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ON CORNER IRREGULARITIES THAT ARISE IN HYPERBOLIC SHELL MEMBRANE THEORY

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Abstract: *We study the regularity of the solution to a two-dimensional linear hyperbolic system that arises in shell membrane theory. We focus on the behavior of the solution at a corner where none of the characteristic lines intersects the domain. In this case there appears an algebraic singularity at the corner, in addition to the usual hyperbolic irregularities that propagate along the characteristic lines. We carry out the analysis on a triangular domain, using as tools the Banach fixed point Theorem and the Mellin transform techniques.*

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1 Introduction

The aim of the present work is to study the regularity properties of the solution to the hyperbolic problem

$$\mathcal{F}(\mathbf{u}) = \int_\omega |AD\mathbf{u}|^2\, dx_1 dx_2 - 2\int_\omega \mathbf{f}\cdot\mathbf{u}\, dx_1 dx_2 = \min!, \tag{1.1}$$

where ω is a polygonal plane domain, $A = (a_{ij})$ is a real constant matrix such that A is non-singular, D is a differential operator defined by

$$D = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \end{pmatrix},$$

and finally $\mathbf{f} = (f_1, f_2)$ and $\mathbf{u} = (u_1, u_2)$ are vector fields on ω, with $\mathbf{f} \in (L_2(\omega))^2$ given and $\mathbf{u}$ to be found such that $\mathbf{u}, D\mathbf{u} \in (L_2(\omega))^2$. As boundary conditions for $\mathbf{u}$ we assume the (essential) conditions $u_i = 0$ when $n_i \neq 0$ throughout $\partial\omega$, $i = 1, 2$, where $\mathbf{n} = (n_1, n_2)$ is the normal to $\partial\omega$. We are interested in the regularity properties of $\mathbf{u}$ when $\mathbf{f}$ is (sufficiently) smooth.

The work is motivated by certain problems of linear elasticity theory. More specifically, problem of the form (1.1) arises in the asymptotic membrane theory of thin hyperbolic shells. In this application, $\mathbf{u}$ is the tangential displacement vector field of the shell mid-surface expressed in characteristic coordinates, see the Appendix where the connection is shown in case of a simplified (shallow) shell geometry. In more general shell geometries, the membrane theory leads to problem (1.1) with $|AD\mathbf{u}|$ replaced by $|AD\mathbf{u} + B\mathbf{u}|$, where matrices A and B are variable, so what we consider here is the localized, frozen-coefficient version of the actual shell problem. For more information on the related shell theory, the reader is referred to [1, 3, 4] and the further references therein.

As concluded already in [1], the regularity properties of $\mathbf{u}$ at a corner P of ω are rather different, depending on whether (a) at least one of the characteristic lines through P intersects $\bar{\omega}$ in the vicinity of P, or (b) the characteristic lines through P only touch $\bar{\omega}$ at P in small neighbourhoods of P. In case (a) there occur at most simple jump discontinuties in the derivatives of $\mathbf{u}$ across the characteristic line(s) through P, whereas in case (b) the behavior of $\mathbf{u}$ is more complicated: An algebraic singularity appears at P in this case. In this work we focus on case (b) which was left as an open problem in [1].

As will be shown, the algebraic singularity at a corner of type (b) can be resolved by applying Mellin transform techniques to a specific system of integral equations that underlies the Euler equations of (1.1) near the corner. A simplified scalar model of this integral equation was studied in [2]. Here we extend this analysis to obtain the regularity theory for problem (1.1), a result of its own interest (as that of [2]), and an essential step towards understanding the underlying shell problem.

Instead of considering a general polygonal domain, we make here a further simplification assuming that the domain is actually a triangle with vertices at $(0, 0)$, $(a, 1)$ and $(b, 1)$ where $b > a > 0$ (Figure 1). Since the characteristic lines are coordinate lines in our model problem, the corner at $(0, 0)$ is of the mentioned type (b), while the other two corners are of type (a). In the assumed geometric setting, when $\mathbf{f}$ in (1.1) is smooth on $\bar{\omega}$, then $\mathbf{u}(x_1, x_2)$ is uniformly smooth on ω, except when $(x_1, x_2) \to (0, 0)$ or when (x_1, x_2) crosses a line S along which a local irregularity generated by the corner at (a,1) may propagate. The line S is a continuous broken line that passes through $(a, 1)$ and undergoes infinitely many reflections while approaching the corner at $(0, 0)$, see Figure 1, where the line segment after k reflections is denoted by S_k. At any point of S, the derivatives of u_1, u_2 of sufficiently high order have simple jump discontinuities.

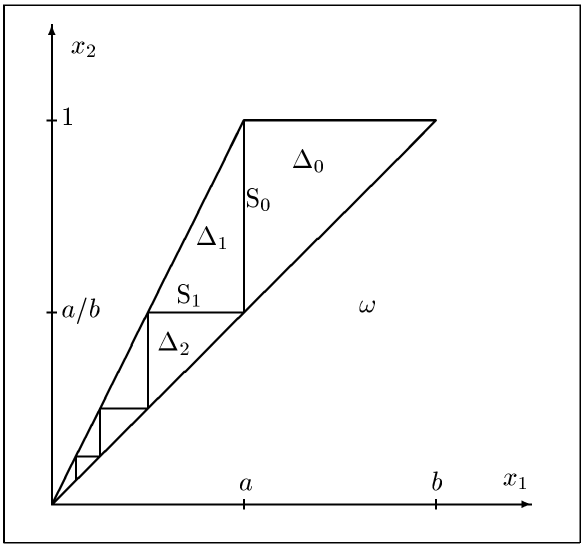

FIGURE 1. Geometry of the problem.

In our analysis we will assume that for some $m \geq 1$, $\mathbf{f} \in [C^m(\bar{\omega})]^2$ in (1.1). We will then first show that the solution $\mathbf{u}$ satisfies $\mathbf{u}_{|\Delta_k} \in [C^m(\bar{\Delta}_k)]^2$ for every triangular subdomain Δ_k generated by the line S (see Figure 1), and that $\mathbf{u}_{|\Delta_k \cup \Delta_{k+1}} \in [C^\ell(\overline{\Delta_k \cup \Delta_{k+1}})]^2$ where $\ell = \min\{k, m\}$ (see Theorem 3.1 ahead). Secondly, we show that the partial derivatives of u_i of order $\leq m$ grow at most algebraically when $(x_1, x_2) \to (0, 0)$. Finally, we resolve the singularity at the origin in more detail, showing in particular that the leading terms in the singular expansion of u_i are of the form

$$\begin{cases} x_i^{\eta+1}, & x_i x_j^\eta, & x_j^{\eta+1} & \text{if } \eta \notin \{1, 2, ..., m\}, \\ x_i^{\eta+1}\ln(x_i), & x_i x_j^\eta \ln(x_j), & x_j^{\eta+1}\ln(x_j) & \text{otherwise,} \end{cases}$$

where $-\eta < 0$ is the smallest real root of the function

$$g(z) = (b-a)^2(z-1)^2 - 4ab\kappa^2 \sinh^2(\frac{1}{2}\ln(b/a)(z-1)), \tag{1.2}$$

where further $\kappa \in [0, 1)$ is a parameter related to matrix A by

$$\kappa^2 = \frac{\alpha^2}{\alpha_1 \alpha_2}, \qquad \begin{pmatrix} \alpha_1 & \alpha \\ \alpha & \alpha_2 \end{pmatrix} = (A^T A)^{-1}. \tag{1.3}$$

Here the case $\kappa = 0$ ($A^T A$ diagonal) is a degenerate case where no singularity arises at the origin.

The plan of the paper is as follows. In Section 2 we introduce the basic notation to be used, prove the solvability of (1.1), and introduce the Mellin transform to be needed in Section 4. In Section 3 we mainly apply the Banach fixed point theorem to get the abovementioned regularity prioperties of $\mathbf{u}$ away from the origin. In Section 4 we obtain the singular resolution of $\mathbf{u}$ at the origin using the Mellin transform. The roots of $g(z)$ that are needed in the singular resolution are located in Section 5.

