed $\mathcal{T}$, the condition in fact holds true either for all $p \in (0, \infty)$ (with C possibly depending on p) or for none. We shall be mostly concerned with the case $p = 1$, and we refer to the smallest C in this inequality as the R-bound of $\mathcal{T}$ and denote it by $\mathcal{R}(\mathcal{T})$.

One of the most standard tools related to R-boundedness is the *contraction principle* (of J.-P. Kahane) stating that

$$\left(\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j\lambda_j x_j\right|_X^p\right)^{\frac{1}{p}} \leq \frac{\pi}{2}\left(\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j x_j\right|_X^p\right)^{\frac{1}{p}}$$

for $p \geq 1$, $N \in \mathbb{Z}_+$, $x_j \in X$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| \leq 1$.

In the literature, one usually finds this with the constant 2 in place of $\pi/2$. Even though the size of constants is quite immaterial for our purposes, we shall use the inequality in this sharper form, which is proved in [PW, §3.5.4].

The *atomic Hardy space* $H^1_{\mathrm{at}}(\mathbb{R}^n; X)$ consists of those $f \in L^1(\mathbb{R}^n; X)$ which have an *atomic decomposition*

$$f = \sum_{k=1}^{\infty} a_k \qquad \text{with} \qquad \sum_{k=1}^{\infty}\left|B^k\right|^{\frac{1}{2}}\|a_k\|_{L^2(\mathbb{R}^n;X)} < \infty,$$

where the $a_k \in L^2(\mathbb{R}^n; X)$ have their support contained in the balls B^k and a vanishing integral: $\int a_k(y)\,\mathrm{d}y = 0$. The norm in $H^1_{\mathrm{at}}(\mathbb{R}^n; X)$ is the infimum of the sums above taken over all such decompositions of f.

We note that in the literature it is customary to impose a size condition on the atoms a_k, so that the decomposition involves extra scale factors $\lambda_k \in \mathbb{C}$, such that $f = \sum \lambda_k a_k$, and the H^1_{at}-norm is defined in terms of $\sum|\lambda_k|$. The present definition is equivalent and simpler to use for our purposes.

Since $\|a_k\|_{L^1} \leq \left|B^k\right|^{\frac{1}{2}}\|a_k\|_{L^2}$, it follows that $\|f\|_{L^1(\mathbb{R}^n;X)} \leq \|f\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)}$; thus $H^1_{\mathrm{at}}(\mathbb{R}^n; X) \hookrightarrow L^1(\mathbb{R}^n; X)$. Let us also observe the following *dilation property* of the H^1_{at}-norm, analogous to that of the L^1-norm: If $f = \sum a_k \in H^1_{\mathrm{at}}(\mathbb{R}^n; X)$ and $r > 0$, then $r^n f(r\cdot) = \sum r^n a_k(r\cdot) =: \sum A_k$. Here A_k is supported in $r^{-1}B^k$ (the ball concentric with B^k and having r^{-1} times the same radius); thus

$$\|r^n f(r\cdot)\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)} \;\leq\; \sum_{k=1}^{\infty}\left|r^{-1}B^k\right|^{\frac{1}{2}}\|r^n a_k(r\cdot)\|_{L^2} \;=\; \sum_{k=1}^{\infty}\left|B^k\right|^{\frac{1}{2}}\|a_k\|_{L^2},$$

and taking the infimum over all decomposition $f = \sum a_k$ we deduce the inequality $\|r^n f(r\cdot)\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)} \leq \|f\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)}$. Writing $f = r^{-n}g(r^{-1}\cdot)$ with $g = r^n g(r\cdot)$ we get the same inequality in the opposite direction, thus in fact an equality.

Below we also consider (in the one-dimensional setting) another Hardy space $H^1_{\mathrm{con}}(\mathbb{R}; X)$ defined in terms of the conjugate function, but we postpone its definition until we need it.

For more information on Hardy spaces of vector-valued functions we refer to [B]; a short review is also given in [H].

2 The main result and its proof

We are going to prove the following theorem:

Theorem 1. *Suppose $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathcal{B}(X;Y))$ is such that the multiplier operator $T_m f := \mathcal{F}^{-1}[m\hat{f}]$ acts boundedly from $H^1_{\mathrm{at}}(\mathbb{R}^n;X)$ to $L^1(\mathbb{R}^n;Y)$.*

Then m is strongly continuous away from the origin and moreover

$$\mathcal{R}\left(\{m(y) : X \to Y \mid y \neq 0\}\right) \leq C_n \left\| T_m : H^1_{\mathrm{at}}(\mathbb{R}^n;X) \to L^1(\mathbb{R}^n;Y)\right\|,$$

where the constant C_n depends only on the dimension n. In particular, $m \in L^\infty(\mathbb{R}^n;\mathcal{B}(X;Y))$.

Before we prove this, we need two lemmata. First of all, we require a tool for estimating the H^1_{at}-norms we will encounter. Let B_r be the ball in $\mathbb{R}^n$ of radius r centered at the origin and $A_{r,R} := B_R \setminus B_r$ the annulus with inner and outer radii r and R, respectively.

Lemma 1. *Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n;X)$ satisfies $\int \varphi(x)\,\mathrm{d}x = 0$. Then $\varphi \in H^1_{\mathrm{at}}(\mathbb{R}^n;X)$, and the norm is estimated by*

$$\|\varphi\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)} \leq \sum_{k=1}^{\infty} |B_k|^{\frac{1}{2}} \left\|\varphi \mathbf{1}_{A_{k-1,k}}\right\|_{L^2(\mathbb{R}^n;X)} + (1+2^{\frac{n}{2}}) \sum_{k=1}^{\infty} \left\|\varphi \mathbf{1}_{B_k^c}\right\|_{L^1(\mathbb{R}^n;X)}$$

It is easy to see that the sum is indeed finite for a rapidly decreasing φ.

