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SOME REMARKS ON THE METHOD OF SUMS

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Abstract: *This note presents a reasonably short but self-contained proof of the method of sums due to DaPrato and Grisvard. Explicit constants for the regularity estimates are given.*

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1 Introduction

In this note we consider the method of sums of operators, devised by DaPrato and Grisvard. The method of sums gives conditions under which the problem $Ay + By = x$ can be solved. Here A and B are linear operators mapping, respectively, $\mathcal{D}(A)$ and $\mathcal{D}(B)$ into X, where X is a Banach space and $x \in X$ is given. In general, only the existence of a mild solution can be guaranteed, but if this solution y belongs to either $\mathcal{D}(A)$ or $\mathcal{D}(B)$, then it is a strong solution. In particular, if x belongs to a certain interpolation space, then one has a strong solution y. Moreover, then Ay and By belong to the same interpolation space, i.e., one has maximal regularity.

Our purpose is twofold. First, the aim is to present a brief but concise and self-contained proof of several previously known results scattered in the literature.

Second, our aim is to make explicit the constants occurring in the estimates for the various interpolation norms of Au and Bu. In addition, we extend the method to give some regularity results for the case where neither A nor B is invertible, but then the existence of a strong solution is assumed.

We make very little claim as to originality; most of the results that we present can, in one form or another, be found in [3]–[6]. See also [1] and [2].

We begin by defining the class of operators considered. If X is a Banach space, then we denote the norm in X by $\|\cdot\|$ (or $\|\cdot\|_X$) and we let $\|\cdot\|$ denote the norm of bounded linear operators on X as well.

Definition 1. *Let X be a (complex) Banach space. A linear operator $L : \mathcal{D}(L) \subset X \to X$ is nonnegative if $(-\infty, 0) \subset \rho(L)$ (the resolvent set of L) and*

$$\sup_{t>0} \|t(L + tI)^{-1}\| < \infty.$$

If L is a nonnegative operator on X, then

$$\phi_L \stackrel{\text{def}}{=} \sup\{\, \phi \in [0, \pi] \mid \sup_{\substack{|\arg(\lambda)| \le \phi \\ \lambda \ne 0}} \|\lambda(L + \lambda I)^{-1}\| < \infty \,\},$$

and

$$M(L, \phi) \stackrel{\text{def}}{=} \sup_{\substack{|\arg(\lambda)| = \phi \\ \lambda \ne 0}} \|\lambda(L + \lambda I)^{-1}\|.$$

In Definition 1 we, of course, take $\|(L + \lambda I)^{-1}\| = \infty$ if $-\lambda$ does not belong to the resolvent set of L, i.e., if $L + \lambda I$ is not invertible. Observe also that if L is a nonnegative operator, then $\phi_L \ge \arcsin(1/M(L, 0))$. One usually says that $\pi - \phi_L$ is the spectral angle of L.

Definition 2. *Let X be a (complex) Banach space and let L be a nonnegative operator on X. If $\gamma \in (0, 1)$ and $p \in [1, \infty]$, then*

$$\mathcal{D}_L(\gamma, p) \stackrel{\text{def}}{=} \{\, x \in X \mid [x]_{\mathcal{D}_L(\gamma,p)} < \infty \,\},$$

where

$$[x]_{\mathcal{D}_L(\gamma,p)} = \begin{cases} \left(\int_0^\infty \left(t^\gamma \|L(L+tI)^{-1}x\|\right)^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup_{t>0} t^\gamma \|L(L+tI)^{-1}x\|, & p = \infty. \end{cases}$$

Moreover,

$$\mathcal{D}_L(\gamma,\infty_0) \stackrel{\text{def}}{=} \{\, x \in \mathcal{D}_L(\gamma,\infty) \mid \lim_{t\to\infty} t^\gamma \|L(L+tI)^{-1}x\| = 0 \,\},$$

with $[\cdot]_{\mathcal{D}_L(\gamma,\infty_0)} = [\cdot]_{\mathcal{D}_L(\gamma,\infty)}$.

It is easy to see that $[\cdot]_{\mathcal{D}_L(\gamma,p)}$ is (at least) a seminorm. Note that for notational convenience we write $\mathcal{D}_L(\gamma,\infty_0) = \mathcal{D}_L(\gamma)$. The interpolation spaces between X and $\mathcal{D}(L)$, defined by, e.g., the K-method, are denoted by $(X,\mathcal{D}(L))_{\gamma,p}$ where $0 < \gamma \le 1$ and $p \in [1,\infty] \cup \{\infty_0\}$, (where again $(X,\mathcal{D}(L))_{\gamma,\infty_0} = (X,\mathcal{D}(L))_\gamma$); see [7, Chap. 1.2] or the proof of Proposition 3 below.