2 Preliminaries

We summarize in this Section the main notation to be used in the sequel. Let ω be a classical polygon and define the norms

$$\|u\|^2_{\mathcal{W}_i} = \|u\|^2_{L_2(\omega)} + \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L_2(\omega)}, \quad i = 1, 2,$$

and the associated function spaces

$$
\begin{array}{rcll}
\mathcal{W}_i & = & \left\{ \text{closure of } C^\infty(\overline{\omega}) \text{ in norm } \|\cdot\|_{\mathcal{W}_i} \right\}, & i = 1,2, \\
\mathcal{W}_i^0 & = & \left\{ \text{closure of } \mathcal{D}(\omega) \text{ in norm } \|\cdot\|_{\mathcal{W}_i} \right\}, & i = 1,2, \\
\mathcal{U} & = & \mathcal{W}_1 \times \mathcal{W}_2, & \\
\mathcal{U}^0 & = & \mathcal{W}_1^0 \times \mathcal{W}_2^0, &
\end{array}
\tag{2.1}
$$

where $\mathcal{D}(\omega) = C_0^\infty(\omega)$ is the space of smooth functions with compact support on ω. We have

Theorem 2.1. *Given* $\mathbf{f} = (f_1, f_2) \in L_2(\omega)^2$*, Problem (1.1) admits a unique solution* $\mathbf{u} \in \mathcal{U}^0$*.*

Proof. By the Riesz representation theorem, and by the usual closure arguments, the statement of the theorem is equivalent to asserting that for some constant $c > 0$,

$$\|AD\mathbf{u}\|_{L_2(\omega)} := \|\mathbf{u}\|_E \geq c\|\mathbf{u}\|_{L_2(\omega)} \quad \forall \mathbf{u} \in \mathcal{D}(\omega)^2. \tag{2.2}$$

To prove this inequality, let $\mathbf{u} \in \mathcal{D}(\omega)^2$ be given and let $\tilde{\mathbf{u}} = \exp(-2x_1 - 2x_2)\mathbf{u}$. Then

$$D\mathbf{u} \cdot \tilde{\mathbf{u}} = \frac{1}{2}\frac{\partial}{\partial x_1}[\exp(-2x_1 - 2x_2)\, u_1^2] + \frac{1}{2}\frac{\partial}{\partial x_2}[\exp(-2x_1 - 2x_2)\, u_2^2] + \mathbf{u} \cdot \tilde{\mathbf{u}}.$$

Since $\mathbf{u}$ vanishes on $\partial\omega$, it follows that

$$\int_\omega D\mathbf{u} \cdot \tilde{\mathbf{u}}\, dx_1 dx_2 \geq e^{-2L}\|\mathbf{u}\|^2_{L_2(\omega)},$$

where L is chosen so that $|x_1 + x_2| \leq L$ for $(x_1, x_2) \in \omega$. Using here finally the Cauchy-Schwarz inequality and noting that $|\tilde{\mathbf{u}}| \leq \exp(2L)|\mathbf{u}|$ and $|A\mathbf{v}| \geq c_0|\mathbf{v}|$ for some $c_0 > 0$, we conclude that

$$\|\mathbf{u}\|_E \geq c_0\, \|D\mathbf{u}\|_{L_2(\omega)} \geq c_0\, e^{-4L}\|\mathbf{u}\|_{L_2(\omega)},$$

so the assertion follows. □

In what follows we assume that ω is the specific triangular domain of Figure 1, i.e.,

$$\omega = \left\{(x_1, x_2) \in \mathcal{R}^2 \mid ax_2 < x_1 < bx_2,\ 0 < x_2 < 1,\ 0 < a < b\right\} \tag{2.3}$$

and that in (1.1)

$$\mathbf{f} \in C^m(\overline{\omega})^2, \quad m \geq 1. \tag{2.4}$$

Further, let us define

$$
\begin{array}{rcl}
U(x_1) & = & \min\{x_1/a,\ 1\}, \\
W_1(x_2) & = & (b-a)x_2, \\
W_2(x_1) & = & U(x_1) - x_1/b,
\end{array}
\tag{2.5}
$$

so that $W_1(x_2)$ and $W_2(x_1)$ are, respectively, the width and height of ω at a given position. In what follows, we shall also use the abbreviations

$$I_1 f = \int_{ax_2}^{x_1} f(s_1, x_2)\, ds_1, \quad \mathcal{I}_1 f(x_2) = I_1 f(bx_2, x_2), \tag{2.6}$$

$$I_2 f = \int_{x_1/b}^{x_2} f(x_1, s_2)\, ds_2, \quad \mathcal{I}_2 f(x_1) = I_2 f(x_1, U(x_1)). \tag{2.7}$$

For later use we also define the following domains and spaces (see Figure 1);

$$
\begin{aligned}
\Delta_i &= \omega \cap \left(\left(\tfrac{a^{l+1}}{b^l},\ \tfrac{a^l}{b^{l-1}} \right) \times \left(\tfrac{a^{j+1}}{b^{j+1}},\ \tfrac{a^j}{b^j} \right) \right), \\
&\qquad l = \left\lfloor \tfrac{i+1}{2} \right\rfloor,\ j = \left\lfloor \tfrac{i}{2} \right\rfloor, \\
\omega_i &= \omega - \textstyle\bigcup_{j=0}^{i-1} \overline{\Delta}_j, \\
C_p^k[0,1] &= \Big\{ \phi \in C[0,1] \mid \phi \in C^k \left[\tfrac{a^{i+1}}{b^{i+1}}, \tfrac{a^i}{b^i} \right], \\
&\qquad \phi \in C^n \left[0, \tfrac{a^j}{b^j} \right], j = \left\lfloor \tfrac{n}{2} \right\rfloor,\ 0 \le n \le k \Big\}, \\
C_p^k[0,b] &= \Big\{ \phi \in C[0,b] \mid \phi \in C^k \left[\tfrac{a^{i+1}}{b^i}, \tfrac{a^i}{b^{i-1}} \right], \\
&\qquad \phi \in C^n \left[0, \tfrac{a^{j+1}}{b^j} \right], j = \left\lfloor \tfrac{n-1}{2} \right\rfloor,\ 0 \le n \le k \Big\}, \\
C_{p1}^k(\overline{\omega}) &= \{ u \in C(\overline{\omega}) \mid u_{|\Delta_i} \in C^k(\overline{\Delta}_i), \\
&\qquad u_{|\omega_j} \in C^n(\overline{\omega}_j),\ j = 2 \left\lfloor \tfrac{n}{2} \right\rfloor,\ 0 \le n \le k \}, \\
C_{p2}^k(\overline{\omega}) &= \{ u \in C(\overline{\omega}) \mid u_{|\Delta_i} \in C^k(\overline{\Delta}_i), \\
&\qquad u_{|\omega_j} \in C^n(\overline{\omega}_j),\ j = 2 \left\lfloor \tfrac{n-1}{2} \right\rfloor + 1,\ 0 \le n \le k \},
\end{aligned}
\tag{2.8}
$$

where $i \in \{0,1,2,...\}$ and $\lfloor x \rfloor$ is the biggest integer n such that $n \le x$.
To characterize the edge behaviour of the elements of $\boldsymbol{\mathcal{U}}$, assume that $\phi \in C^\infty(\overline{\omega})$ and apply the Green formula to obtain

$$
\begin{aligned}
\int_\omega (bx_2 - x_1)\phi \,\frac{\partial \phi}{\partial x_1}\, dx_1 dx_2 &= (b-a) \int_0^1 x_2 \phi(ax_2, x_2)\, dx_2 \\
&\quad - \int_\omega \left((bx_2 - x_1)\phi \,\frac{\partial \phi}{\partial x_1} - \phi^2 \right) dx_1 dx_2.
\end{aligned}
$$