Proof. Let us denote

$$\varphi_k := \left(\varphi - \frac{1}{|B_k|}\int_{B_k} \varphi(y)\,\mathrm{d}y\right)\mathbf{1}_{B_k} = \left(\varphi + \frac{1}{|B_k|}\int_{B_k^c} \varphi(y)\,\mathrm{d}y\right)\mathbf{1}_{B_k},$$

where it is clear from the first form that $\int \varphi_k(x)\,\mathrm{d}x = 0$, and the latter equality follows from the assumption that the total integral of φ vanishes. Then

$$\begin{aligned}|\varphi(x) - \varphi_k(x)|_X &\leq |\varphi(x)|_X\, \mathbf{1}_{B_k^c}(x) + \frac{1}{|B_k|}\int_{B_k^c} |\varphi(y)|_X\,\mathrm{d}y \\ &\leq \max_{|y|\geq k} |\varphi(y)|_X + \frac{1}{|B_k|}\int_{B_k^c} |\varphi(y)|_X\,\mathrm{d}y \underset{k\to\infty}{\longrightarrow} 0;\end{aligned}$$

thus $\varphi_k \to \varphi$ uniformly as $k \to \infty$.

We then define $\phi_1 := \varphi_1$ and

$$\phi_k := \varphi_k - \varphi_{k-1} \text{ for } k > 1 \qquad \text{so that} \qquad \sum_{k=1}^{N} \phi_k = \varphi_N \underset{N\to\infty}{\longrightarrow} \varphi \text{ uniformly.}$$

Thus we have $\varphi = \sum_{k=1}^{\infty} \phi_k$, where $\operatorname{supp} \phi_k \subset B_k$ and $\int \phi_k(x)\,\mathrm{d}x = 0$. This is hence an atomic decomposition of φ, and we have

$$\|\varphi\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)} \leq \sum_{k=1}^{\infty} |B_k|^{\frac{1}{2}} \|\phi_k\|_{L^2(\mathbb{R}^n;X)} .$$

Hence it remains to estimate the L^2-norm of

$$\phi_k = \varphi \mathbf{1}_{A_{k-1,k}} + \frac{\mathbf{1}_{B_k}}{|B_k|} \int_{B_k^c} \varphi(y)\,\mathrm{d}y - \frac{\mathbf{1}_{B_{k-1}}}{|B_{k-1}|} \int_{B_{k-1}^c} \varphi(y)\,\mathrm{d}y,$$

where the last term is interpreted as 0 for $k = 1$, and this yields

$$\begin{aligned}\|\phi_k\|_{L^2(\mathbb{R}^n;X)} \leq \left\|\varphi \mathbf{1}_{A_{k-1,k}}\right\|_{L^2(\mathbb{R}^n;X)} &+ \frac{1}{|B_k|^{\frac{1}{2}}} \left\|\varphi \mathbf{1}_{B_k^c}\right\|_{L^1(\mathbb{R}^n;X)} \\ &+ \frac{1}{|B_{k-1}|^{\frac{1}{2}}} \left\|\varphi \mathbf{1}_{B_{k-1}^c}\right\|_{L^1(\mathbb{R}^n;X)} .\end{aligned}$$

Multiplying by $|B_k|^{\frac{1}{2}}$, observing that $|B_k|^{\frac{1}{2}} / |B_{k-1}|^{\frac{1}{2}} = (k/(k-1))^{\frac{n}{2}} \leq 2^{\frac{n}{2}}$ and summing over k we obtain the asserted estimate. □

The following simple result handles the easy part of the main theorem. It is not really crucial for the proof of the assertion concerning the R-boundedness of the multiplier m, since the strong continuity at $y \neq 0$ is only exploited via the fact that these points are strong Lebesgue points of m, and in any case we know that almost every point is a Lebesgue point. Nevertheless, we obtain a somewhat neater form of the theorem without the need for almost-every-qualifications.

Lemma 2. *If $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{B}(X;Y))$ defines a bounded multiplier operator $T_m f := \mathcal{F}^{-1}[m\hat{f}]$, which maps $H^1_{\mathrm{at}}(\mathbb{R}^n;X)$ boundedly into $L^1(\mathbb{R}^n;Y)$, then m is strongly continuous at every $y \neq 0$. In particular, every $y \neq 0$ is a strong Lebesgue point of m.*

Proof. Let $y_0 \neq 0$. Then there exists a test function $\hat{\varphi} \in \mathcal{D}(\mathbb{R})$, which is supported away from the origin and equals unity in a neighbourhood of y_0. Then for $x \in X$ we have $\varphi(\cdot)x \in \mathcal{S}(\mathbb{R}^n;X)$ and $\int \varphi(y)x\,\mathrm{d}y = \hat{\varphi}(0)x = 0$. Hence $\varphi(\cdot)x \in H^1_{\mathrm{at}}(\mathbb{R};X)$, and thus $T_m[\varphi(\cdot)x] \in L^1(\mathbb{R};Y)$. The Fourier transform of this latter function is $m(y)\hat{\varphi}(y)x$, and in a neighbourhood of y_0, this is just $m(y)x$. But the Fourier transform of an L^1-function is continuous, thus $y \mapsto m(y)x$ is continuous in a neighbourhood of y_0, and this being true for every $x \in X$ the assertion is established. □

Now we are ready to prove the main result.

