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GROWTH OF RESOLVENTS OF CERTAIN INFINITE MATRICES

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Abstract: *The convergence of iterative methods with meromorphic resolvents is related to the growth of the resolvent of the iteration operator. Here we analyze the growth of the resolvent for a set of nonnormal operators in a separable Hilbert space. These operators are written as infinite matrices. The aim is to gain insight on the behavior of iterative methods associated with these infinite matrices and their submatrices.*

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1 Introduction

Consider the fixed point problem $x = Ax + b$, where A is a bounded linear operator in a separable complex Hilbert space H. This problem can be solved iteratively by applying some Krylov space method, e.g. GMRES. In [7] it is shown that for operators with meromorphic resolvents the convergence of GMRES is related to the growth of the resolvent of A. The growth of the resolvent $(1 - zA)^{-1}$ is measured by a function

$$T_\infty(r, (1 - zA)^{-1}),$$

which is the straight forward generalization [9] of the Nevanlinna characteristic function [13], [14], used to measure the growth of scalar valued meromorphic functions. The function $T_\infty(r, (1 - zA)^{-1})$ consists of two parts:

$$T_\infty(r, (1 - zA)^{-1}) = N_\infty(r, (1 - zA)^{-1}) + m_\infty(r, (1 - zA)^{-1}),$$

where $N_\infty(r, (1 - zA)^{-1})$ is determined by the spectrum of A alone, while $m_\infty(r, (1 - zA)^{-1})$ contains the nonspectral information. For bounded, self-adjoint A with a meromorphic resolvent it has been shown in [9] that

$$m_\infty(r, (1 - zA)^{-1}) \leq \log 2,$$

in which case

$$T_\infty(r, (1 - zA)^{-1}) = N_\infty(r, (1 - zA)^{-1}) + \mathcal{O}(1).$$

On the other hand, it is not difficult to give examples of operators such that

$$N_\infty(r, (1 - zA)^{-1}) = 0$$

and

$$T_\infty(r, (1 - zA)^{-1}) = m_\infty(r, (1 - zA)^{-1}),$$

see e.g. [4] and Section 6 in this paper. To gain further insight as to how $T_\infty(r, (1-zA)^{-1})$ grows for different A and how this growth is divided between $N_\infty(r, (1 - zA)^{-1})$ and $m_\infty(r, (1 - zA)^{-1})$, we have studied here a number of examples and answered this question in these cases.

This work is motivated by the desire to understand the convergence behavior of iterative methods. In practice the problems that are solved are finite dimensional. However, the number of iteration steps we can afford to take is small compared to the size of the problem. So though the iteration always terminates in a number of steps less or equal to the dimension of the problem, this is something we do not expect to witness in practice. Therefore it is reasonable to discuss the more general case of infinite matrices. This way we get a better picture of the behavior of iterative methods. In particular we get a clear picture of the asymptotic behavior of the iteration error, as termination at the point where the iteration step equals the dimension of the problem does not occur.

In Section 2 we introduce the generalization $T_\infty(r, F)$ of the Nevanlinna characteristic function and state some of its properties. In Section 3 we formulate a class of model problems, for which we estimate $N_\infty(r,(1-zA)^{-1})$ in Section 4 and $m_\infty(r,(1-zA)^{-1})$ in Section 5. We discuss an example which does not fit our class of model problems in Section 6. Finally, in Section 7 we present some numerical experiments. The results of this work are summarized in Section 8.

2 The Nevanlinna characteristic function

Consider now the operator valued function

$$F : z \to F(z)$$

in a separable Hilbert space H with the norm $\|\cdot\|$.

Definition 1 *A function $F(z)$ is called meromorphic for $|z| < R \leq \infty$ if around each z_0 in $|z_0| < R \leq \infty$ it has a representation of the form*

$$F(z) = \sum_{k=-h}^{\infty} F_k (z - z_0)^k. \tag{1}$$

Here F_k is a bounded linear operator in H and F_{-h} is nontrivial. If $-h < 0$, then F has a pole at z_0 of order h, otherwise F is analytic at z_0. If F is analytic in the whole complex plane, it is said to be entire.