This implies that

$$
\int_0^1 W_1(x_2)\phi(ax_2, x_2)^2\, dx_2 \le c \|\phi\|^2_{\boldsymbol{\mathcal{W}}_1}
$$

for some positive constant c. By this reasoning, we see that the trace operator

$$
\begin{aligned}
\gamma :\quad & \left(C^\infty(\overline{\omega})^2, \|\cdot\|_E \right) \to L_2^{\sqrt{w_1}}(\partial\omega) \times L_2^{\sqrt{w_2}}(\partial\omega), \\
& (\phi_1, \phi_2) \mapsto (n_1 \phi_1|_{\partial\omega}, n_2 \phi_2|_{\partial\omega})
\end{aligned}
\tag{2.9}
$$

is bounded, where $\mathbf{n} = (n_1, n_2)$ is the unit outward normal on the boundary $\partial\omega$ and L_2^g stands for the weighted L_2–space supplied with the norm $f \mapsto \|gf\|_{L_2(\partial\omega)}$. We conclude that the bounded trace operator γ can be extended onto $\boldsymbol{\mathcal{U}}$ and thus the minimizer $\mathbf{u}$ of Equation (1.1) satisfies

$$
\gamma \mathbf{u} = \mathbf{0} \tag{2.10}
$$

in the described distributional sense.
In the analysis of Section 3 below we proceed from the Euler equations of (1.1),

$$
-DA^T AD\mathbf{u} = \mathbf{f},
$$

rewritten as the first order system

$$
\left\{ \begin{array}{rcl} D\mathbf{u} &=& M\mathbf{v}, \\ -D\mathbf{v} &=& \mathbf{f}, \end{array} \right. \tag{2.11}
$$

where

$$
M := (A^T A)^{-1} = \begin{pmatrix} \alpha_1 & \alpha \\ \alpha & \alpha_2 \end{pmatrix} \tag{2.12}
$$

and $\mathbf{v} = (v_1, v_2)$ is defined by the upper equation (2.11). Accordingly, $\mathbf{v} \in \mathcal{U}$, and by (2.9) we may define the trace rectrictions

$$\begin{cases} \phi_1(x_2) & := & v_1(ax_2, x_2) \in L_2^{\sqrt{W_1}}(0,1), \\ \phi_2(x_1) & := & v_2(x_1, x_1/b) \in L_2^{\sqrt{W_2}}(0,b). \end{cases} \tag{2.13}$$

The analysis of Section 4 is based on the Mellin transform

$$\mathcal{M}\ f(z) = \int_0^\infty x^{z-1} f(x)\ dx = \tilde{f}(z). \tag{2.14}$$

which is known to have the following properties (see e.g. [5]).

Lemma 2.1. *(i)* $\mathcal{M} : L_2^{\lambda - 1/2}(0,\infty) \to L_2\{z \in \mathcal{C} \mid \Re(z) = \lambda\}$ *isometrically, where* $L_2^{\lambda}(0,\infty)$, $\lambda \in \mathcal{R}$, *is the weighted* L_2*–space supplied with the norm* $f \mapsto \|x^\lambda f\|_{L_2(0,\infty)}$.

(ii) Let $f \in L^1_{loc}(\mathcal{R}_+)$, *and let the numbers* λ_1 *and* λ_2 *be given by*

$$\begin{aligned} \lambda_1 &= \sup\{\tau_1 \mid f(x) = \mathcal{O}(x^{-\tau_1}) \text{ as } x \to 0+\}, \\ \lambda_2 &= \sup\{\tau_2 \mid f(x) = \mathcal{O}(x^{-\tau_2}) \text{ as } x \to \infty\}. \end{aligned}$$

If $\lambda_2 > \lambda_1$, *then the Mellin transform integral* (2.2) *converges uniformly for* $\lambda_1 < \Re(z) < \lambda_2$ *and defines an analytic function there.*

(iii) For any compact subinterval I *of* (λ_1, λ_2), *the function*

$$K(f, I, y) = \sup_{x \in I} |\tilde{f}(x + iy)|$$

is continuous with respect to y *and satisfies*

$$\lim_{y \to \pm\infty} K(f, I, y) = 0.$$

(iv) For $\lambda_1 < \lambda < \lambda_2$, *the inversion formula*

$$f(x) = \frac{1}{2\pi i} \int_{\Re(z) = \lambda} x^{-z} \tilde{f}(z)\ dz = M^{-1}\tilde{f}(x)$$

is valid. □

3 Regularity of u away from the origin

The aim of this Section is to give some basic continuity results and growth estimates for $\mathbf{u}$ and its partial derivates under assumptions (2.3) and (2.4). Here the analysis is mainly based on the Banach fixed point principle. The regularity result below can be improved by deriving a full singular resolution of (u_1, u_2) at the origin. This is done separately in Chapter 4 by applying the Mellin transform techniques.

Theorem 3.1. $r^{k-1-\nu}\mathbf{u} \in C^k_{p1}(\varpi) \times C^k_{p2}(\varpi)$ *for any* $k = 0, 1, ..., m$ *and* $\nu \in [0, \eta) \cap [0, 1]$, *where* $r = (x_1^2 + x_2^2)^{1/2}$ *and* $-\eta < 0$ *is the smallest real root of* g *as defined by (1.2)-(1.3).*

*P*roof. In what follows, $i, j \in \{1, 2\}, i \neq j$. Applying this notation, the lower equation (2.11) can be rewritten as (see (2.6), (2.7) and (2.13))

$$v_i = \phi_i(x_j) + I_i f_i. \tag{3.1}$$

Next, applying similarly the upper equation (2.11) together with (2.10) and (2.12), we get

$$u_i = I_i\left(\alpha_i v_i + \alpha v_j\right).$$

Substituting then (3.1) into this equation, we find that

$$u_i = \alpha_i W_i \phi_i + \alpha I_i \phi_j - I_i \left(\alpha_i I_i f_i + \alpha I_j f_j\right). \tag{3.2}$$

Imposing next the boundary conditions $u_1(bx_2, x_2) = u_2(x_1, U(x_1)) = 0$ above, we get

$$\phi_i(x_j) + \frac{\alpha}{\alpha_i W_i(x_j)} \mathcal{I}_i \phi_j(x_j) = g_i(x_j), \tag{3.3}$$

where thus

$$g_i(x_j) = \frac{1}{\alpha_i W_i(x_j)} \mathcal{I}_i \left(\alpha_i I_i f_i + \alpha I_j f_j\right)(x_j). \tag{3.4}$$

In the following analysis we apply repeatedly the following inclusion, easily proven by Taylor expansions:

$$\xi \in C^k([0,1]\times[0,1]) \Rightarrow \zeta(y) := y^{-1}\int_0^y \xi(x,y)\, dx \in C^k[0,1], \quad k \geq 0. \tag{3.5}$$

Accordingly, and since obviously $g_i(0) = 0$, we see that

$$x_2^{-1+k} g_1 \in C^k[0,1], \quad x_1^{-1} g_2 \in \left\{ g \in C[0,b] \mid g \in C^k[0,a],\ g \in C^k[a,b] \right\} \tag{3.6}$$

for $k = 0, 1, ..., m$. Next, set (see 1.3)

$$K_i^\nu\ \phi(x_j) = x_j^{-\nu} \frac{\kappa^2}{W_i} \mathcal{I}_i \left(\frac{1}{W_j} \mathcal{I}_j (x_j^\nu \phi) \right)(x_j), \tag{3.7}$$

and

$$\mathcal{G}_i(x_j) = g_i(x_j) - \frac{\alpha}{\alpha_i W_i} \mathcal{I}_i g_j(x_j), \tag{3.8}$$

so that by (3.3),

$$x_j^{-\nu} \phi_i(x_j) - K_i^\nu \left(x_j^{-\nu} \phi_i\right)(x_j) = x_j^{-\nu} \mathcal{G}_i(x_j). \tag{3.9}$$

It first follows from (3.5), (3.6) and (3.8), that

$$x_2^{-1} \mathcal{G}_1 \in C[0,1], \quad x_1^{-1} \mathcal{G}_2 \in C[0,b]. \tag{3.10}$$