For completeness we state (and prove) the following well-known facts:

Proposition 3. *Let X be a (complex) Banach space and let L be a nonnegative operator on X with domain $\mathcal{D}(L)$. Let the norm in $\mathcal{D}(L)$ be either $\|\underline{x}\|_{\mathcal{D}(L)} = \|L\underline{x}\| + \|\underline{x}\|$ or $\|\underline{x}\|_{\mathcal{D}(L)} = \|L\underline{x}\|$ (if L is invertible). Suppose that $\gamma \in (0,1)$ and $p \in [0,\infty] \cup \{\infty_0\}$. Then $\mathcal{D}_L(\gamma,p) = (X,\mathcal{D}(L))_{\gamma,p}$ and for each $x \in X$,*

$$\frac{1}{1+M(L,0)}[x]_{\mathcal{D}_L(\gamma,p)} \le \|x\|_{(X,\mathcal{D}(L))_{\gamma,p}}$$
$$\le 2[x]_{\mathcal{D}_L(\gamma,p)} + \begin{cases} 0, & \text{if } \|\underline{x}\|_{\mathcal{D}(L)} = \|L\underline{x}\|, \\ M(L,0)^{1-\gamma}\left(p\gamma(1-\gamma)\right)^{-\frac{1}{p}}\|x\|, & \text{if } \|\underline{x}\|_{\mathcal{D}(L)} = \|L\underline{x}\| + \|\underline{x}\|. \end{cases}$$

Next we state a theorem on the method of sums.

Theorem 4. *Let X be a (complex) Banach space and assume that*

(i) *A and B are two linear operators on X with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, respectively, and there are numbers α and β in the resolvent sets $\rho(A)$ and $\rho(B)$ of A and B, respectively, such that*

$$(A-\alpha I)^{-1}(B-\beta I)^{-1} = (B-\beta I)^{-1}(A-\alpha I)^{-1}.$$

(ii) *A and B are nonnegative operators on X and*

$$\phi_A + \phi_B > \pi.$$

(iii) *$0 \in \rho(A) \cup \rho(B)$, i.e., at least one of the operators A and B is invertible.*

Then the following statements hold true:

(a) *There is a bounded linear operator* $S : X \to X$ *such that*

$$\begin{aligned} S + BA^{-1}S = A^{-1} &\quad \text{if } A \text{ is invertible,} \\ AB^{-1}S + S = B^{-1} &\quad \text{if } B \text{ is invertible.} \end{aligned}$$

(b) *If* $y \in \mathcal{D}(A) \cap \mathcal{D}(B)$*, then* $S(Ay + By) = y$.

(c) *If* $Sx \in \mathcal{D}(A) \cup \mathcal{D}(B)$ *for some* $x \in X$*, then* $Sx \in \mathcal{D}(A) \cap \mathcal{D}(B)$ *and* $ASx + BSx = x$.

(d) *The operator* $A + B$ *with domain* $\mathcal{D}(A) \cap \mathcal{D}(B)$ *is closable in* X *and if* $\mathcal{D}(A) + \mathcal{D}(B)$ *is dense in* X*, then* $S = (\overline{A + B})^{-1}$.

(e) *If* $x \in \mathcal{D}_A(\gamma, p)$ *for some* $\gamma \in (0, 1)$ *and* $p \in [1, \infty] \cup \{\infty_0\}$*, then* $Sx \in \mathcal{D}(A) \cap \mathcal{D}(B)$*,* $ASx \in \mathcal{D}_A(\gamma, p) \cap \mathcal{D}_B(\gamma, p)$ *and* $BSx \in \mathcal{D}_A(\gamma, p)$*. Moreover*

$$\begin{aligned} [ASx]_{\mathcal{D}_A(\gamma,p)} &\le c_1 [x]_{\mathcal{D}_A(\gamma,p)}, \\ [BSx]_{\mathcal{D}_A(\gamma,p)} &\le (1 + c_1)[x]_{\mathcal{D}_A(\gamma,p)}, \\ [ASx]_{\mathcal{D}_B(\gamma,p)} &\le c_2 [x]_{\mathcal{D}_A(\gamma,p)}, \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{1}{\pi} M(B, \pi - \theta)\Big(1 + 2\sin(\tfrac{\theta}{2}) M(A, \theta)\Big) \int_0^\infty \frac{s^{\gamma-1}}{|s - \mathrm{e}^{\mathrm{i}\theta}|}\, \mathrm{d}s, \\ c_2 &= \frac{1}{\pi} M(B, \pi - \theta)\Big(1 + 2\sin(\tfrac{\theta}{2}) M(A, \theta)\Big) \int_0^\infty \frac{s^{\gamma-1}}{|s + \mathrm{e}^{\mathrm{i}\theta}|}\, \mathrm{d}s, \end{aligned} \tag{1}$$

and $\theta \in (\pi - \phi_B, \phi_A)$.