Rolf Nevanlinna [13], [14], introduced in 1925 a characteristic function $T(r, f)$ to measure the growth of meromorphic functions. This can be generalized to operator valued functions. The obvious way is to replace the absolute value by the norm of the operator valued function. This leads to a generalization denoted here by $T_\infty(r, F)$ [11], [12], [9], [7], defined as follows:

Definition 2 *Let $F(z)$ be a meromorphic operator valued function as above. Denote $h(z_0) := \max\{h, 0\}$ and define*

$$n_\infty(r, F) := \sum_{|b| \leq r} h(b).$$

Thus n_∞ counts the poles in $\{z \mid |z| < r\}$ together with their orders. Furthermore define

$$N_\infty(r, F) := \int_0^r \frac{n_\infty(t, F) - n_\infty(0, F)}{t} dt + n_\infty(0, F) \log r$$

and

$$m_\infty(r, F) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \|F(re^{i\varphi})\| d\varphi.$$

Finally define

$$T_\infty(r, F) = m_\infty(r, F) + N_\infty(r, F).$$

If F is an entire function, then the growth of F can also be measured by looking at the maximum modulus

$$M_\infty(r, F) := \sup_{|z|=r} \|F(z)\|.$$

When no confusion arises, we write $T_\infty(r)$, $m_\infty(r)$, $N_\infty(r)$ and $M_\infty(r)$ for $T_\infty(r, F)$, $m_\infty(r, F)$, $N_\infty(r, F)$ and $M_\infty(r, F)$ respectively.

Definition 3 *Let $F(z)$ be a meromorphic operator valued function for $|z| < \infty$. The function F is of Nevanlinna order ω and (if $0 < \omega < \infty$) of Nevanlinna type τ where*

$$\omega := \limsup_{r\to\infty} \frac{\log T_\infty(r, F)}{\log r}$$

$$\tau := \limsup_{r\to\infty} r^{-\omega} T_\infty(r, F).$$

Similarly, let $F(z)$ be an operator valued entire function. Then $F(z)$ is of order ω and (if the order is positive) type τ, where the order and type are defined as follows:

$$\omega := \limsup_{r\to\infty} \frac{\log\log M_\infty(r, F)}{\log r}$$

$$\tau := \limsup_{r\to\infty} r^{-\omega} \log M_\infty(r, F).$$

With a slight abuse of language we say that in the above case $T_\infty(r, F)$ grows with order ω_{mer} and type τ_{mer}, and $M_\infty(r, F)$ grows with order ω and type τ. Furthermore, we sometimes wish to distinguish between the growth of $N_\infty(r, F)$ and $m_\infty(r, F)$. Then we say that $N_\infty(r, F)$ grows with order ω_N and type τ_N, where

$$\omega_N := \limsup_{r\to\infty} \frac{\log N_\infty(r, F)}{\log r}$$

$$\tau_N := \limsup_{r\to\infty} r^{-\omega_N} N_\infty(r, F),$$

and $m_\infty(r, F)$ grows with order ω_m and type τ_m, where ω_m and τ_m are obtained by replacing N by m in the expressions above.

For an entire function the growth of $T_\infty(r, F)$ and $M_\infty(r, F)$ are related in the following way [7], [15, pp. 181–182].

Theorem 1 *If $F(z)$ is an operator valued entire function, then its orders as an entire and a meromorphic function are equal: $\omega_{mer} = \omega$. Furthermore the types satisfy the following inequality:*

$$\tau_{mer} \le \tau \le (2\omega + 1)\mathrm{e}\tau_{mer}.$$

Moreover, the order and type of an entire function can be read from the decay of the coefficients of its power series representation:

Theorem 2 *Assume $F(z) = \sum_{k=0}^{\infty} F_k z^k$ is entire of order ω. Then*

$$\omega = \limsup_{k\to\infty} \frac{k \log k}{\log \frac{1}{\|F_k\|}}.$$

If $F(z)$ is of finite positive order ω and of finite type τ, then

$$\tau = \frac{1}{e\omega} \limsup_{k\to\infty} k\|F_k\|^{\omega/k}.$$

For a proof modify that of the scalar case in e.g. [1, pp.9–12].