Accordingly, and by (3.7) and (3.9), the results above hold with $\mathcal{G}_i$ replaced by ϕ_i provided that $\kappa = 0$. In general, the remaining analysis is quite trivial in the case $\kappa = 0$. Therefore we assume next and always in the sequel that $\kappa > 0$. Obviously the norms of K_1^ν and K_2^ν in $C[0,1]$ and $C[0,b]$, respectively, are given by

$$\|K_i^\nu\| = \frac{\kappa^2}{(ab)^\nu} \left(\frac{b^{\nu+1} - a^{\nu+1}}{(b-a)(\nu+1)} \right)^2 = \frac{4ab\kappa^2 \sinh^2\left((\nu+1)\ln\left(\frac{b}{a}\right)/2\right)}{(b-a)^2(\nu+1)^2}.$$

Hence, K_i^ν is a contraction whenever (see (1.2))

$$g(\nu+2) := (b-a)^2(\nu+1)^2 - 4ab\kappa^2 \sinh^2\left((\nu+1)\ln\left(\frac{b}{a}\right)/2 \right) > 0.$$

In Chapter 5 we show that this inequality holds on the interval

$$\nu \in [0, \eta), \quad \eta = \frac{2x_0 - \ln(b/a)}{\ln(b/a)},$$

where $x_0 > \ln(b/a)/2$ is the only positive real root of

$$x = \frac{\sqrt{ab}\ \kappa}{b-a}\ \ln(b/a)\sinh(x).$$

Accordingly, K_i^ν is a contraction if $\nu \in [0,\eta)$ and we conclude by the Banach fixed point theorem and by (3.10), that

$$x_2^{-\nu+k}\phi_1 \in C_p^k[0,1], \quad x_1^{-\nu+k}\phi_2 \in C_p^k[0,b] \tag{3.11}$$

holds with $k=0$ for any $\nu \in [0,\eta)\cap[0,1]$. Actually (3.11) holds for any $k=0,...,m$, which can be proved by the induction principle. Indeed, assume (3.11) with $k \le n-1 \le m-1$. Applying the recursive differentiation in (3.3) (with $i=1$), we have

$$\begin{aligned}\alpha_1(b-a)\left(x_2\phi_1\right)^{(n)}(x_2)+ \quad &\alpha\left(b^n\phi_2^{(n-1)}(bx_2) - a^n\phi_2^{(n-1)}(ax_2)\right) = \\ &\alpha_1(b-a)\left(x_2 g_1\right)^{(n)}(x_2)\end{aligned}$$

for any $x_2 \in (0,1)$. The equation above, (3.6) and the induction assumption imply (3.11) for ϕ_1 with $k \le n$. Hence, ϕ_1 is a C^n-function for $x_2 > a/b$ and so it follows from (3.3), (3.5) and (3.6), that ϕ_2 is another C^n-function on (a,b). To see what happens on the interval $(0,a)$, we can differentiate again in (3.3)(with $i=2$) to find that

$$\begin{aligned}\alpha_2\tfrac{b-a}{ab}\left(x_1\phi_2\right)^{(n)}(x_1)+ \quad &\alpha\left(a^{-n}\phi_1^{(n-1)}(a^{-1}x_1) - b^{-n}\phi_1^{(n-1)}(b^{-1}x_1)\right) = \\ &\alpha_2\tfrac{b-a}{ab}\left(x_1 g_2\right)^{(n)}(x_1).\end{aligned}$$

The equation above, (3.6) and the induction assumption complite the proof of (3.11) with $k \le n$. Accordingly, (3.11) holds for any $k=0,...,m$ which together with (3.2) finally implies the assertion. $\square$

4 Singular Resolution of (u_1,u_2) at the Origin

In this Section, where the analysis is mainly based on the Mellin transform, we are going to improve the results of Theorem 3.1. To this end, we see that since $\mathbf{f} \in C^m(\overline{\varpi})^2$, we have the Taylor expansions (see (3.4))

$$g_i(x_j) = \sum_{k=1}^{m} c_{ik}x_j^k + \mathcal{O}(x_j^{m+1}) \quad \text{as} \quad x_j \to 0+, \tag{4.1}$$

where c_{ik} depends on a, b, M and on the derivatives of f_1, f_2 of order $\le k-1$ at the origin. Let us define

$$\phi_i^R(x_j) = \begin{cases} \sum_{k=1}^{m} A_{ik}x_j^k & \text{if } \eta \notin \{1,2,...,m\}, \\ \sum_{1\le k\le m}^{k\ne\eta} A_{ik}x_j^k \; + B_{i\eta}x_j^\eta\ln(x_j) & \text{otherwise,}\end{cases} \tag{4.2}$$

where for any $k=1,...,m$,

$$\begin{aligned} A_{ik} &= \tfrac{(b-a)^2(k+1)^2C_{ik}}{g(k+2)}, & B_{i\eta} &= \tfrac{(\eta+1)\sinh(X_\eta)C_{i\eta}}{2(\sinh(X_\eta)-X_\eta\cosh(X_\eta))}, \\ C_{ik} &= c_{ik} - \tfrac{\alpha(b^{k+1}-a^{k+1})c_{jk}}{\alpha_i(b-a)(ab)^{k(i-1)}(k+1)}, & X_\eta &= \tfrac{\eta+1}{2}\ln\left(\tfrac{b}{a}\right). \end{aligned} \tag{4.3}$$

In Lemma 5.1 we prove that $\eta \notin \{1,2,...,m\}$ implies $g(k+2) \ne 0$, so, A_{ik} above is well defined under this assumption. Further, if $F(X) = \sinh(X) - X\cosh(X)$, then $F(0)=0$ and $F'(X) = -X\sinh(X) < 0$ for any $X > 0$. Hence, $F(X) < 0$ for any $X > 0$ and thus also $B_{i\eta}$ is well defined. Based on these definitions we now find that (recall (3.3))

$$\phi_i^R(x_j) + \frac{\alpha}{\alpha_i W_i(x_j)}\mathcal{I}_i\phi_j^R(x_j) = \sum_{k=1}^{m} c_{ik}x_j^k \tag{4.4}$$

for $x_1 \in [0,a]$ and $x_2 \in [0,1]$. Hence, setting

$$\phi_i^S := \phi_i - \phi_i^R, \quad g_i^S := \phi_i^S(x_j) + \frac{\alpha}{\alpha_i W_i(x_j)}\mathcal{I}_i(\phi_j^S)(x_j), \tag{4.5}$$

it first follows from (3.6), (4.1), (4.4) and (4.5) that

$$x_2^{-m-1+k} g_1^S \in C^k[0,1], \quad x_1^{-m-1+k} g_2^S \in \{g \in C[0,b] \,| g \in C^k[0,a], \; g \in C^k[a,b]\} \tag{4.6}$$

for any $k = 0, ..., m$. Then, proceeding as in the proof of (3.11), we see that

$$x_2^{-\nu+k} \phi_1^S \in C_p^k[0,1], \quad x_1^{-\nu+k} \phi_2^S \in C_p^k[0,b] \tag{4.7}$$

holds for any $k = 0, ..., m$ and $\nu \in [0, \eta) \cap [0, m+1]$. Accordingly and by (3.2) and (4.2),

$$\eta > m-1 \Rightarrow \mathbf{u} \in C_{p1}^m(\overline{\omega}) \times C_{p2}^m(\overline{\omega}). \tag{4.8}$$

Therefore we assume in the sequel that $\eta \leq m-1$.
To get a singular expansion of ϕ_i^S at the origin, let θ be a smooth cut-off function defined on $[0, \infty)$ such that $\theta(x) = 1$ near the origin and $\theta(x) = 0$ for $x > \sigma = \min\{a^k/b^k, a^{k+1}/b^k\}$, $k = \lfloor m/2 \rfloor$. Multiplying then the right side equation (4.5) by $\theta(x_j)$, we get

$$\varphi_1(x_2) + \frac{\alpha}{\alpha_1(b-a)x_2} \int_{ax_2}^{bx_2} \varphi_2(x_1)\, dx_1 = \xi_1(x_2), \tag{4.9}$$

$$\varphi_2(x_1) + \frac{\alpha ab}{\alpha_2(b-a)x_1} \int_{x_1/b}^{x_1/a} \varphi_1(x_2)\, dx_2 = \xi_2(x_1), \tag{4.10}$$

where

$$\begin{aligned} \varphi_i &= \theta(x_j)\phi_i^S \quad \text{and} \\ \xi_1 &= \theta(x_2)g_1^S - \frac{\alpha}{\alpha_1(b-a)x_2} \int_{ax_2}^{bx_2} (\theta(x_2) - \theta(x_1))\phi_2^S(x_1)\, dx_1, \\ \xi_2 &= \theta(x_1)g_2^S - \frac{\alpha ab}{\alpha_1(b-a)x_2} \int_{x_1/b}^{x_1/a} (\theta(x_1) - \theta(x_2))\phi_1^S(x_2)\, dx_2. \end{aligned} \tag{4.11}$$