The statements (a)–(c) have, in the form stated here, previously been formulated in [2, Thm. 3.3 and Prop. 3.4]. Related results can be found in [4]–[6]. For (d) and for the claims $ASx \in \mathcal{D}_A(\gamma, p)$ and $BSx \in \mathcal{D}_A(\gamma, p)$ in (e), see [6, Thm. 2.7, p. 315], where however $\overline{\mathcal{D}(A)} = \overline{\mathcal{D}(B)} = X$ is assumed, and [4, Thm. 3.7, p. 324 and Thm. 3.11, p. 328]. In [3] a cross-regularity result ($ASx \in \mathcal{D}_B(\gamma, p)$) is proved for the case where both $-A$ and $-B$ generate bounded semigroups.

In the case where neither A nor B is invertible, we have the following result:

Corollary 5. *Let* X *be (complex) Banach space and suppose that assumptions* (i) *and* (ii) *of Theorem* 4 *hold true. If* $x \in \mathcal{D}_A(\gamma, p)$ *for some* $\gamma \in (0, 1)$ *and* $p \in [1, \infty] \cup \{\infty_0\}$ *and if* $y \in \mathcal{D}(A) \cap \mathcal{D}(B)$ *is a solution to the equation* $Ay + By = x$*, then* $Ay \in \mathcal{D}_A(\gamma, p) \cap \mathcal{D}_B(\gamma, p)$ *and* $By \in \mathcal{D}_A(\gamma, p)$*. Moreover*

$$\begin{aligned} [Ay]_{\mathcal{D}_A(\gamma,p)} &\le c_1 [x]_{\mathcal{D}_A(\gamma,p)}, \\ [By]_{\mathcal{D}_A(\gamma,p)} &\le (1 + c_1)[x]_{\mathcal{D}_A(\gamma,p)}, \\ [Ay]_{\mathcal{D}_B(\gamma,p)} &\le c_2 [x]_{\mathcal{D}_A(\gamma,p)}, \end{aligned}$$

where c_1 *and* c_2 *are as in* (1).

We shall repeatedly make use of the following lemma, and for completeness we give a proof below.

Lemma 6. *Let X be a (complex) Banach space and let assumption* (i) *of Theorem* 4 *hold true. Then*

(a) *If $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$, $Ax \in \mathcal{D}(B)$ and $Bx \in \mathcal{D}(A)$, then $ABx = BAx$.*

(b) *$A^m(A-\mu I)^{-1}B^n(B-\nu I)^{-1} = B^n(B-\nu I)^{-1}A^m(A-\mu I)^{-1}$ for all $\mu \in \rho(A)$ and $\nu \in \rho(B)$ and all m, $n \in \{0,1\}$.*

2 Proofs

Proof of Proposition 3. Recall that if X and Y are two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and if $Y \subset X$, then one defines $K(\tau,x) = \inf_{\substack{a+b=x\\ a\in X, b\in Y}} \big(\|a\|_X + \tau\|b\|_Y\big)$, where $x \in X$ and $\tau > 0$, and if $p \in [1,\infty]$, then $(X,Y)_{\gamma,p} \stackrel{\text{def}}{=} \{\, x \in X \mid \|x\|_{(X,Y)_{\gamma,p}} < \infty \,\}$ where

$$\|x\|_{(X,Y)_{\gamma,p}} \stackrel{\text{def}}{=} \begin{cases} \big(\int_0^\infty \big(\tau^{-\gamma}K(\tau,x)\big)^p \frac{d\tau}{\tau}\big)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup_{\tau>0} \tau^{-\gamma}K(\tau,x), & p = \infty. \end{cases}$$

Moreover, $(X,Y)_{\gamma,\infty_0} \stackrel{\text{def}}{=} \{\, x \in (X,Y)_{\gamma,\infty} \mid \lim_{\tau\downarrow 0} \tau^{-\gamma}K(\tau,x) = 0 \,\}$, with norm $\|\cdot\|_{(X,Y)_{\gamma,\infty_0}} = \|\cdot\|_{(X,Y)_{\gamma,\infty}}$.