Proof of Theorem 1. Let $N \in \mathbb{Z}_+$ and $x_1, \ldots, x_N \in X$, and let first

$$y_1, \ldots, y_N \in \{y = (y^1, \ldots, y^n) \in \mathbb{R}^n \mid y^n \geq 0,\ y \neq 0\},$$

i.e., the points are taken from the closed upper half-space, excluding the origin. Let us choose a (real-valued) test-function $\psi \in \mathcal{D}(\mathbb{R}^n)$ with support strictly contained in the lower half-space $\{y \in \mathbb{R}^n \mid y^n < 0\}$ and such that

$$\int_{\mathbb{R}^n} \psi^2(y)\,\mathrm{d}y = 1.$$

This function will be exploited in building an appropriate approximation of the identity; the reason for the support condition will become clear later. Since y_j is a Lebesgue point of $y \mapsto m(y)x_j$ by Lemma 2, we have

$$m(y_j)x_j = \lim_{k\to\infty} \int_{\mathbb{R}^n} m(y)x_j\psi^2(k(y_j - y))k^n\,\mathrm{d}y,$$

the convergence being in the norm of Y. Thus

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j m(y_j)x_j\right|_Y \\
&\qquad = \lim_{k\to\infty} k^n\mathbb{E}\left|\int_{\mathbb{R}^n}\sum_{j=1}^N \varepsilon_j m(y)\psi(k(y_j-y))x_j\psi(k(y_j-y))\,\mathrm{d}y\right|_Y .
\end{aligned}$$

Note that since the Rademacher functions ε_j are simple random variables, the expectation $\mathbb{E}$ is nothing but a weighted finite sum, and thus it certainly commutes with limits. (Of course, for more general random variables we could have simply invoked Fatou's lemma to yield the above result with "$= \lim$" replaced by "$\leq \liminf$", and the rest of the proof would run in exactly the same way.)

We then write

$$\begin{aligned}
m(y)\psi(k(y_j - y))x_j &= m(y)\mathcal{F}\mathcal{F}^{-1}[\psi(k(y_j - \cdot))x_j](y) \\
&= m(y)\mathcal{F}[e^{\mathrm{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)/k^n = \mathcal{F}T_m[e^{\mathrm{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)/k^n,
\end{aligned}$$

and using the duality equality $\int \hat{g}f\,\mathrm{d}y = \int g\hat{f}\,\mathrm{d}y$ of the Fourier transform we arrive at

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j m(y_j)x_j\right|_Y \\
&= \lim_{k\to\infty} k^{-n}\mathbb{E}\left|\int_{\mathbb{R}^n}\sum_{j=1}^N \varepsilon_j T_m[e^{\mathrm{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)e^{-\mathrm{i}2\pi y_j\cdot y}\hat{\psi}(-y/k)\,\mathrm{d}y\right|_Y \\
&\leq \liminf_{k\to\infty} k^{-n}\left\|\hat{\psi}\right\|_{L^\infty}\mathbb{E}\int_{\mathbb{R}^n}\left|\sum_{j=1}^N \varepsilon_j e^{-\mathrm{i}2\pi y_j\cdot y}T_m[e^{\mathrm{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)\right|_Y \mathrm{d}y.
\end{aligned}$$

We now invoke the contraction principle to get rid of the exponential factors $e^{-\mathrm{i}2\pi y_j\cdot y}$ and then the assumed boundedness of the operator T_m to yield

$$
\begin{aligned}
&\leq \frac{\pi}{2}\left\|\hat{\psi}\right\|_{L^\infty}\left\|T_m : H^1_{\mathrm{at}}\to L^1\right\|\liminf_{k\to\infty} k^{-n}\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j\right\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)} \\
&\qquad = \frac{\pi}{2}\left\|\hat{\psi}\right\|_{L^\infty}\left\|T_m\right\|\liminf_{k\to\infty}\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}(\cdot)x_j\right\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)}, \quad (1)
\end{aligned}
$$

where the last equality follows from the dilation property of the H^1_{at}-norm.

So far the proof has been completely parallel to that in [CP] concerning the L^p situation, except for the choice of our auxiliary function ψ, but now we are faced with the H^1_{at}-norm, with which the contraction principle can no longer be applied. Instead, we invoke Lemma 1 for the evaluation of this norm. Let us first check that the assumptions of the lemma are satisfied by

$$
\varphi(y) := \sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot y}\hat{\psi}(y)x_j :
$$

Certainly $\hat{\psi}\in\mathcal{S}(\mathbb{R}^n)$ since $\psi\in\mathcal{D}(\mathbb{R}^n)$, and since the exponential factors are C^∞ with bounded derivatives of all orders, the entire function φ belongs to $\mathcal{S}(\mathbb{R}^n;X)$. Moreover, recognizing the formula of the inverse Fourier transform, we have

$$
\int_{\mathbb{R}^n} e^{\mathbf{i}2\pi k y_j\cdot y}\hat{\psi}(y)\,\mathrm{d}y = \psi(ky_j) = 0,
$$

since $k>0$ and y_j is in the upper half-space, whereas ψ is supported in the lower half-space.

Hence we get, for the H^1_{at}-norm appearing in (1), the estimate

$$
\begin{aligned}
&\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}(\cdot)x_j\right\|_{H^1_{\mathrm{at}}(\mathbb{R}^n;X)} \\
&\qquad \leq \sum_{\ell=1}^\infty |B_\ell|^{\frac{1}{2}}\,\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}(\cdot)x_j\right\|_{L^2(\mathbb{R}^n;X)} \\
&\qquad\qquad + (1+2^{\frac{n}{2}})\sum_{\ell=1}^\infty \mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}\mathbf{1}_{B_\ell^c}(\cdot)x_j\right\|_{L^1(\mathbb{R}^n;X)}.
\end{aligned}
$$

Now we are back to L^p-norms, and the contraction principle applies again:

$$\le \frac{\pi}{2}\sum_{\ell=1}^{\infty}|B_\ell|^{\frac{1}{2}}\left(\mathbb{E}\left\|\sum_{j=1}^{N}\varepsilon_j\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}(\cdot)x_j\right\|^2_{L^2(\mathbb{R}^n;X)}\right)^{\frac{1}{2}}$$
$$+(1+2^{\frac{n}{2}})\frac{\pi}{2}\sum_{\ell=1}^{\infty}\mathbb{E}\left\|\sum_{j=1}^{N}\varepsilon_j\hat{\psi}\mathbf{1}_{B_\ell^c}(\cdot)x_j\right\|_{L^1(\mathbb{R}^n;X)}$$
$$=\frac{\pi}{2}\sum_{\ell=1}^{\infty}|B_\ell|^{\frac{1}{2}}\left\|\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}\right\|_{L^2(\mathbb{R}^n)}\left(\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j x_j\right|^2_X\right)^{\frac{1}{2}}$$
$$+(1+2^{\frac{n}{2}})\frac{\pi}{2}\sum_{\ell=1}^{\infty}\left\|\hat{\psi}\mathbf{1}_{B_\ell^c}\right\|_{L^1(\mathbb{R}^n)}\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j x_j\right|_X.$$