Finally, it is easy to show that the following holds:

Lemma 1 *Let F be an entire function such that $F(z) = G(|z|)$ for some G. Then*

$$m_\infty(r, F) = \log^+ M(r, F).$$

3 Problem formulation

Let A be a bounded linear operator in a separable Hilbert space H. Then A admits a matrix representation. Assume there is a bounded S with a bounded inverse S^{-1} such that $B = SAS^{-1}$. Then

$$(1 - zA)^{-1} = S^{-1}(1 - zB)^{-1}S,$$

and it is easy to see that the growth of the resolvents of A and B are related in the following way:

Theorem 3 *Assume there is a bounded S with a bounded inverse S^{-1} such that $B = SAS^{-1}$. Then*

$$\frac{1}{\kappa(S)}\|(1 - zB)^{-1}\| \le \|(1 - zA)^{-1}\| \le \kappa(S)\|(1 - zB)^{-1}\|. \tag{2}$$

Furthermore,

$$N_\infty(r, (1 - zA)^{-1}) = N_\infty(r, (1 - zB)^{-1}),$$

while

$$m_\infty(r, (1-zB)^{-1}) - \log^+ \kappa(S) \le m_\infty(r, (1-zA)^{-1}) \le m_\infty(r, (1-zB)^{-1}) + \log^+ \kappa(S).$$

Let us introduce the following model classes, which are motivated by the Jordan canonical forms of matrices. Note, that for infinite matrices not all operators can be transformed into an operator belonging to one of the model classes, which, nevertheless, provide interesting examples to study.

Assume first that A is diagonalizable in the sense, that there is a bounded S with a bounded inverse S^{-1} such that $SAS^{-1} = \Lambda$, where Λ is a diagonal operator the diagonal elements of which are the eigenvalues λ_j of A. Then

$$\|(1 - z\Lambda)^{-1}\| = \frac{1}{\inf_j |1 - z\lambda_j|},$$

so

$$\frac{1}{\kappa(S)} \frac{1}{\inf_j |1 - z\lambda_j|} \leq \|(1 - zA)^{-1}\| \leq \kappa(S) \frac{1}{\inf_j |1 - z\lambda_j|}. \tag{3}$$

Assume now that there is a bounded S with a bounded inverse S^{-1} such that $SAS^{-1} = J$, where $J = diag(J_j)$ and J_j are Jordan blocks, each of dimension $\dim(J_j) = d_j$. We have

$$\frac{1}{\kappa(S)} \|(1 - zJ)^{-1}\| \leq \|(1 - zA)^{-1}\| \leq \kappa(S) \|(1 - zJ)^{-1}\|, \tag{4}$$

where

$$\|(1 - zJ)^{-1}\| = \sup_j \|(1 - zJ_j)^{-1}\|. \tag{5}$$

For all j

$$(1 - zJ_j)^{-1} = \frac{1}{z} \begin{pmatrix} \frac{-z}{1-z\lambda_j} & \cdots & (\frac{-z}{1-z\lambda_j})^{d_j} \\ & \ddots & \vdots \\ & & \frac{-z}{1-z\lambda_j} \end{pmatrix}$$

is a Toeplitz matrix so we know that

$$\frac{1}{|z|} \sqrt{\sum_{k=1}^{d_j} |\frac{z}{1 - z\lambda_j}|^{2k}} \leq \|(1 - zJ_j)^{-1}\| \leq \frac{1}{|z|} \sum_{k=1}^{d_j} |\frac{z}{1 - z\lambda_j}|^k. \tag{6}$$

Now if the dimensions of the Jordan blocks are not bounded, i.e. there exists no d_{max} such that $\sup_j d_j \leq d_{max}$, then the resolvent of J is not meromorphic in the whole complex plane, see Section 5.2.1. It is natural to look at a slightly different model class. Namely, let $\tilde{J} = diag(\tilde{J}_j)$, where $\dim(\tilde{J}_j) = d_j$ and $\tilde{J}_j$ is defined by

$$\tilde{J}_j = \begin{pmatrix} \lambda_j & \varepsilon_j & & \\ & \ddots & \ddots & \\ & & & \varepsilon_j \\ & & & \lambda_j \end{pmatrix} = \varepsilon_j \begin{pmatrix} \hat{\lambda}_j & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \hat{\lambda}_j \end{pmatrix} = \varepsilon_j \hat{J}_j, \tag{7}$$

where $\hat{\lambda}_j = \frac{\lambda_j}{\varepsilon_j}$. So

$$(1 - z\tilde{J}_j)^{-1} = (1 - \varepsilon_j z \hat{J}_j)^{-1} = (1 - \hat{z}_j \hat{J}_j)^{-1},$$

where $\hat{z}_j = \varepsilon_j z$. Now

$$\frac{1}{|\hat{z}_j|}\sqrt{\sum_{k=1}^{d_j} |\frac{\hat{z}_j}{1-\hat{z}_j\hat{\lambda}_j}|^{2k}} \leq \|(1-z\tilde{J}_j)^{-1}\| \leq \frac{1}{|\hat{z}_j|}\sum_{k=1}^{d_j} |\frac{\hat{z}_j}{1-\hat{z}_j\hat{\lambda}_j}|^{k}. \tag{8}$$