Based on the definition of θ and on (2.8), (4.6) and (4.7), we have

Lemma 4.1. *Functions φ_i and ξ_i have compact support on $[0, \infty)$ and*

$$x_j^{-\nu+k} \varphi_i \in C^k[0, \infty), \quad x_j^{-m-1+k} \xi_i \in C^k[0, \infty), \; k = 0, 1, 2, ..., m,$$

for any $\nu \in [0, \eta)$. $\square$

Note by Lemmas 2.1 and 4.1, that the corresponding Mellin transforms $\tilde{\varphi}_i$ are analytic when $\Re(z) > -\eta$ and $\tilde{\xi}_i$ are analytic when $\Re(z) > -m-1$. Applying now the Mellin transform to (4.9) and (4.10), it follows easily from the Fubini theorem that

$$\begin{aligned} \tilde{\varphi}_1 + \frac{\alpha(ab)^{1-z}(b^{z-1} - a^{z-1})}{\alpha_1(b-a)(z-1)} \tilde{\varphi}_2 &= \tilde{\xi}_1, \\ \tilde{\varphi}_2 + \frac{\alpha \;\; ab \quad (b^{z-1} - a^{z-1})}{\alpha_2(b-a)(z-1)} \tilde{\varphi}_1 &= \tilde{\xi}_2, \end{aligned}$$

or equivalently

$$\tilde{\varphi}_i = \frac{(b-a)^2(z-1)^2 \tilde{F}_i}{g(z)} =: \frac{p_i(z)}{g(z)}, \tag{4.12}$$

where g is defined by (1.2) and

$$\begin{aligned} F_1 &= \xi_1 - \frac{\alpha}{\alpha_1(b-a)x_2} \int_{ax_2}^{bx_2} \xi_2(x_1)\, dx_1, \\ F_2 &= \xi_2 - \frac{\alpha ab}{\alpha_2(b-a)x_1} \int_{x_1/b}^{x_1/a} \xi_1(x_2)\, dx_2. \end{aligned}$$

Note that obviously (F_1, F_2) satisfies the same promerties as (ξ_1, ξ_2) in Lemma 4.1 and thus $\tilde{F}_i$ are analytic when $\Re(z) > -m-1$. Accordingly, so are p_i in (4.12) and so all the poles of $\tilde{\varphi}_i$ in this region are the roots of g. These are carefully analyzed in Theorem 5.1 where we show that in addition to the simple roots at $z = -\eta$ and at $z = \eta + 2$, there exists a double root at $z = 1$ and a numerable set of nonreal simple roots

$$-z_k^+, \quad -\bar{z}_k^+, \quad z_k^+ + 2, \quad \bar{z}_k^+ + 2, \quad k \;=\; 1, 2, 3, \dots$$

$$-z_k^-, \quad -\bar{z}_k^-, \quad z_k^- + 2, \quad \bar{z}_k^- + 2, \quad k \;=\; 0, 1, 2, \dots$$

where $-\alpha_k^{\pm} = \Re(-z_k^{\pm}) < -\eta < 0$ for any k and $-\alpha_k^{\pm} \to -\infty$ as $k \to \infty$. Hence, by Lemmas 2.1 and 4.1, the inversion formula

$$\varphi_l(x_j) = \frac{1}{2\pi i} \int_{\Re(z)=\gamma_2} x_j^{-z} \tilde{\varphi}_l(z)\, dz, \quad l = 1, 2, \tag{4.13}$$

is valid if $\gamma_2 > -\eta$. Assume now that $\gamma_2 \in (-\eta, 0)$ so that there are no roots of $g(z)$ with $\Re(z) = \gamma_2$. The asymptotic expansion for φ_l will be derived by pushing the integration line to the left. To this end, let $\Gamma = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$ be the closed positively oriented curve where

$$\begin{aligned} \Gamma_1 &= \{z = \gamma_2 + ix_2 \mid |x_2| \le M\}, \\ \Gamma_2 &= \{z = x_1 + iM \mid \gamma_1 < x_1 < \gamma_2\}, \\ \Gamma_3 &= \{z = \gamma_1 + ix_2 \mid |x_2| \le M\}, \\ \Gamma_4 &= \{z = x_1 - iM \mid \gamma_1 < x_1 < \gamma_2\}, \end{aligned}$$

where further $\gamma_1 \in (-m-1, -m-1/2)$ is such that $\gamma_1 \neq \Re(-z_k^{\pm})$ for any k and $M > 0$ is such that the set

$$\{z \mid \gamma_1 < \Re(z) < \gamma_2,\ |\Im(z)| > M\}$$

contains no roots of g (see Theorem 5.1 (iv)). In Fig. 2 the curve Γ for $m = 2$ and the leading roots of g are plotted in the example case

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad a = 1, \quad b = e^2, \tag{4.14}$$

where $\kappa = 0.8$, $-\eta \simeq -0.577$ and the remaining roots lie in the set $\Re(z) < -1.669$.

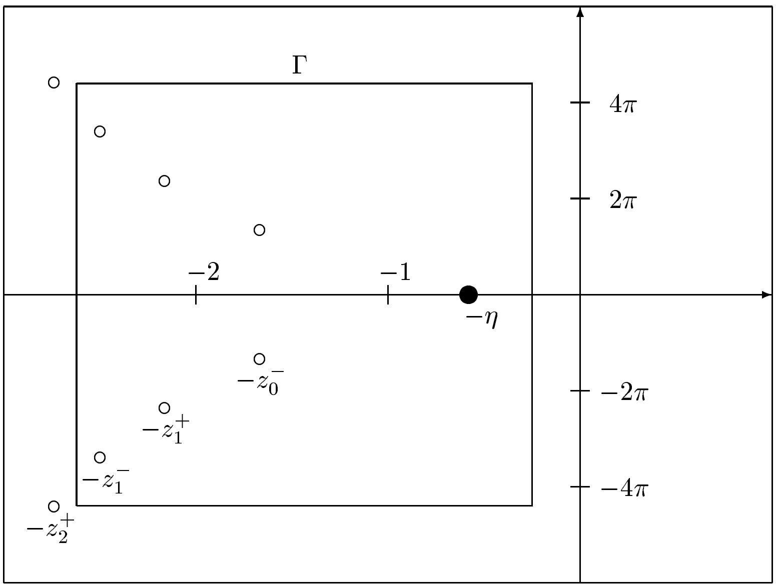

FIGURE 2. Roots of g and the curve Γ for $m = 2$ in the example case (4.14).