First suppose that $x \in (X,\mathcal{D}(L))_{\gamma,p}$ and that $\tau > 0$. If $\epsilon > 0$ there are $a \in X$ and $b \in \mathcal{D}(L)$ such that $x = a+b$, $\|a\|_X \le (1+\epsilon)K(\tau,x)$ and $\tau\|Lb\|_X \le \tau\|b\|_{\mathcal{D}(L)} \le (1+\epsilon)K(\tau,x)$. If $t = \frac{1}{\tau}$ we get

$$\begin{aligned} &\|L(L+tI)^{-1}x\|_X \le \|L(L+tI)^{-1}a\|_X + \|(L+tI)^{-1}Lb\|_X \le \|a\|_X \\ &+ \|L(L+tI)^{-1}a\|_X + \|t(L+tI)^{-1}\tau Lb\|_X \le (1+M(L,0))(1+\epsilon)K(\tau,x). \end{aligned}$$

This inequality shows that $x \in \mathcal{D}_L(\gamma,p)$. Since $\epsilon > 0$ is arbitrary, a change of variables in the integral shows that $[x]_{\mathcal{D}_L(\gamma,p)} \le (1+M(L,0))\|x\|_{(x,\mathcal{D}(L))_{\gamma,p}}$.

Next suppose that $x \in \mathcal{D}_L(\gamma,p)$ and first assume that the norm in $\mathcal{D}(L)$ is $\|\underline{x}\|_{\mathcal{D}(L)} = \|L\underline{x}\|$. If $\tau > 0$ is given we take $t = \frac{1}{\tau}$, $b = t(L+tI)^{-1}x$ and $a = x - b$. Then

$$K(\tau,x) \le \|L(L+tI)^{-1}x\| + \frac{1}{t}\|t(L+tI)^{-1}Lx\| = 2\|L(L+tI)^{-1}x\|.$$

Thus we conclude that $x \in (X,\mathcal{D}(L))_{\gamma,p}$ and that $\|x\|_{(x,\mathcal{D}(L))_{\gamma,p}} \le 2[x]_{\mathcal{D}_L(\gamma,p)}$.

Finally we consider the case where the norm in $\mathcal{D}(L)$ is $\|\underline{x}\|_{\mathcal{D}(L)} = \|L\underline{x}\|_X + \|x\|_X$. By the same choice of a and b as above we get

$$\begin{aligned} K(\tau,x) &\le 2\|L(L+tI)^{-1}x\| + \tau\|t(L+tI)^{-1}x\| \\ &\le 2\|L(L+tI)^{-1}x\| + \tau M(L,0)\|x\|. \end{aligned}$$

Since $K(\tau,x) \le \|x\|$ we get $K(\tau,x) \le 2\|L(L+tI)^{-1}x\| + \min\{\tau M(L,0),1\}\|x\|$. This shows that $x \in (X,\mathcal{D}(L))_{\gamma,p}$ and a calculation gives $\|x\|_{(x,\mathcal{D}(L))_{\gamma,p}} \le 2[x]_{\mathcal{D}_L(\gamma,p)} + M(L,0)^{1-\gamma}(p\gamma(1-\gamma))^{-\frac{1}{p}}\|x\|$. □

Proof of Lemma 6. (a) First let us assume that A and B are invertible and $A^{-1}B^{-1} = B^{-1}A^{-1}$. Then

$$A^{-1}B^{-1}\big(ABx - BAx\big) = B^{-1}A^{-1}ABx - A^{-1}B^{-1}BAx = 0,$$

and we get the claim since $A^{-1}B^{-1}$ is an injection. Since $\mathcal{D}(A-\alpha I) = \mathcal{D}(A)$ and $\mathcal{D}(B-\beta I) = \mathcal{D}(B)$ we have

$$ABx - BAx = (A-\alpha I)(B-\beta I)x - (B-\beta I)(A-\alpha I)x,$$

and we can use the calculation above with A replaced by $A-\alpha I$ and B replaced by $B-\beta I$ to get the claim.

(b) We use case (a) and we have only to observe that $((A-\mu I)^{-1} - \frac{1}{\alpha-\mu}I)^{-1} = -(\alpha-\mu)^2(A-\alpha I)^{-1} - (\alpha-\mu)I$ and $((B-\nu I)^{-1} - \frac{1}{\beta-\nu}I)^{-1} = -(\beta-\nu)^2(B-\beta I)^{-1} - (\beta-\nu)I$ so that the assumptions of case (a) are satisfied with A replaced by $(A-\mu I)^{-1}$ and B replaced by $(B-\nu I)^{-1}$. Thus we get the desired claim when $m = n = 0$. If m or $n = 1$ we have only to use the facts that $A(A-\mu I)^{-1} = I + \mu(A-\mu I)^{-1}$ and $B(B-\nu I)^{-1} = I + \nu(B-\nu I)^{-1}$ and the case already proved. □

Proof of Theorem 4. Since $\phi_A + \phi_B > \pi$ we can choose a number $\theta \in (\pi - \phi_B, \phi_A)$. Let $r > 0$ and let γ_r be a path in $\mathbb{C}$ with range consisting of the rays $\rho e^{\pm i\theta}$ with $\rho \geq r$ and the part of the circle re^{it} with $|t| \leq \theta$ if B is invertible and $|\pi - t| \leq \pi - \theta$ if A is invertible. We can choose r so small that the range of γ_r lies in the intersection of the resolvent sets of $-A$ and B and we take the direction of γ_r to be such that the imaginary part increases on the rays.