Furthermore note that

$$\sum_{k=1}^{d_j} |\frac{\hat{z}_j}{1-\hat{z}_j\hat{\lambda}_j}|^{k} = |\frac{\hat{z}_j}{1-\hat{z}_j\hat{\lambda}_j}| \frac{1-|\frac{\hat{z}_j}{1-\hat{z}_j\hat{\lambda}_j}|^{d_j}}{1-|\frac{\hat{z}_j}{1-\hat{z}_j\hat{\lambda}_j}|}. \tag{9}$$

We assume in Sections 4 and 5 that there exists a bounded S with a bounded inverse S^{-1} such that $A = S^{-1}\tilde{J}S$, where $\tilde{J}$ is as above. In practice it is not realistic to assume this form to be known. For analytical purposes, however, this is a enlightening form to look at. The choice of the ε_j in $\tilde{J}$ depends on the choice of S, which of course should be made so that $\kappa(S)$ is as small as possible. Consider the following example.

Example 1 Assume

$$A = S^{-1}JS, \quad J = diag(J_j)$$

where J_j are the Jordan blocks, $\dim J_j = d_j$. Fix a sequence $\{\varepsilon_j\}$. Then

$$J_j = T_j^{-1}\tilde{J}_j T_j,$$

where $\tilde{J}_j$ are as in (7), $\dim\tilde{J}_j = d_j$, and

$$T_j = diag(\varepsilon_j^{d_j-1}, \varepsilon_j^{d_j-2}, \ldots, 1).$$

Now

$$A = (TS)^{-1}\tilde{J}(TS),$$

where $\tilde{J} = diag(\tilde{J}_j)$ and $T = diag(T_j)$. In this case the sequence $\{\varepsilon_j\}$ should be chosen so that $\kappa(TS)$ is as small as possible.

4 Estimating $N_\infty(r, (1-zA)^{-1})$

Let A be an operator, for which there exists a bounded S with a bounded inverse S^{-1} such that

$$A = S^{-1}\tilde{J}S,$$

where $\tilde{J} = diag(\tilde{J}_j)$, $\dim\tilde{J}_j = d_j$ and $\tilde{J}_j$ is given by (7). Let the eigenvalues of A satisfy

$$\lambda_n = (\frac{\tilde{\tau}\tilde{\omega}}{n})^{1/\tilde{\omega}}.$$

The pole $z_n = \frac{1}{\lambda_n}$ of the resolvent is inside the disk $|z| < r$ when

$$|z_n| = (\frac{n}{\tilde{\tau}\tilde{\omega}})^{1/\tilde{\omega}} < r,$$

that is, $n < \tilde{\tau}\tilde{\omega}r^{\tilde{\omega}}$. So the number of separate poles in the disk $|z| < r$ is $\lfloor \tilde{\tau}\tilde{\omega}r^{\tilde{\omega}} \rfloor$.

Assume that the dimension of the j^{th} block $\tilde{J}_j$ satisfies

$$d_j = aj^b, \quad a, b \geq 0,$$

then

$$n_\infty(r, (1 - zA)^{-1}) = \sum_{j=1}^{n} d_j \approx \int_0^{\tilde{\tau}\tilde{\omega}r^{\tilde{\omega}}} aj^b \mathrm{d}j = \frac{a}{b+1}(\tilde{\tau}\tilde{\omega}r^{\tilde{\omega}})^{b+1}$$

and

$$N_\infty(r, (1 - zA)^{-1}) \approx \frac{a(\tilde{\tau}\tilde{\omega})^{b+1}}{\tilde{\omega}(b+1)^2} r^{\tilde{\omega}(b+1)}.$$

If $T_\infty(r, (1 - zA)^{-1})$ is of finite order, then we must have $\tilde{\omega} < \infty$ and $b < \infty$, for the order of $T_\infty(r, (1 - zA)^{-1})$ is greater or equal to that of $N_\infty(r, (1 - zA)^{-1})$.