Finally, combining (1) with the estimate above and applying the Khintchine–Kahane inequality $\sqrt{\mathbb{E}|\sum\varepsilon_j x_j|_X^2}\le\sqrt{2}\mathbb{E}|\sum\varepsilon_j x_j|_X$ (see [PW, §4.1.10]), we get

$$\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j m(y_j)x_j\right|_Y\le C_n\left\|T_m:H^1_{\mathrm{at}}(\mathbb{R}^n;X)\to L^1(\mathbb{R}^n;Y)\right\|\left|\sum_{j=1}^{N}\varepsilon_j x_j\right|_X,$$

where the constant

$$C_n=\frac{\pi^2}{4}\left\|\hat{\psi}\right\|_{L^\infty(\mathbb{R}^n)}\left(\sqrt{2}\sum_{\ell=1}^{\infty}|B_\ell|^{\frac{1}{2}}\left\|\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}\right\|_{L^2(\mathbb{R}^n)}\right.$$
$$\left.+(1+2^{\frac{n}{2}})\sum_{\ell=1}^{\infty}\left\|\hat{\psi}\mathbf{1}_{B_\ell^c}\right\|_{L^1(\mathbb{R}^n)}\right)<\infty$$

depends only on the dimension n and the choice of the auxiliary function ψ, thus fixing one ψ once and for all, only on the dimension n.

It is clear that we can repeat the same argument for points $y_1,\dots,y_N$ in the lower half-space, exploiting another auxiliary function $\tilde{\psi}\in\mathcal{D}(\mathbb{R}^n)$ supported in the upper half-space (e.g., the reflection of ψ about the hyperplane $\{y\in\mathbb{R}^n|\ y^n=0\}$). Thus we get the R-boundedness of $\{m(y)|\ y\ne 0\}$ with an R-bound of the asserted form. □

3 New characterization of maximal regularity

It turns out that the main theorem proved above leads directly to some interesting implications related to the problem of maximal regularity. We consider the abstract Cauchy problem (ACP)

$$y'(t)=Ay(t)+f(t),\qquad t\ge 0,\qquad y(0)=0,$$

with A the generator of a bounded analytic semigroup on a UMD-space X, and f a given function taking values in this space. If $\mathfrak{F}$ is a normed function space (e.g., $\mathfrak{F} = L^p$ or $\mathfrak{F} = H^1_{\text{at}}$), the problem is said to have maximal $\mathfrak{F}$-regularity if for every $f \in \mathfrak{F}(\mathbb{R}; X)$ supported on the positive half-line, the mild solution y (with zero-fill on the negative half-line) of ACP satisfies $y', Ay \in \mathfrak{F}(\mathbb{R}; X)$, and moreover $\|Ay\|_{\mathfrak{F}(\mathbb{R};X)} \leq C \|f\|_{\mathfrak{F}(\mathbb{R};X)}$ with C independent of f.

Let us first recall what is known so far. The maximal regularity of this problem is equivalent to the boundedness of the multiplier operator whose multiplier is $m(t) := A(\mathbf{i}t - A)^{-1}$. Indeed, if y satisfies ACP at least in the sense of tempered distributions, then by taking the Fourier transform one finds that $\mathcal{F}[Ay](t) = A(\mathbf{i}t - A)^{-1}\hat{f}(t)$, hence $Ay = T_m f$. Thus, if m is a bounded Fourier-multiplier on the appropriate space, then we immediately get the desired estimate for Ay. Conversely, if ACP has maximal $\mathfrak{F}$-regularity, then for every $f \in \mathfrak{F}(\mathbb{R}; X)$ supported on $[0, \infty)$ we know that $\|Ay\|_{\mathfrak{F}(\mathbb{R};X)} = \|T_m f\|_{\mathfrak{F}(\mathbb{R};X)} \leq C \|f\|_{\mathfrak{F}(\mathbb{R};X)}$. But it is well-known and easy to see that the multiplier operators commute with translations; thus this implies for $\mathfrak{F} = L^p$ or $\mathfrak{F} = H^1_{\text{at}}$, whose norms are translation-invariant, that $\|T_m f\|_{\mathfrak{F}(\mathbb{R};X)} \leq C \|f\|_{\mathfrak{F}(\mathbb{R};X)}$ for every $f \in \mathfrak{F}(\mathbb{R}; X)$ supported on some half-line $[a, \infty)$. But such functions are dense in $L^p(\mathbb{R}; X)$ for all $p \in [1, \infty)$ as well as in $H^1_{\text{at}}(\mathbb{R}; X)$ (where this density follows, e.g., from the density of finite sums of atoms, which are compactly supported). Thus T_m is a bounded operator on $\mathfrak{F}(\mathbb{R}; X)$.

A special feature of the particular multiplier $m(t) = A(\mathbf{i}t - A)^{-1}$ is the fact that once it satisfies the R-boundedness condition $\mathcal{R}(\{m(t) |\ t \neq 0\}) < \infty$, it already satisfies the infinity of the conditions $\mathcal{R}(\{t^j (D^j m)(t) |\ t \neq 0\}) < \infty$, $j = 0, 1, 2, \ldots$ This follows by a direct computation from the form of the derivatives. (A similar property is actually shared by various multipliers related to maximal regularity problems; see e.g. [H].)

Now that we agree that the question of maximal regularity is a multiplier problem, we can deduce various facts. In the L^p setting, $1 < p < \infty$, we know from the theorem of L. Weis that

ACP has maximal regularity *if* $\mathcal{R}(\{m(t), tm'(t) |\ t \neq 0\}) < \infty$.

On the other hand, the result of Ph. Clément and J. Prüss shows that

ACP has maximal regularity *only if* $\mathcal{R}(\{m(t) |\ t \neq 0\}) < \infty$.