By the Residy theorem

$$\frac{1}{2\pi i}\int_{\Gamma} x_j^{-z}\frac{p_l(z)}{g(z)}\,dz = \frac{p_l(-\eta)}{g'(-\eta)}x_j^{\eta} + 2\Re\left(\sum_{\gamma_1<-\alpha_k^{\pm}<-\eta}\frac{p_l(-z_k^{\pm})}{g'(-z_k^{\pm})}x_j^{z_k^{\pm}}\right). \tag{4.15}$$

Applying Lemma 2.1 (iii) and noting that $p_l = (b-a)^2(z-1)^2\tilde{F}_l$, we conclude that

$$\lim_{M\to\infty}\frac{1}{2\pi i}\int_{\Gamma_k} x_j^{-z}\frac{p_l(z)}{g(z)}\,dz = 0, \quad k = 2, 4. \tag{4.16}$$

Further, by Lemma 2.1 (i),

$$R_l := -\lim_{M\to\infty}\frac{1}{2\pi i}\int_{\Gamma_3} x_j^{-z}\frac{p_l(z)}{g(z)}\,dz \in L^{2,\gamma_1-\ 1/2}(0,\infty). \tag{4.17}$$

Collecting finally the results (4.13)-(4.17), we get the expansion

$$\varphi_i = \frac{p_i(-\eta)}{g'(-\eta)}x_j^{\eta} + 2\Re\left(\sum_{\gamma_1<-\alpha_k^{\pm}<-\eta}\frac{p_i(-z_k^{\pm})}{g'(-z_k^{\pm})}x_j^{z_k^{\pm}}\right) + R_i, \quad R_i \in L^{2,\gamma_1-\ 1/2}(0,\infty). \tag{4.18}$$

Clearly above R_i are continuos functions since φ_i are. Further, noting that the exponent terms above satisfy the homogeneous equations (4.9) and (4.10), we conclude that actually

$$\begin{aligned}
\alpha_1(b-a)x_2R_1(x_2) \;+\; \alpha\int_{ax_2}^{bx_2} R_2(x_1)\,dx_1 \;&=\; \alpha_1(b-a)x_2\xi_1(x_2),\\
\alpha_2(b-a)x_1R_2(x_1) + \alpha ab\int_{x_1/b}^{x_1/a} v_1(x_2)\,dx_2 \;&=\; \alpha_2(b-a)x_1\xi_2(x_1),
\end{aligned}$$

so that by differentating both equations $k+1$ times and solving then $R_i^{(k+1)}$, we get

$$R_i^{(k+1)} = \frac{(x_j\xi_i)^{(k+1)} - c_i\left(b_i^{k+1}R_j^{(k)}(b_ix_j) - a_i^{k+1}R_j^{(k)}(a_ix_j)\right) - (k+1)R_i^{(k)}(x_j)}{x_j}, \tag{4.19}$$

where

$$c_1 = \frac{\alpha}{\alpha_1(b-a)},\ c_2 = \frac{\alpha ab}{\alpha_2(b-a)},\ a_1 = a,\ b_1 = b,\ a_2 = b^{-1},\ b_2 = a^{-1}.$$

Accordingly, R_i is a C^m-function away from the origin. To analyse what happens near the origin, it first follows from Lemma 4.1, (4.18) and (4.19), that

$$x_j^{-n}R_i^{(k)} \in L^{2,\gamma_1-1/2+k+n}(0,1) \subset L^{2,k+n-m-1}(0,1), \quad k = 0,1,2,...,m.$$

Hence, this regularity result together with the Sobolev imbedding theorem implies

$$x_j^{-n}R_i \in C^{k-1}[0,1], \quad k = 0,...,m,\ k+n \le m+1.$$

Then, applying this (with $k = m,\ n = 1$) and (4.19), we get

$$R_i \in C^m[0,\infty), \quad R_i^{(k)}(0) = 0,\ k = 0,1,...,m-1. \tag{4.20}$$

Collecting finally the results from (3.2), Theorem 3.1, (4.2), (4.5), (4.18) and (4.20), we can summarize the main results of the paper in the following regularity result for (u_1, u_2).

Theorem 4.1. *Assume (2.3) and (2.4). Then there are coefficients*

$$A_i,\ B_i,\ C_i,\ D_i,\ E_i,\ F_i \in \boldsymbol{\mathcal{R}} \text{ and } A_{ik}^{\pm},\ B_{ik}^{\pm},\ C_{ik}^{\pm} \in \boldsymbol{\mathcal{C}}$$

such that

$$\begin{aligned} u_i &- A_i x_i^{\eta+1} - B_i x_i x_j^{\eta} - C_i x_j^{\eta+1} - D_i x_i^{\eta+1}\ln(x_i) - E_i x_i x_j^{\eta}\ln(x_j) - F_i x_j^{\eta+1}\ln(x_j) \\ &- \Re\left(\sum_{-m+1\le -\alpha_k^{\pm} < -\eta}\left[A_{ik}^{\pm} x_i^{z_k^{\pm}+1} + B_{ik}^{\pm} x_i x_j^{z_k^{\pm}} + C_{ik}^{\pm} x_j^{z_k^{\pm}+1}\right]\right) \in C_{pi}^{m}(\varpi), \\ & i,j\in\{1,2\},\ i\neq j, \end{aligned}$$

where $-\eta$ *and* $-z_k^{\pm}$ *are those roots of* g *defined by (1.2)-(1.3), whose real part lie in interval* $[-m+1,0)$, $-\alpha_k^{\pm} = \Re(-z_k^{\pm})$, *and where further* $D_i = E_i = F_i = 0$ *if* $\eta \notin \{1,2,...,m\}$.
□

5 Roots of $g(z)$

This final technical section is motivated by Theorem 4.1. For notational convenience we write in what follows (x,y) instead of (x_1,x_2).

Theorem 5.1. *Assume that* a, b *and* κ *are real numbers such that* $b>a>0$ *and* $0<\kappa<1$.
(i) Then the roots of $g(z) = (b-a)^2(z-1)^2 - 4ab\kappa^2\sinh^2((z-1)\ln(b/a)/2)$ *are*

$$\begin{aligned} -\eta &= \frac{\ln(b/a)-2x_0}{\ln(b/a)}, \quad 1, \quad \eta+2, \\ -z_k^{+} &= -\alpha_k^{+} - i\beta_k^{+} = \frac{\ln(b/a)-2x_k^{+}}{\ln(b/a)} - i\frac{2y_k^{+}}{\ln(b/a)}, \quad z_k^{+}+2, \quad k=1,2,3,\dots, \\ -\overline{z}_k^{+} &= -\alpha_k^{+} + i\beta_k^{+}, \quad \overline{z}_k^{+}+2, \quad k=1,2,3,\dots, \\ -z_k^{-} &= -\alpha_k^{-} - i\beta_k^{-} = \frac{\ln(b/a)-2x_k^{-}}{\ln(b/a)} - i\frac{2y_k^{-}}{\ln(b/a)}, \quad z_k^{-}+2, \quad k=0,1,2,\dots, \\ -\overline{z}_k^{-} &= -\alpha_k^{-} + i\beta_k^{-}, \quad \overline{z}_k^{-}+2, \quad k=0,1,2,\dots, \end{aligned}$$

where x_0 *is the only positive real root of* $x = R\sinh(x)$, $y_k^{\pm}$ *are the unique roots of*

$$g_{\pm}(y) = y \mp R\sin(y)\cosh\left(\sqrt{y^2 - R^2\sin^2(y)}\ \cot(y)\right) \tag{5.1}$$

on the intervals $\left(2k\pi, (2k+\ 1/2)\pi\right)$, $k=1,2,3,\dots$ *and*
$\left((2k+1)\pi, (2k+\ 3/2)\pi\right)$, $k=0,1,2,\dots$, *respectively, and*

$$x_k^{+} = \cosh^{-1}\left(\frac{y_k^{+}}{R\sin(y_k^{+})}\right), \qquad x_k^{-} = \cosh^{-1}\left(-\frac{y_k^{-}}{R\sin(y_k^{-})}\right), \tag{5.2}$$

where finally

$$R = \frac{\sqrt{ab}\ \kappa}{b-a}\ln(b/a) \in (0,\kappa).$$

(ii) $0<\eta<2K\ln[\kappa^{-1}(2K+1)]$ *where* $K = 2[\ln(b/a)]^{-1}$. *Moreover* $\alpha_k^{\pm} > \eta$ *and* $\alpha_k^{\pm}\to\infty$ *as* $k\to\infty$.
(iii) All the roots have multiplicity 1 except $z=1$ *which is a double root.*
(iv) For any γ_1, $\gamma_2 \in \mathcal{R}$, $\gamma_1<\gamma_2$, *there exists* $M = M(\gamma_1,\gamma_2)$ *such that the set*

$$\{x+iy \mid \gamma_1 < x < \gamma_2,\ |y|\ge M\}$$

contains no roots.