In fact we have the following.

Theorem 4 *Let the eigenvalues of A satisfy*

$$\alpha_1 j^{-\beta_1} \leq \lambda_j \leq \alpha_2 j^{-\beta_2}$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ such that $\beta_1 \leq \beta_2$ and if $\beta_1 = \beta_2$ then $\alpha_1 \leq \alpha_2$. Furthermore, let the dimension d_j of each block $\tilde{J}_j$ satisfy

$$a_1 j^{b_1} \leq d_j \leq a_2 j^{b_2}$$

for some $a_1, a_2, b_1, b_2 \geq 0$ such that $b_1 \leq b_2$ and if $b_1 = b_2$ then $a_1 \leq a_2$. Then $N_\infty(r, (1 - zA)^{-1})$ grows with order ω and type τ, where

$$\frac{b_1 + 1}{\beta_2} \leq \omega \leq \frac{b_2 + 1}{\beta_1}$$

and

$$\frac{a_1\beta_2}{(b_1 + 1)^2}\alpha_2^{\frac{b_1+1}{\beta_2}} \leq \tau \leq \frac{a_2\beta_1}{(b_2 + 1)^2}\alpha_1^{\frac{b_2+1}{\beta_1}}.$$

Remark 1 Note that

$$\lambda_j = (\frac{\tilde{\tau}\tilde{\omega}}{j})^{1/\tilde{\omega}} = \alpha j^{-\beta}$$

where $\alpha = (\tilde{\tau}\tilde{\omega})^{\frac{1}{\tilde{\omega}}}$ and $\beta = 1/\tilde{\omega}$. Throughout this text we use both representations.

The growth of $N_\infty(r, (1 - zA)^{-1})$ is easily determined in the above cases. But what can we say about that of $m_\infty(r, (1 - zA)^{-1})$?

5 Estimating $m_\infty(r, (1 - zA)^{-1})$

5.1 Dimensions of Jordan blocks are bounded

Assume that the dimensions of the Jordan blocks are bounded, so that $d_{max} := \sup_j d_j < \infty$. Then by (6)

$$\max_{1 \le k \le d_j} \frac{1}{|z|} |\frac{z}{1 - z\lambda_j}|^k \le \|(1 - zJ_j)^{-1}\| \le \max_{1 \le k \le d_j} \frac{d_j}{|z|} |\frac{z}{1 - z\lambda_j}|^k$$

so by (5)

$$\sup_j |\frac{1}{1 - z\lambda_j}| \le \|(1 - zJ)^{-1}\| \le \sup_j \frac{d_{max}}{|z|} \max_{1 \le k \le d_j} |\frac{z}{1 - z\lambda_j}|^k.$$

Note that as long as $|\frac{z}{1-z\lambda_j}| > 1$ it is true that

$$\max_{1 \le k \le d_j} |\frac{z}{1 - z\lambda_j}|^k = |\frac{z}{1 - z\lambda_j}|^{d_j} \le |\frac{z}{1 - z\lambda_j}|^{d_{max}}.$$

On the other hand if $|\frac{z}{1-z\lambda_j}| \le 1$ then

$$\log^+ \max_{1 \le k \le d_j} |\frac{z}{1 - z\lambda_j}|^k = \log^+ |\frac{z}{1 - z\lambda_j}| = \log^+ |\frac{z}{1 - z\lambda_j}|^{d_{max}} = 0.$$

So for all z

$$\log^+ \max_{1 \le k \le d_j} |\frac{z}{1 - z\lambda_j}|^k = \log^+ |\frac{z}{1 - z\lambda_j}|^{d_j} \le \log^+ |\frac{z}{1 - z\lambda_j}|^{d_{max}}.$$

Thus, for $|z| > d_{max}$ we have

$$\sup_j \log^+ |\frac{1}{1 - z\lambda_j}| \le \log^+ \|(1 - zJ)^{-1}\| \le \sup_j \log^+ |\frac{z}{1 - z\lambda_j}|^{d_{max}}$$

and

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \|(1 - re^{i\varphi} J)^{-1}\| d\varphi &\le \frac{d_{max}}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{r}{\inf_j |1 - re^{i\varphi}\lambda_j|} d\varphi \\ &\le \frac{d_{max}}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{1}{\inf_j |1 - re^{i\varphi}\lambda_j|} d\varphi + d_{max} \log r. \end{aligned}$$

So we have the following.