But now that the R-boundedness of $\{m(t) |\ t \neq 0\}$ already implies that of $\{tm'(t) |\ t \neq 0\}$, the two statement combine to give

ACP has maximal regularity *if and only if* $\mathcal{R}(\{m(t) |\ t \neq 0\}) < \infty$.

So much for the (by now well-known) L^p situation. But now we also have at our disposal analogues of all the theorems quoted above valid for H^1_{at} in place of L^p; indeed, the necessary condition for a Fourier-multiplier on H^1_{at} was just shown in the previous section, and the sufficient condition (the Mikhlin-type theorem) was proved in [H]. All these combined we can state the result:

Theorem 2. *Let X be a UMD-space and A the generator of a bounded analytic semigroup on X. Then the following conditions are equivalent:*

1. *ACP has maximal L^p-regularity for all $p \in (1, \infty)$.*

2. *ACP has maximal L^p-regularity for some $p \in (1, \infty)$.*

3. *ACP has maximal H^1_{at}-regularity.*

4. *The collection $\{A(\mathbf{i}t - A)^{-1} | \ t \neq 0\}$ is R-bounded.*

What appears to be new here is the fact that maximal L^p-regularity is implied by the maximal H^1_{at}-regularity. That L^p-regularity (for some $p \in (1, \infty)$) implies H^1_{at}-regularity was shown in [H]; the characterization of L^p-regularity in terms of the R-boundedness condition above (and similar equivalent conditions) was obtained independently by N. Kalton and by L. Weis (see [W]). From the R-boundedness characterization the equivalence of the various L^p-regularities is evident, but several results in this direction were known even before.

We should note that the treatment of the abstract Cauchy problem only required the multiplier theorems for functions of one variable (i.e., with domain $\mathbb{R}$). However, the necessary condition above was proved for $\mathbb{R}^n$, and the Mikhlin-type theorem giving sufficient condition for H^1_{at}-multipliers also has a version for several variables in [H]. Therefore, there is no obstacle against treating abstract PDE's as well. As an example in this direction, let us mention the abstract Laplace equation

$$-\triangle u(y) + Au(y) = f(y), \qquad y \in \mathbb{R}^n,$$

which was also treated in [H]. With the multiplier theorems at hand, it is straighforward to state and prove a result analogous to Theorem 2 for this problem.

4 Another necessity proof for $n = 1$

For the one-dimensional domain, the result of Theorem 1, and actually a little more, can be derived with a simpler argument, which is only a slight modification of the proof of [CP] for the necessity of R-boundedness for L^p-multipliers.

Here we consider the Hardy space H^1_{con} defined in terms of the conjugate operation or the Hilbert transform $\mathcal{H}$, which is the Fourier-multiplier operator with multiplier $-\mathbf{i}\,\mathrm{sgn}(\xi)$. We set

$$H^1_{\mathrm{con}}(\mathbb{R}; X) := \{f \in L^1(\mathbb{R}; X) | \ \mathcal{H}f \in L^1(\mathbb{R}; X)\}$$

equipped with the graph norm

$$\|f\|_{H^1_{\mathrm{con}}(\mathbb{R};X)} := \|f\|_{L^1(\mathbb{R};X)} + \|\mathcal{H}f\|_{L^1(\mathbb{R};X)} .$$

Our assumption in the following will be the boundedness of a multiplier operator T_m from $H^1_{\text{con}}(\mathbb{R};X)$ to $L^1(\mathbb{R};X)$, and we shall show the R-boundedness of $\{m(t)|\ t \neq 0\}$. This result reproduces Theorem 1 in the case $n = 1$ and is a slight extension of it for non-UMD Banach spaces. Namely, in general we have $H^1_{\text{con}}(\mathbb{R};X) \hookrightarrow H^1_{\text{at}}(\mathbb{R};X)$, and if X is UMD, there is an equality of spaces with equivalence of norm. This has been shown by O. Blasco [B] for the unit circle $\mathbb{T}$ in place of the real line $\mathbb{R}$, but the two results quoted are proved by methods which have direct analogues in the case of the line. Thus the assumption that T_m be bounded from the smaller space $H^1_{\text{con}}(\mathbb{R};X)$ to $L^1(\mathbb{R};Y)$ is clearly weaker than the boundedness from the possibly larger space $H^1_{\text{at}}(\mathbb{R};X)$.

What makes the one-dimensional proof for H^1_{con} so simple, is the existence of a large class of functions for which the evaluation of the graph norm of the Hilbert transform is particularly easy: If the Fourier transform of f is supported only on the positive (resp. negative) half-axis, then $\mathcal{H}f$ is simply $-\mathbf{i}f$ (resp. $\mathbf{i}f$), and therefore $\|f\|_{H^1_{\text{con}}(\mathbb{R};X)} = 2\|f\|_{L^1(\mathbb{R};X)}$

Now let us state and prove the result:

Proposition 1. *Suppose $m \in L^1_{\text{loc}}(\mathbb{R};\mathcal{B}(X;Y))$ is such that the multiplier operator $T_m f := \mathcal{F}^{-1}[m\hat{f}]$ acts boundedly from $H^1_{\text{con}}(\mathbb{R};X)$ to $L^1(\mathbb{R};Y)$.*

Then m is strongly continuous away from the origin, and the collection $\{m(t)|\ t \neq 0\}$ is R-bounded in terms of an absolute constant times the operator norm of T_m.

Proof. Let $N \in \mathbb{Z}_+$, $t_1,\ldots,t_N > 0$ and $x_1,\ldots,x_N \in X$. The fact that m is strongly continuous outside $t = 0$ follows from Lemma 2 and the above mentioned embedding, or one can also give a direct proof parallel to Lemma 2. Indeed, if $t_0 \neq 0$ and $\hat{\psi} \in \mathcal{D}(\mathbb{R})$ is equal to unity in a neighbourhood of t_0 and supported on one half axis only, it is clear that $\psi(\cdot)x \in H^1_{\text{con}}(\mathbb{R};X)$ for all $x \in X$, and the rest of the proof is just like Lemma 2.