Our choice of of θ implies that we have the following estimates for $|\arg(z)| = \theta$:

$$\begin{aligned} \|(A+zI)^{-1}\| &\leq |z|^{-1}M(A,\theta), \\ \|(B-zI)^{-1}\| &\leq |z|^{-1}M(B,\pi-\theta). \end{aligned} \tag{2}$$

Since $(A+zI)^{-1}$ and $(B-zI)^{-1}$ are continuous on the range of γ_r we see that if we define the operator S by

$$S = \frac{1}{2\pi i}\int_{\gamma_r}(A+zI)^{-1}(B-zI)^{-1}\,dz, \tag{3}$$

then the integral converges absolutely, and S is a well-defined bounded operator.

Suppose now that A is invertible. Because $A^{-1}(A+zI)^{-1} = \frac{1}{z}A^{-1} - \frac{1}{z}(A+zI)^{-1}$ we get

$$A^{-1}S = A^{-1}\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z}(B-zI)^{-1}\,dz - \frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z}(A+zI)^{-1}(B-zI)^{-1}\,dz.$$

By "closing" the curve γ_r through infinity with increasing argument we see by Cauchy's theorem that

$$\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{z}(B-zI)^{-1}\,dz = 0.$$

Hence we conclude that

$$A^{-1}S = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{1}{z}(A+zI)^{-1}(B-zI)^{-1}\,\mathrm{d}z.$$

Next we note that

$$BA^{-1}S = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{1}{z}B(B-zI)^{-1}(A+zI)^{-1}\,\mathrm{d}z,$$

where the fact that the integral on the right-hand side converges absolutely implies that $A^{-1}S$ maps X into $\mathcal{D}(B)$. Finally, because $\frac{1}{z}B(B-zI)^{-1} = (B-zI)^{-1} + \frac{1}{z}I$ we get

$$BA^{-1}S = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} (B-zI)^{-1}(A+zI)^{-1}\,\mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{1}{z}(A+zI)^{-1}\,\mathrm{d}z,$$

and by Cauchy's theorem, when we "close" the curve at infinity through decreasing argument, we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{1}{z}(A+zI)^{-1}\,\mathrm{d}z = -A^{-1}.$$

Thus we have obtained the formula

$$S + BA^{-1}S = A^{-1}, \tag{4}$$

which is what we wanted to prove. In order to treat the case where B is invertible it suffices to observe that interchanging A and B is equivalent to changing the variable in the integral defining S.

We proceed to the proof of (b). Because

$$\begin{aligned} (A+zI)^{-1} &= \frac{1}{z}\Big(I - A(A+zI)^{-1}\Big), \\ (B-zI)^{-1} &= \frac{1}{z}\Big(B(B-zI)^{-1} - I\Big), \end{aligned} \qquad z \in \rho(-A) \cap \rho(B).$$

we have by Lemma 6,

$$\begin{aligned} &(A+zI)^{-1}(B-zI)^{-1}(Ay+By) \\ &\quad = (B-zI)^{-1}(A+zI)^{-1}Ay + (A+zI)^{-1}(B-zI)^{-1}By \\ &\qquad = \frac{1}{z}B(B-zI)^{-1}A(A+zI)^{-1}y - \frac{1}{z}A(A+zI)^{-1}y \\ &\qquad\quad + \frac{1}{z}B(B-zI)^{-1}y - \frac{1}{z}A(A+zI)^{-1}B(B-zI)^{-1}y \\ &\qquad = \frac{1}{z}B(B-zI)^{-1}y - \frac{1}{z}A(A+zI)^{-1}y. \end{aligned}$$

By the definition of S we therefore get that

$$S(Ay+By) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{1}{z}(B-zI)^{-1}By \,\mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int_{\gamma_r} \frac{1}{z}(A+zI)^{-1}Ay \,\mathrm{d}z.$$

If, for example A is invertible, then we can complete the path γ_r at infinity with increasing argument and the first integral becomes 0 by Cauchy's theorem. In the second integral we complete the path γ_r at infinity with decreasing argument and the integral is seen to be $-y$ by Cauchy's formula. Thus we get $S(Ay+By) = y$ as claimed.