Theorem 5 *Assume that A can be transformed into Jordan form $J = diag(J_j)$, where J_j are the Jordan blocks, each of dimension $\dim(J_j) = d_j$, and $d_{max} := \sup_j d_j < \infty$. Then for $|z| > d_{max}$*

$$m_\infty(r, (1 - zA)^{-1}) \le \frac{d_{max}}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{1}{\inf_j |1 - re^{i\varphi}\lambda_j|} d\varphi + d_{max} \log r + \log^+ \kappa(S)$$

and

$$m_\infty(r, (1 - zA)^{-1}) \ge \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \frac{1}{\inf_j |1 - re^{i\varphi}\lambda_j|} d\varphi - \log^+ \kappa(S).$$

Our problem is reduced to finding estimates for

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}\frac{1}{\inf_j|1-re^{i\varphi}\lambda_j|}\mathrm{d}\varphi.$$

We do this for two special cases.

Case 5.1.1. If the spectrum lies on the real axis then

$$\inf_j|1-re^{i\varphi}\lambda_j|\geq\inf_{\lambda\in\mathbb{R}}|1-re^{i\varphi}\lambda|=|\sin\varphi|$$

so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}\frac{1}{\inf_j|1-re^{i\varphi}\lambda_j|}\mathrm{d}\varphi\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\frac{1}{|\sin\varphi|}\mathrm{d}\varphi=\log 2$$

and

$$m_\infty(r,(1-zA)^{-1})\leq d_{max}(\log 2+\log r)+\log^{+}\kappa(S).$$

Case 5.1.2. If the spectrum lies on the lines $l_k=\{z\mid z=re^{i\theta_k},\ r\in\mathbb{R}\}$, $k=1,\ldots,K$, then

$$\sup_j\frac{1}{|1-re^{i\varphi}\lambda_j|}\leq\sum_k\sup_{\lambda\in l_k}\frac{1}{|1-re^{i\varphi}\lambda|}=\sum_k\frac{1}{|\sin(\varphi+\theta_k)|}$$

so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}\frac{1}{\inf_j|1-re^{i\varphi}\lambda_j|}\mathrm{d}\varphi\leq\sum_k\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\frac{1}{|\sin(\varphi+\theta_k)|}\mathrm{d}\varphi=K\log 2$$

and

$$m_\infty(r,(1-zA)^{-1})\leq d_{max}(K\log 2+\log r)+\log^{+}\kappa(S).$$

So we have the following.

Theorem 6 *Assume that A can be transformed into Jordan form $J=diag(J_j)$, where J_j are the Jordan blocks, each of dimension $\dim(J_j)=d_j$. Assume furthermore that*

1. $d_{max}:=\sup_j d_j<\infty$.
2. *the spectrum of A lies on K lines l_k of the form $l_k=\{z\mid z=re^{i\theta_k},\ r\in\mathbb{R}\}$,*

Then

$$m_\infty(r,(1-zA)^{-1})\leq d_{max}(K\log 2+\log r)+\log^{+}\kappa(S).\tag{10}$$

This means that the term $m_\infty(r,(1-zA)^{-1})$ does not affect the type nor the order of the function $T_\infty(r,(1-zA)^{-1})$.

As discussed in Section 5.2.1, the violation of the first requirement leads to operators the resolvents of which are not meromorphic outside the unit disc, in which case the function $T_\infty(r,(1-zA)^{-1})$ no longer is a useful tool. Numerical experiments presented in Section 7 show that the second requirement is not essential.

5.2 Dimensions of Jordan blocks are not bounded

5.2.1 The resolvent of J is not meromorphic outside the unit disc

From (6) we have

$$|\frac{z}{1-z\lambda_j}|^{d_j} \le |z|\,\|(1-zJ_j)^{-1}\| \le \left|\frac{z}{1-z\lambda_j}\right| \frac{1-|\frac{z}{1-z\lambda_j}|^{d_j}}{1-|\frac{z}{1-z\lambda_j}|}.$$