We choose a real-valued test-function $\psi \in \mathcal{D}(\mathbb{R})$ supported on $(-\infty,0)$ and with the same integral condition as in the proof of Theorem 1. The proof runs in exactly the same fashion as there until we reach the estimate

$$\begin{aligned}
&\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j m(t_j)x_j\right|_Y \\
&\leq \frac{\pi}{2}\left\|\hat{\psi}\right\|_{L^\infty}\left\|T_m : H^1_{\text{con}} \to L^1\right\| \liminf_{k\to\infty}\frac{1}{k}\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi t_j \cdot}\hat{\psi}(\cdot/k)x_j\right\|_{H^1_{\text{con}}(\mathbb{R};X)}.
\end{aligned}$$

We then observe that the Fourier transform of the function whose H^1_{con}-norm is to be evaluated is given by $\sum \varepsilon_j k\psi(k(t_j-\xi))x_j$, and for this to be non-zero, recalling the support condition imposed on ψ, we must have $t_j - \xi < 0$, i.e., $\xi > t_j > 0$.

Thus the support of the Fourier transform is contained on $(0,\infty)$, and so the H^1_{con} norm is just twice the L^1 norm. Using this and the contraction principle, which is valid once we get back to L^1, we have

$$\leq 2\left(\frac{\pi}{2}\right)^2 \left\|\hat{\psi}\right\|_{L^\infty} \|T_m\| \liminf_{k\to\infty} \frac{1}{k}\mathbb{E}\left\|\sum_{j=1}^{N} \varepsilon_j \hat{\psi}(\cdot/k) x_j\right\|_{L^1(\mathbb{R};X)}$$

$$= \frac{\pi^2}{2}\left\|\hat{\psi}\right\|_{L^\infty}\left\|\hat{\psi}\right\|_{L^1}\left\|T_m : H^1_{\text{con}}(\mathbb{R};X) \to L^1(\mathbb{R};Y)\right\| \mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j x_j\right|_X ,$$

and a parallel argument can be used to handle the negative half-axis. □

Of course, one should note in Proposition 1 that the origin has to be excluded. Indeed, for $A \in \mathcal{B}(X;Y)\setminus\{0\}$, the operator $A\mathcal{H}$ maps $H^1_{\text{con}}(\mathbb{R};X)$ boundedly into $L^1(\mathbb{R};Y)$, but the corresponding multiplier $-\mathbf{i}\,\text{sgn}(t)A$ is certainly not even weakly continuous at $t = 0$; the origin of the frequency domain has a genuinely special meaning in the spaces $H^1_{\text{con}}(\mathbb{R};X)$. Whereas it is trivial to derive a shifted version of the Mikhlin theorem, say, in the L^p setting, with the possible discontinuity at a point other than the origin, the very construction of the space H^1_{con} is made so as to allow for bad behaviour of multipliers at the origin, but not elsewhere. For this reason one should not regard the method of proof of the above result, with one half-axis handled at a time, as something artificial, but rather it is intimately connected with the structure of the space $H^1_{\text{con}}(\mathbb{R};X)$.

The proof of Proposition 1 does not seem to be easily generalized to conjugate Hardy spaces of several variables. Namely, the n Riesz transforms R_j, $j = 1,\ldots,n$, which play the same rôle on $H^1_{\text{con}}(\mathbb{R}^n;X)$ as the Hilbert transform $\mathcal{H}$ has on $H^1_{\text{con}}(\mathbb{R};X)$, have multipliers $\xi_j/|\xi|$, which are not locally constant like the multiplier of $\mathcal{H}$, and thus do not allow us to use the trick that was here employed to overcome the difficulty of dealing with the H^1_{con}-norms.

In the one-dimensional setting, however, the result can be slightly generalized:

A sharpened necessary condition for L^p-multipliers, $p > 1$. The method of proof of Proposition 1 also applies to give a slightly sharpened form of the original result of [CP] concerning the L^p-multipliers. To see how this comes out, consider the spaces

$$H^p_{\text{con}}(\mathbb{R};X) := \{f \in L^p(\mathbb{R};X) |\ \mathcal{H}f \in L^p(\mathbb{R};X)\}$$

with the graph norm, in analogy with the case $p = 1$. Of course, for a UMD-space X, we have $H^p_{\text{con}}(\mathbb{R};X) = L^p(\mathbb{R};X)$ with equivalence of norms for $1 < p < \infty$, and this condition actually characterizes UMD-spaces, but our intention is now to provide a piece of insight into the multiplier theory in non-UMD Banach spaces.

Now we observe the following: The proof of the result concerning the R-boundedness of $\{m(t)|\ t \neq 0 \text{ a strong Lebesgue point of } m\}$ goes through with the assumption $T_m : H^1_{\text{con}}(\mathbb{R}; X) \to L^1(\mathbb{R}; Y)$ replaced by $T_m : H^p_{\text{con}}(\mathbb{R}; X) \to L^p(\mathbb{R}; Y)$. We just use the (equivalent) definition of R-boundedness in terms of the pth moment rather than the first, so that we can freely interchange the order of the L^p-norm with respect to the Lebesgue measure on the real line and the probability measure related to the Rademacher variables. Where we extracted the L^∞-norm of $\hat{\psi}$ from the integral, we now invoke the Hölder inequality to extract $\left\|\hat{\psi}\right\|_{L^{p'}}$, so that in place of the L^1 norm of the rest of the integrand we now have the L^p norm and we can apply the assumption. (This is also what was done in [CP].) Due to the choice of the auxiliary function ψ, the evaluation of the H^p_{con}-norm also reduces to that of the L^p-norm, and we arrive at a similar conclusion as before but with $\left\|\hat{\psi}\right\|_{L^{p'}} \left\|\hat{\psi}\right\|_{L^p}$ instead of $\left\|\hat{\psi}\right\|_{L^\infty} \left\|\hat{\psi}\right\|_{L^1}$ in the constant. We formulate this result as a corollary, but it is a consequence of the proof rather than Proposition 1 itself.