Next we prove claim (c) and again we may without loss of generality assume that A is invertible. First suppose that $Sx \in \mathcal{D}(A)$. We have by (4) and Lemma 6

$$\begin{aligned}(B+I)^{-1}BA^{-1}Sx &= (B+I)^{-1}A^{-1}x - (B+I)^{-1}A^{-1}ASx \\ &= A^{-1}(B+I)^{-1}\big(x - ASx\big).\end{aligned}$$

On the other hand we have, again by Lemma 6,

$$(B+I)^{-1}BA^{-1}Sx = \big(I-(B+I)^{-1}\big)A^{-1}Sx = A^{-1}\big(I-(B+I)^{-1}\big)Sx.$$

Combining the two previous results, we see because A^{-1} is an injection, that

$$Sx = (B+I)^{-1}\big(Sx + x - ASx\big).$$

It follows that $Sx \in \mathcal{D}(B)$.

Next suppose that $Sx \in \mathcal{D}(B)$. Since $(A^{-1}+I)^{-1} = A(A+I)^{-1} = I-(A+I)^{-1}$, we see that the assumptions of Lemma 6 are satisfied with A replaced by A^{-1}. Since $\mathcal{D}(A^{-1}) = X$ and $Sx \in \mathcal{D}(B)$ we therefore conclude that

$$BA^{-1}Sx = A^{-1}BSx,$$

and by (4) we then have

$$Sx = A^{-1}x - A^{-1}BSx,$$

and it follows that $Sx \in \mathcal{D}(A)$ and in addition that

$$ASx + BSx = x.$$

For the proof of claim (e) we no longer make the assumption that A is invertible, only that A or B is invertible. Since $x \in \mathcal{D}_A(\gamma,p)$ we know that $x \in \mathcal{D}_A(\gamma,\infty)$ which implies that

$$\sup_{t>0} t^{\gamma}\|A(A+tI)^{-1}x\| = [x]_{\mathcal{D}_A(\gamma,\infty)} < \infty. \tag{5}$$

Because

$$A(A+se^{\pm i\theta}I)^{-1} - A(A+sI)^{-1} = (e^{\mp i\theta}-1)se^{\pm i\theta}(A+se^{\pm i\theta}I)^{-1}A(A+sI)^{-1},$$

we have

$$\|A(A+zI)^{-1}x\| \le \Big(1+2\sin(\tfrac{\theta}{2})M(A,\theta)\Big)\|A(A+|z|I)^{-1}x\|, \quad |\arg(z)|=\theta. \tag{6}$$

An immediate consequence is that $Sx \in \mathcal{D}(A)$ with

$$ASx = \frac{1}{2\pi i}\int_{\gamma_r} A(A+zI)^{-1}(B-zI)^{-1}x\,dz \tag{7}$$

because the integral converges absolutely by Lemma 6, (5) and (6). By claim (c) we know that $Sx \in \mathcal{D}(B)$ as well.

Now let $t > r$ be arbitrary. Because

$$A(A+tI)^{-1}A(A+zI)^{-1} = \frac{t}{t-z}A(A+tI)^{-1} - \frac{z}{t-z}A(A+zI)^{-1}, \tag{8}$$

we have by (7)

$$\begin{aligned}A(A+tI)^{-1}ASx = A(A+tI)^{-1}\frac{1}{2\pi i}\int_{\gamma_r}\frac{t}{t-z}(B-zI)^{-1}x\,dz\\ -\frac{1}{2\pi i}\int_{\gamma_r}\frac{z}{t-z}A(A+zI)^{-1}(B-zI)^{-1}x\,dz.\end{aligned}$$

When we "close" the path γ_r at infinity by increasing argument, we see that the first integral is 0 by Cauchy's theorem and we get from Lemma 6 that

$$A(A+tI)^{-1}ASx = -\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{t-z}z(B-zI)^{-1}A(A+zI)^{-1}x\,dz. \tag{9}$$

In this integral we may let $r \downarrow 0$ without changing the value of the integral, because the function we integrate is analytic and the integral over a part of the circle with radius r goes to 0 by the assumption that $\pi - \phi_B < \theta < \phi_A$, the definition of γ_r and by the assumption that A or B is invertible.