Proof. Writing $w = (z-1)\ln(b/a)/2$, $g(z)=0$ is equivalent to $w^2 = R^2\sinh^2(w)$, which further may be split into two equations;

$$w = R\sinh(w), \tag{5.3}$$
$$w = -R\sinh(w). \tag{5.4}$$

Let us first search for the roots for (5.3), writing $w = x + iy$, so that

$$\begin{cases} x &= R\sinh(x)\cos(y), \\ y &= R\cosh(x)\sin(y). \end{cases} \tag{5.5}$$

The solution to (5.5) occur in the groups of four; (x, y), $(x, -y)$, $(-x, y)$ and $(-x, -y)$, so we may assume in the sequel that $x \geq 0$, $y \geq 0$. If $y = 0$, (5.5) is equivalent to $x = R\sinh(x)$, which clearly holds if $x = 0$, but also has one extra root $x_0 > 0$ since

$$0 < R < \kappa < 1, \tag{5.6}$$

as follows from

$$\frac{R}{\kappa} = \frac{\sqrt{t}\ \ln(t)}{t-1} < 1,\ t = \frac{b}{a} > 1.$$

When $x = 0$, we find no other solutions than $y = 0$, so we may assume next that $x > 0$ and $y > 0$. Then by (5.5) and (5.6),

$$\cos(y) = \frac{x}{R\sinh(x)} > 0, \qquad \sin(y) = \frac{y}{R\cosh(x)} > 0,$$

and thus

$$y \in \big(2k\pi, (2k+1/2)\pi\big) \text{ for some } k \in \mathcal{N} = \{0, 1, 2, \ldots\}. \tag{5.7}$$

By the lower equation (5.5), we have

$$x = \cosh^{-1}\left(\frac{y}{R\sin(y)}\right), \tag{5.8}$$

and when this is substituted to the upper equation (5.5), we obtain $g_+(y) = 0$ (see (5.1)). Let us show next that there is a unique root of g_+ on each interval (5.7) except on the first one ($k = 0$). Indeed, if $y \in (0, \pi/2)$, we have $\cos(y) < \sin(y)/y$, and thus

$$\begin{aligned} \frac{g_+(y)}{\sin(y)} &= \frac{y}{\sin(y)} - R\cosh\left(\left(\frac{y^2}{\sin^2(y)} - R^2\right)^{1/2}\cos(y)\right) \\ &> \frac{y}{\sin(y)} - R\cosh\left(\left(\frac{y^2}{\sin^2(y)} - R^2\right)^{1/2}\frac{\sin(y)}{y}\right) \\ &= R\big((1-p^2)^{-1/2} - \cosh(p)\big) = R\sum_{n=2}^{\infty} a_{2n}p^{2n}, \end{aligned}$$

where

$$\begin{aligned} p &= \left(\frac{y^2}{\sin^2(y)} - R^2\right)^{1/2}\frac{\sin(y)}{y} \in (0,1) \text{ and} \\ a_{2n} &= \frac{\big(1\cdot 3\cdot 5\cdots(2n-1)\big)^2 - 1}{(2n)!} > 0 \text{ for any } n \geq 2. \end{aligned}$$

This and (5.6) imply that $g_+(y) > 0$ for any $y \in (0, \pi)$.
Assume next (5.7) with $k \geq 1$. Since

$$\begin{aligned} \lim_{y\to 2k\pi+} g_+(y) &= -\infty,\ k = 1, 2, 3, \ldots, \\ \lim_{y\to(2k+1/2)\pi-} g_+(y) &= (2k+1/2)\pi - R > 0,\ k = 1, 2, 3, \ldots, \end{aligned} \tag{5.9}$$

g_+ has at least one zero on each interval considered. To guarantee the uniqueness, write

$$Y_1 = \sqrt{y^2 - R^2\sin^2(y)}\ \cot(y) > 0,\ Y_2 = \frac{y\sin(y)\cos(y) + R^2\sin^4(y) - y^2}{\sin^2(y)Y_1}.$$

Then

$$g'_+(y) = 1 - R\cos(y)\big(\cosh(Y_1) + Y_2 \sinh(Y_1)\big),$$

so that

$$Y_2 < -1 \Rightarrow g'_+(y) > R\cos(y)\big(1 + \sinh(Y_1) - \cosh(Y_1)\big) > 0, \tag{5.10}$$

and so it suffices for us to prove that $Y_2 < -1$. But this is equivalent to

$$y^2 - \sin(y)\cos(y)\ y - R^2\sin^4(y) > \sin(y)\cos(y)\sqrt{y^2 - R^2\sin^2(y)},$$

where $y > \sqrt{y^2 - R^2\sin^2(y)}$, so we are done provided

$$\begin{aligned} & y^2 - 2\sin(y)\cos(y)\ y - R^2\sin^4(y) > 0 \Leftrightarrow \\ & y > \sin(y)\left(\cos(y) + \sqrt{\cos^2(y) + R^2\sin^2(y)}\right). \end{aligned}$$

However, by (5.6) the right side above is less than 2 for any y, and thus

$$y > 2 \Rightarrow Y_2 < -1. \tag{5.11}$$

It now follows from (5.9), (5.10) and (5.11), that g_+ has the unique root y_k^+ on each interval $\big(2k\pi, (2k+\ 1/2)\pi\big),\ k = 1, 2, \ldots$ Further, by (5.8) the first equation (5.2) holds.
To find the roots for (5.4), rewrite this as

$$\left\{ \begin{array}{lcl} x & = & -R\sinh(x)\cos(y), \\ y & = & -R\cosh(x)\sin(y), \end{array} \right. \qquad w = x + iy. \tag{5.12}$$

Again by symmetry, we may consider only the set where $x \geq 0,\ y \geq 0$, and since the only root on the coordinate axes is $(x, y) = (0, 0)$, we may further assume that $x > 0,\ y > 0$. Then it follows from (5.6) and (5.12), that both $\sin(y)$ and $\cos(y)$ are negative, so we must have

$$y \in \big((2k+1)\pi, (2k\ +3/2)\pi\big) \ \text{ for some } \ k \in \mathcal{N} = \{0, 1, 2, \ldots\}. \tag{5.13}$$

>From the lower equation (5.12), we have

$$x = \cosh^{-1}\left(-\frac{y}{R\sin(y)}\right), \tag{5.14}$$

and if this is substituted to the upper equation (5.12), we obtain $g_-(y) = 0$ (see (5.1)). Using the same notation as above, we have for any $k \in \mathcal{N}$ that

$$\lim_{y \to (2k+1)\pi+} g_-(y) = -\infty, \qquad \lim_{y \to (2k+3/2)\pi-} g_-(y) = (2k+\ 3/2)\pi - R > 0. \tag{5.15}$$

Further, $g'_-(y) = 1 + R\cos(y)\big(\cosh(Y_1) + Y_2\sinh(Y_1)\big)$, so that also this time

$$Y_2 < -1 \Rightarrow g'_-(y) > 0. \tag{5.16}$$

By (5.11), (5.15) and (5.16), g_- has the unique root y_k^- on each interval (5.13) and by (5.14), also the second equation (5.2) holds.
Since the roots of g are

$$\left\{ z = \frac{\ln(b/a) + 2w}{\ln(b/a)} \mid w \in \mathcal{C},\ w^2 = R^2\sinh^2(w) \right\},$$

we are done with the proof of (i).

(ii) Write $\eta = Kw_0$ where w_0 is the largest real root of

$$h(w) = R\sinh(w + K^{-1}) - (w + K^{-1}).$$

Then since

$$h(0) = \frac{1}{2}R\sqrt{\frac{b}{a}}\,\left(1 - \frac{a}{b}\right) - K^{-1} = K^{-1}(\kappa - 1) < 0,$$

and since for $w > 0$,

$$\begin{aligned} h(w) &> \kappa K^{-1}e^{w} - w - K^{-1} \\ &\geq (2 + K^{-1})\big[w - \ln[\kappa^{-1}(2K+1)] + 1\big] - w - K^{-1} \\ &> (1 + K^{-1})\big[w - 2\ln[\kappa^{-1}(2K+1)]\big], \end{aligned}$$

it follows that $0 < w_0 < 2\ln[\kappa^{-1}(2K+1)]$, so η is bounded as asserted. Further, by the upper equations in (5.5), (5.12), $x_k^{\pm} > x_0$ for any k, and by (5.2), $x_k^{\pm} \to \infty$ as $k \to \infty$, so assertion (ii) is proved.