Corollary 1. *If $m \in L^1_{\text{loc}}(\mathbb{R}; \mathcal{B}(X;Y))$ gives rise to a bounded multiplier operator $T_m = \mathcal{F}^{-1} m \mathcal{F} : H^p_{\text{con}}(\mathbb{R}; X) \to L^p(\mathbb{R}; Y)$ for some $p \in (1,\infty)$, then*

$$\mathcal{R}(\{m(t) |\ t \neq 0 \text{ a strong Lebesgue point of } m\}) \\ \leq C \left\| T_m : H^p_{\text{con}}(\mathbb{R}; X) \to L^p(\mathbb{R}; Y) \right\|.$$

Very roughly speaking, this result could be interpreted as follows: Even if we restrict the action of our multipliers to a function class on which non-trivial scalar-valued multipliers act boundedly (according to the condition of the boundedness of the Hilbert transform, which lies at the heart of $H^p_{\text{con}}(\mathbb{R}; X)$), we cannot help the fact that operator-valued multipliers will not be bounded unless the multiplier function is R-bounded.

5 Some variations and further results

The periodic case. Let us have a brief look at the analogue of Proposition 1 in the periodic case, i.e., with the unit-circle $\mathbb{T}$ in place of $\mathbb{R}$. The space $H^1_{\text{con}}(\mathbb{T}; X)$ is defined simply by replacing $\mathbb{R}$ by $\mathbb{T}$ throughout above, and the observations concerning the computation of the norm are equally valid.

The periodic case is instructive in the sense that it contains all the main ideas but there are less technical points to be considered. We note that whereas the proof of Proposition 1 was nothing but an adaptation of the proof of [CP] and a trick, the periodic proof below is nothing but the proof of [AB] plus the same trick.

Proposition 2. *Suppose the collection of operators $m(k) \in \mathcal{B}(X;Y)$, $k \in \mathbb{Z}$, define a multiplier T_m by*

$$T_m \left(\sum_{k=-\infty}^{\infty} e^{\mathrm{i}k\cdot} x_k \right) := \sum_{k=-\infty}^{\infty} e^{\mathrm{i}k\cdot} m(k) x_k \tag{2}$$

which is bounded from $H^1_{\text{con}}(\mathbb{T};X)$ *to* $L^1(\mathbb{T};Y)$.

Then this collection is R-bounded, and more precisely

$$\mathcal{R}(\{m(k) : X \to Y|\ k \in \mathbb{Z}\}) \leq (\pi^2+1) \left\|T_m : H^1_{\text{con}}(\mathbb{T};X) \to L^1(\mathbb{T};Y)\right\|.$$

Proof. Let first $N \in \mathbb{Z}_+$, $x_1,\ldots,x_N \in X$ and $k_1,\ldots,k_N \in \mathbb{Z}_+$ be given. Let us write

$$\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j m(k_j)x_j\right|_Y = \int_{-\pi}^{\pi} \mathbb{E}\left|\sum_{j=1}^N \varepsilon_j e^{-\mathbf{i}k_jt} m(k_j)x_j e^{\mathbf{i}k_jt}\right|_Y \frac{\mathrm{d}t}{2\pi},$$

and we recognize $m(k_j)x_je^{\mathbf{i}k_jt}$ as $T_m[x_je^{\mathbf{i}k_j\cdot}](t)$. Now first an application of the contraction principle and then the assumed boundedness of $T_m : H^1_{\text{con}}(\mathbb{T};X) \to L^1(\mathbb{T};Y)$ give

$$\leq \frac{\pi}{2}\left\|T_m : H^1_{\text{con}}(\mathbb{T};X) \to L^1(\mathbb{T};Y)\right\| \frac{1}{2\pi}\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}k_j\cdot}x_j\right\|_{H^1_{\text{con}}(\mathbb{T};X)}$$

Finally, by the assumption $k_j > 0$, the function whose H^1_{con}-norm appears above contains only positive frequencies, so the H^1_{con}-norm is simply twice the L^1-norm. With this observation and one more application of the contraction principle we get

$$= \frac{\pi}{2}\|T_m\| \, 2\frac{1}{2\pi}\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}k_j\cdot}x_j\right\|_{L^1(\mathbb{T};X)} \leq 2\left(\frac{\pi}{2}\right)^2 \|T_m\| \, \mathbb{E}\left|\sum_{j=1}^N \varepsilon_j x_j\right|_X,$$

and hence we have shown that

$$\mathcal{R}\left(\{m(k) : X \to Y|\ k > 0\}\right) \leq \frac{\pi^2}{2}\left\|T_m : H^1_{\text{con}}(\mathbb{T};X) \to L^1(\mathbb{T};Y)\right\|.$$

The assertion now follows, once we observe that analogous reasoning proves the R-boundedness of $\{m(k)|\ k < 0\}$ with the same bound, and moreover, since the constant function x is mapped into $m(0)x$, it is clear that $\|m(0) : X \to Y\| \leq \|T_m\|$. (We recall that the R-bound of a union of sets is at most the sum of the individual R-bounds, and the R-bound of a singleton coincides with the operator norm. This yields the overall constant $\pi^2/2 + \pi^2/2 + 1 = \pi^2 + 1$.) □

A necessary condition for L^∞–BMO-multipliers. Before concluding, we would like to mention one more necessary condition for multipliers, this time for those acting boundedly between $L^\infty(\mathbb{T};X)$ and $\mathrm{BMO}(\mathbb{T};Y)$. Although the (rather modest) argument does not give R-boundedness here, it is still in the same spirit as the results above; namely, we once again see that a collection of operators giving rise to a bounded Fourier-multiplier needs to be more than just bounded.