Thus we have by (5), (6), and (9)

$$\begin{aligned}t^\gamma\|A(A+tI)^{-1}ASx\| &\le c_3\int_0^\infty \frac{t^\gamma}{|t-se^{i\theta}|}\|A(A+sI)^{-1}x\|\,ds\\ &= c_3\int_0^\infty\frac{(\frac{t}{s})^\gamma}{|\frac{t}{s}-e^{i\theta}|}s^\gamma\|A(A+sI)^{-1}x\|\,\frac{ds}{s},\end{aligned} \tag{10}$$

where

$$c_3 = \frac{1}{\pi}M(B,\pi-\theta)\Big(1+2\sin(\tfrac{\theta}{2})M(A,\theta)\Big). \tag{11}$$

Let $f(\tau) \stackrel{\text{def}}{=} \mathrm{e}^{\tau\gamma}\|A(A+\mathrm{e}^{\tau}I)^{-1}x\|$, $g(\tau) \stackrel{\text{def}}{=} \mathrm{e}^{\tau\gamma}\|A(A+\mathrm{e}^{\tau}I)^{-1}ASx\|$, and $h(\tau) \stackrel{\text{def}}{=} \mathrm{e}^{\tau\gamma}/|\mathrm{e}^{\tau} - \mathrm{e}^{\mathrm{i}\theta}|$ where $\tau \in \mathbb{R}$. By changing variables ($s = \mathrm{e}^{\sigma}$) in the integral in (10) we conclude that

$$g(\tau) \leq c_3 \int_{-\infty}^{\infty} h(\tau - \sigma) f(\sigma)\,\mathrm{d}\sigma. \tag{12}$$

Since convolution with an integrable function is a bounded mapping from $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, into itself and because a change of variable shows that $\|f\|_{L^p(\mathbb{R})} = [x]_{\mathcal{D}_A(\gamma,p)}$ and $\|g\|_{L^p(\mathbb{R})} = [ASx]_{\mathcal{D}_A(\gamma,p)}$, we conclude after another change of variables that

$$[ASx]_{\mathcal{D}_A(\gamma,p)} \leq c_3 \int_0^{\infty} \frac{s^{\gamma-1}}{|s - \mathrm{e}^{\mathrm{i}\theta}|}\,\mathrm{d}s [x]_{\mathcal{D}_A(\gamma,p)}.$$

Because convolution with an integrable function is a bounded mapping from the space of bounded functions converging to 0 at $+\infty$ into itself, the claim for the case $p = \infty_0$ follows as well.

Since $x \in \mathcal{D}_A(\gamma, p)$ and $BSx = x - ASx$ we see that $BSx \in \mathcal{D}_A(\gamma, p)$.

Finally we observe that if we instead of (8) use the equation

$$B(B + tI)^{-1}(B - zI)^{-1} = \frac{t}{t+z}(B + tI)^{-1} + \frac{z}{t+z}(B - zI)^{-1},$$

in (7), then we get

$$\begin{aligned} B(B + tI)^{-1}ASx = (B + tI)^{-1}\frac{1}{2\pi\mathrm{i}}&\int_{\gamma_r} \frac{t}{t+z}A(A + zI)^{-1}x\,\mathrm{d}z \\ &+ \frac{1}{2\pi\mathrm{i}}\int_{\gamma_r} \frac{z}{t+z}A(A + zI)^{-1}(B - zI)^{-1}x\,\mathrm{d}z. \end{aligned}$$

When we "close" the path at infinity with decreasing argument and use the fact that $t > r$, we see that the first integral is 0 and we conclude that we have instead of (9)

$$B(B + tI)^{-1}ASx = \frac{1}{2\pi\mathrm{i}}\int_{\gamma_r} \frac{1}{t+z}z(B - zI)^{-1}A(A + zI)^{-1}x\,\mathrm{d}z.$$

We see that the right-hand side of this equation only differs from the right-hand side of (9) by two minus signs and it follows that we get

$$t^{\gamma}\|B(B + tI)^{-1}ASx\| \leq c_3 \int_0^{\infty} \frac{(\frac{t}{s})^{\gamma}}{|\frac{t}{s} + \mathrm{e}^{\mathrm{i}\theta}|} s^{\gamma}\|A(A + sI)^{-1}x\|\,\frac{\mathrm{d}s}{s}.$$

Proceeding in the same way as above we conclude that

$$[ASx]_{\mathcal{D}_B(\gamma,p)} \leq c_3 \int_0^{\infty} \frac{s^{\gamma-1}}{|s + \mathrm{e}^{\mathrm{i}\theta}|}\,\mathrm{d}s [x]_{\mathcal{D}_A(\gamma,p)}.$$

It is also clear that if $x \in \mathcal{D}_A(\gamma, \infty_0)$ then $ASx \in \mathcal{D}_B(\gamma, \infty_0)$.