(iii) Since the only common root of

$$h_1(w) = R\sinh(w) - w \text{ and } \ h_2(w) = R\sinh(w) + w$$

is $w = 0$, it obviously suffices to show that all the roots of $h_1(w) = 0$ and $h_2(w) = 0$ are simple. But

$$h_1'(w) = 0 \Rightarrow w = \cosh^{-1}(1/R) + i2k\pi, \quad k \in \boldsymbol{\mathcal{Z}},$$

so that the only common root of h_1 and h_1' could be $w = \cosh^{-1}(1/R)$. However, $h_1\left(\cosh^{-1}(1/R)\right) = 0$ is equivalent to

$$(1 - p^2)^{-1/2} - \cosh(p) = 0, \quad p = \sqrt{1 - R^2} \in (0, 1),$$

where the first equation above is false for any $p \in (0,1)$ as we proved earlier by expanding the left side into Taylor series.
On the other hand,

$$h_2'(w) = 0 \Rightarrow w = \cosh^{-1}(1/R) + i(2k+1)\pi, \quad k \in \boldsymbol{\mathcal{Z}},$$

and since $h_2(w) \neq 0$ for all such w, the roots $(x, y) \neq (0, 0)$ are simple as asserted.

(iv) For given $\gamma_1 < \gamma_2$, set

$$K = \left\{ k \in \boldsymbol{\mathcal{Z}} \mid \frac{\ln(b/a) \pm 2x_k^{\pm}}{\ln(b/a)} \in [\gamma_1, \gamma_2] \right\}.$$

By (ii), K is a finite set. Further, set

$$N = \max_{k \in K}\{x_k^{\pm}\}, \qquad M = \frac{2\cosh(N)}{\ln(b/a)}.$$

By (5.5) and (5.12),

$$y_k^{\pm} \leq R\cosh(x_k^{\pm}) < \cosh(N), \quad k \in K,$$

which further implies that

$$\frac{2y_k^{\pm}}{\ln(b/a)} < M \ \text{ for any } \quad k \in K.$$

This completes the proof. $\square$

A Membrane theory of a shallow hyperbolic shell

In the membrane theory of a thin shell, the forces due to bending deformations are considered negligible. Assuming that the shell consists of homogeneous isotropic material with Poisson ratio ν ($0 \le \nu \le 1/2$), the deformation energy of the shell may be expressed as [3]

$$\begin{aligned} F(u,v,w) = &\int_\Omega \left[\nu(\beta_{11}+\beta_{22})^2 + (1-\nu)(\beta_{11}^2 + 2\beta_{12}^2 + \beta_{22}^2)\right] dxdy \\ &- 2\int_\Omega (g_1 u + g_2 v + g_3 w)\, dxdy, \end{aligned} \tag{A.1}$$

where Ω is the midsurface of the shell, u, v are the tangential and w the normal component of the displacement field that represents the deformation of the midsurface under the loads g_i, and β_{ij} are the membrane strains arising from the change of metric in the deformation. The membrane strains relate to the displacements u, v, w via consitutive laws that depend on the geometry of Ω and on the coordinates x, y chosen. Assuming here shallow shell theory with approximate principal curvature coordinates as x, y, we have the relations

$$\beta_{11} = \frac{\partial u}{\partial x} + \frac{w}{R_1}, \quad \beta_{22} = \frac{\partial v}{\partial y} + \frac{w}{R_2}, \quad \beta_{12} = \frac{1}{2}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big), \tag{A.2}$$

where R_1^{-1}, R_2^{-1} are the localized (constant) principal curvatures [3].
Denoting

$$\frac{1}{R} = \Big(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{2\nu}{R_1 R_2}\Big)^{1/2}, \quad c_1 = \frac{R}{R_1}, \quad c_2 = \frac{R}{R_2},$$

and rearranging (A.1)–(A.2) as

$$\begin{aligned} F(u,v,w) = &\int_\Omega \Big\{(1-\nu^2)\Big(c_2\frac{\partial u}{\partial x} - c_1\frac{\partial v}{\partial y}\Big)^2 + \frac{1}{2}(1-\nu)\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2\Big\}\, dxdy \\ &+ \int_\Omega \Big[R^{-1}w + (c_1 + \nu c_2)\frac{\partial u}{\partial x} + (\nu c_1 + c_2)\frac{\partial v}{\partial y}\Big]^2 dxdy \\ &- 2\int_\Omega (g_1 u + g_2 v + g_3 w)\, dxdy, \end{aligned} \tag{A.3}$$

we may eliminate w from the Euler equations of the energy principle $F(u,v,w) = min!$, thus obtaining the reduced energy principle

$$\begin{aligned} \tilde{F}(u,v) = &\int_\Omega \Big\{(1-\nu^2)\Big(c_2\frac{\partial u}{\partial x} - c_1\frac{\partial v}{\partial y}\Big)^2 + \frac{1}{2}(1-\nu)\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2\Big\}\, dxdy \\ &- 2\int_\Omega (\tilde{g}_1 u + \tilde{g}_2 v)\, dxdy = \min!, \end{aligned} \tag{A.4}$$

where

$$\tilde{g}_1 = g_1 + (c_1 + \nu c_2)R\frac{\partial g_3}{\partial x}, \quad \tilde{g}_2 = g_2 + (\nu c_1 + c_2)R\frac{\partial g_3}{\partial y}.$$

Assume now that $c_1 c_2 < 0$, so that the shell is hyperbolic. Then passing in (A.4) to the characteristic coordinates

$$x_1 = \frac{1}{\sqrt{2}}(|c_2|^{-1/2}x + |c_1|^{-1/2}y), \quad x_2 = \frac{1}{\sqrt{2}}(|c_2|^{-1/2}x - |c_1|^{-1/2}y),$$

and transforming the displacements accordingly as

$$u = |c_1|^{1/2}(u_1 + u_2), \quad v = |c_2|^{1/2}(u_1 - u_2),$$

we find that $\tilde{F}(u,v) = \mathcal{F}(u_1,u_2) = \mathcal{F}(\mathbf{u})$, where

$$\mathcal{F}(\mathbf{u}) = \int_\omega \Big\{ 2(1-\nu^2)|c_1c_2|^{1/2}\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)^2 + (1-\nu)|c_1c_2|^{-1/2}\Big(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\Big)^2 - 2\mathbf{f}\cdot\mathbf{u}\Big\}\, dx_1 dx_2, \tag{A.5}$$

where now ω is the transformed domain and $\mathbf{f} = (f_1, f_2)$ is given by

$$f_1 = |c_2|^{-1/2}\tilde{g}_1 + |c_1|^{-1/2}\tilde{g}_2, \quad f_2 = |c_2|^{-1/2}\tilde{g}_1 - |c_1|^{-1/2}\tilde{g}_2.$$

We observe that (A.5) is of the form (1.1) with $A = (a_{ij})$ defined by

$$a_{11} = a_{12} = 2(1-\nu^2)^{1/2}|c_1c_2|^{1/4},$$

$$-a_{21} = a_{22} = (1-\nu^2)^{1/2}|c_1c_2|^{1/4}.$$

The parameter κ^2 in (1.3) is then given by

$$\kappa^2 = \left(\frac{2(1+\nu)|c_1c_2| - 1}{2(1+\nu)|c_1c_2| + 1}\right)^2.$$

We see that $0 \le \kappa^2 < 1$ when $0 < |c_1c_2| \le (2+2\nu)^{-1}$, and $0 \le \kappa^2 \le 1/4$ when $(2+2\nu)^{-1} \le |c_1c_2| \le (2-2\nu)^{-1}$ (the maximal value of $|c_1c_2|$). The degenerate case $\kappa = 0$ is obtained when $|R_1/R_2| = 1 + 2\nu \pm 2\sqrt{\nu+\nu^2}$.

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