More precisely, we find that the collection in question must satisfy an R-boundedness type condition but with the maximum norm instead of the pth moment. This also implies a condition of *unconditional boundedness* or *U-boundedness*, due to N. Kalton and L. Weis [KW], which is defined by the requirement that, for some $C < \infty$, we have

$$\sum_{j=1}^{N} \left|\left\langle T_j x_j, y_j' \right\rangle\right| \leq C \max_{\epsilon_j=\pm 1} \left|\sum_{j=1}^{N} \epsilon_j x_j\right|_X \max_{\epsilon_j=\pm 1} \left|\sum_{j=1}^{N} \epsilon_j y_j'\right|_{Y'}$$

for all $N \in \mathbb{Z}_+$, all $T_1, \ldots, T_N \in \mathcal{T}$, $x_1, \ldots, x_N \in X$ and $y_1', \ldots, y_N'$ in the dual space Y'.

We have the following result:

Proposition 3. *Suppose that the sequence $m(k) \in \mathcal{B}(X;Y)$, $k \in \mathbb{Z}$, defines a Fourier-multiplier T_m as in (2) acting boundedly from $L^\infty(\mathbb{T};X)$ to $\mathrm{BMO}(\mathbb{T};Y)$. Then the collection $\{m(k) |\ k \neq 0\}$ satisfies the inequality*

$$\max_{\epsilon_j=\pm 1} \left|\sum_{j=1}^{N} \epsilon_j m(k_j) x_j\right|_Y \leq \frac{\pi^2}{4} \left\|T_m : L^\infty(\mathbb{T};X) \to \mathrm{BMO}(\mathbb{T};Y)\right\| \max_{\epsilon_j=\pm 1} \left|\sum_{j=1}^{N} \epsilon_j x_j\right|_X . \quad (3)$$

In particular, this collection is U-bounded.

Proof. Proceeding as with $H^1_{\mathrm{con}}(\mathbb{T};X)$ and $L^1(\mathbb{T};Y)$, now with $k_1, \ldots, k_N \neq 0$ but otherwise arbitrary integers, we get

$$\max_{\epsilon_j=\pm 1} \left|\sum_{j=1}^{N} \epsilon_j m(k_j) x_j\right|_Y \leq \frac{\pi}{2} \max_{\epsilon_j=\pm 1} \int_{-\pi}^{\pi} \left| T_m \left[\sum_{j=1}^{N} \epsilon_j e^{\mathbf{i}k_j \cdot} x_j\right](t)\right|_Y \frac{\mathrm{d}t}{2\pi}$$

We then observe that the above integral without the norm would vanish, since each of the terms in the sum vanishes

$$\int_{-\pi}^{\pi} T_m[\epsilon_j e^{\mathbf{i}k_j \cdot} x_j](t)\,\mathrm{d}t = \int_{-\pi}^{\pi} \epsilon_j e^{\mathbf{i}k_j t} m(k_j) x_j \,\mathrm{d}t = 0.$$

Thus the integral is of the form $(2\pi)^{-1}\int_{-\pi}^{\pi} |F(t) - F_{\mathbb{T}}|_Y \,\mathrm{d}t \leq \|F\|_{\mathrm{BMO}(\mathbb{T};Y)}$, where $F_{\mathbb{T}}$ denotes the average of F over $\mathbb{T}$, which now vanishes. By the assumed boundedness of T_m we then have

$$\begin{aligned}\max_{\epsilon_j=\pm 1} &\left|\sum_{j=1}^{N} \epsilon_j m(k_j) x_j\right|_Y \\ &\leq \frac{\pi}{2} \left\|T_m : L^\infty(\mathbb{T};X) \to \mathrm{BMO}(\mathbb{T};Y)\right\| \max_{\epsilon_j=\pm 1} \left\|\sum_{j=1}^{N} \epsilon_j e^{\mathbf{i}k_j \cdot} x_j\right\|_{L^\infty(\mathbb{T};X)} \\ &\leq \left(\frac{\pi}{2}\right)^2 \|T_m\| \max_{\epsilon_j=\pm 1} \left|\sum_{j=1}^{N} \epsilon_j x_j\right|_X ,\end{aligned}$$

where the last inequality is proved by a similar convexity argument as the usual contraction principle with p-norm ($p < \infty$) instead of the maximum norm. (The constant $\pi/2$ instead of 2 is obtained using [PW, §4.11.5].)

U-boundedness follows readily from the stronger condition (3) established; indeed, for appropriate $\alpha_j \in \mathbb{C}$ of unit length,

$$\sum_{j=1}^{N} \left| \langle T_j x_j, y_j' \rangle \right| = \sum_{j=1}^{N} \langle T_j x_j, \alpha_j y_j' \rangle = \mathbb{E} \left\langle \sum_{i=1}^{N} \varepsilon_i T_i x_i, \sum_{j=1}^{N} \varepsilon_j \alpha_j y_j' \right\rangle$$
$$\leq \max_{\epsilon_i = \pm 1} \left| \sum_{i=1}^{N} \epsilon_i T_i x_i \right|_X \max_{\epsilon_j = \pm 1} \left| \sum_{j=1}^{N} \epsilon_j \alpha_j y_j' \right|_{Y'},$$

and then we can apply the inequality already established to the first factor and the contraction principle to the second. □

One should observe that the type of boundedness established for the operators $T_j = m(k_j)$ above is just like that of R-boundedness, except that we have the L^∞-norm instead of an L^p-norm with $p < \infty$. As was mentioned above, in the linear span of constant vector multiples of the Rademacher functions, all the L^p-norms are equivalent for $0 < p < \infty$ (by the Khintchine–Kahane inequality), which gives great flexibility to the notion of R-boundedness and makes it quite compatible with the various L^p-spaces. However, this equivalence does not extend to $p = \infty$. (For $x_j \equiv x$, the ∞-norm grows like N, whereas the p-norms for $p < \infty$ grow like $\sqrt{N}$ as $N \to \infty$.)

Nevertheless, the R-boundedness of L^∞-type encountered here appears to be to a large extent similar to the usual R-boundedness. Indeed, it is easy to see that the same proofs of the contraction principle (which was already applied above), the convexity property etc. go through for this new kind of notion of boundedness.

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