Finally we prove (d). First suppose that $\{y_n\}_{n=1}^{\infty} \subset \mathcal{D}(A) \cap \mathcal{D}(B)$ is such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty}(Ay_n + By_n) = x$. Then it follows from (b) and the continuity of S that $Sx = y$. If $y = 0$ it follows from (c) that $x = ASx + BSx = 0$ and we conclude that $A + B$ is closable. The general case (where we do not assume that $y = 0$) implies that $S(\overline{A+B})y = y$ for $y \in \mathcal{D}(\overline{A+B})$.

If $\mathcal{D}(A) + \mathcal{D}(B)$ is dense in X and $x \in X$ then there are sequences $\{a_n\}_{n=1}^{\infty} \subset \mathcal{D}(A)$ and $\{b_n\}_{n=1}^{\infty} \subset \mathcal{D}(B)$ such that $\lim_{n\to\infty}(a_n + b_n) = x$. Because clearly $\mathcal{D}(A) \subset \mathcal{D}_A(\frac{1}{2}, \infty)$ and $\mathcal{D}(B) \subset \mathcal{D}_B(\frac{1}{2}, \infty)$ we know by (c) and (e) (where we also interchange A and B) that Sa_n and $Sb_n \in \mathcal{D}(A+B)$ and $(A+B)S(a_n + b_n) = a_n + b_n \to x$ as $n \to \infty$. Because S is continuous we have $\lim_{n\to\infty} S(a_n + b_n) = Sx$, and so $(\overline{A+B})Sx = x$. □

Proof of Corollary 5. Let $\epsilon > 0$ be arbitrary and define $B_\epsilon = B + \epsilon I$. Since B_ϵ is invertible, we can apply Theorem 4, (and we can choose θ independent of ϵ). Let S_ϵ be the operator that exists according to Theorem 4.(a). Since $Ay + By + \epsilon y = x + \epsilon y$ we see from Theorem 4.(b) that $y = S_\epsilon(x + \epsilon y)$. Thus we conclude by Theorem 4.(e) that $Ay \in \mathcal{D}_A(\gamma, p)$ with

$$[Ay]_{\mathcal{D}_A(\gamma,p)} \leq c_4 \frac{1}{\pi} M(B_\epsilon, \pi - \theta)\Big(1 + 2\sin(\tfrac{\theta}{2})M(A,\theta)\Big)[x + \epsilon y]_{\mathcal{D}_A(\gamma,p)},$$

where $c_4 = \int_0^\infty \frac{s^{\gamma-1}}{|s-\mathrm{e}^{\mathrm{i}\theta}|}\,\mathrm{d}s$. Since $y \in \mathcal{D}(A)$ we have $y \in \mathcal{D}_A(\gamma, p)$ and $[\epsilon y]_{\mathcal{D}_A(\gamma,p)} \to 0$ when $\epsilon \downarrow 0$. It is also clear that $\lim_{\epsilon\downarrow 0} M(B_\epsilon, \pi-\theta) = M(B, \pi-\theta)$ and we get the desired inequality for $[Ay]_{\mathcal{D}_A(\gamma,p)}$. Since $By = x - Ay$ we get the claim about By as well.

By Theorem 4 we also know that $Ay \in \mathcal{D}_{B_\epsilon}(\theta, p)$ and

$$[Ay]_{\mathcal{D}_{B_\epsilon}(\gamma,p)} \leq c_5 \frac{1}{\pi} M(B_\epsilon, \pi - \theta)\Big(1 + 2\sin(\tfrac{\theta}{2})M(A,\theta)\Big)[x + \epsilon y]_{\mathcal{D}_A(\gamma,p)},$$

where $c_5 = \int_0^\infty \frac{s^{\gamma-1}}{|s+\mathrm{e}^{\mathrm{i}\theta}|}\,\mathrm{d}s$. Since $\mathcal{D}(B) = \mathcal{D}(B_\epsilon)$ we have $\mathcal{D}_B(\theta, p) = \mathcal{D}_{B_\epsilon}(\theta, p)$ by Proposition 3 (since the interpolation space does not depend on the choice of norms), and because $B_\epsilon(B_\epsilon + tI)^{-1} - B(B+tI)^{-1} = \epsilon t(B + (t+\epsilon)I)^{-1}(B + tI)^{-1}$ we get

$$\Big|[x]_{\mathcal{D}_{B_\epsilon}(\gamma,p)} - [x]_{\mathcal{D}_B(\gamma,p)}\Big| \leq \epsilon^\gamma M(B,0)^2 [x]_{\mathcal{D}_I(\gamma,p)},$$

and we see that $\lim_{\epsilon\downarrow 0}[Ay]_{\mathcal{D}_{B_\epsilon}(\gamma,p)} = [Ay]_{\mathcal{D}_B(\gamma,p)}$. This completes the proof.
□

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