To get a lower bound for the norm of the resolvent of J we need an estimate for

$$\inf_j |\frac{1}{r}\mathrm{e}^{-i\varphi} - \lambda_j|^{d_j}.$$

Assume that

$$\lambda_j = (\frac{\tilde{\tau}\tilde{\omega}}{j})^{1/\tilde{\omega}}$$

and that

$$d_j = aj^b$$

for some $0 < \tilde{\omega}, \tilde{\tau}, a, b < \infty$. Now

$$\begin{aligned} |\frac{1}{r}\mathrm{e}^{-i\varphi} - (\frac{\tilde{\tau}\tilde{\omega}}{j})^{1/\tilde{\omega}}|^{aj^b} &= \Big(\frac{1}{r}\,|1 - r\mathrm{e}^{i\varphi}(\frac{\tilde{\tau}\tilde{\omega}}{j})^{1/\tilde{\omega}}|\Big)^{aj^b} \\ &= \Big(\frac{1}{r^n}\,|(1 - \frac{r\mathrm{e}^{i\varphi}(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}}{n})^n|\Big)^{an^{\tilde{\omega}b-1}} \end{aligned}$$

where $n = j^{1/\tilde{\omega}}$, which tends to infinity as $j \to \infty$. Now as $n \to \infty$

$$(1 - \frac{r\mathrm{e}^{i\varphi}(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}}{n})^n \to \mathrm{e}^{-r\mathrm{e}^{i\varphi}(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}}$$

and

$$|(1 - \frac{r\mathrm{e}^{i\varphi}(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}}{n})^n| \to \mathrm{e}^{-r\cos\varphi(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}},$$

which for any $r > 0$, $\varphi \in [-\pi, \pi]$ is bounded from above and below by

$$\mathrm{e}^{-r(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}} \le \mathrm{e}^{-r\cos\varphi}(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}} \le \mathrm{e}^{r(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}}.$$

So for large n and $r > 1$

$$\Big(\frac{1}{r^n}\,|(1 - \frac{r\mathrm{e}^{i\varphi}(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}}{n})^n|\Big)^{an^{\tilde{\omega}b-1}} \approx \Big(\frac{\mathrm{e}^{-r\cos\varphi(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}}}{r^n}\Big)^{an^{\tilde{\omega}b-1}}.$$

If $b \ge \frac{1}{\tilde{\omega}}$ then clearly this tends to zero as $n \to \infty$. On the other hand if $0 < \tilde{\omega}b < 1$, then this can be written as

$$\Big(\frac{\mathrm{e}^{-r\cos\varphi(\tilde{\tau}\tilde{\omega})^{1/\tilde{\omega}}an^{\tilde{\omega}b-1}}}{r^{an^{\tilde{\omega}b}}}\Big).$$

Here as $n \to \infty$ the denominator tends to $\mathrm{e}^0 = 1$ and the nominator tends to infinity, so again the above expression tends to zero. So we have the following.

Lemma 2 *Assume that*

$$\lambda_j = (\frac{\tilde{\tau}\tilde{\omega}}{j})^{1/\tilde{\omega}}$$

and that

$$d_j = aj^b$$

for some $0 \le \tilde{\omega}, \tilde{\tau}, a, b < \infty$. *Then for* $r > 1$

$$\inf_j |\frac{1}{r}\mathrm{e}^{-i\varphi} - \lambda_j|^{d_j} = 0.$$

But this means that the norm of the resolvent of J is not bounded outside the unit disc, and indeed that the resolvent is not meromorphic outside the unit disc.

5.2.2 Estimating $m_\infty(r, (1 - z\tilde{J})^{-1})$

Now let $\tilde{J} = diag(\tilde{J}_j)$, where $\tilde{J}_j$ is given by (7).

If $\lambda_j = 0$ for all j then (9) together with (8) yields

$$\sup_j \sqrt{\frac{1 - |\hat{z}_j|^{2d_j}}{1 - |\hat{z}_j|^2}} \le \|(1 - z\tilde{J})^{-1}\| \le \sup_j \frac{1 - |\hat{z}_j|^{d_j}}{1 - |\hat{z}_j|}$$

so

$$\sup_j \sqrt{\frac{1 - (\varepsilon_j r)^{2d_j}}{1 - (\varepsilon_j r)^2}} \le \|(1 - z\tilde{J})^{-1}\| \le \sup_j \frac{1 - (\varepsilon_j r)^{d_j}}{1 - \varepsilon_j r}.$$

We have $N_\infty(r) = 0$ and

$$\frac{1}{2}\log^+ \sup_j \frac{1 - (\varepsilon_j r)^{2d_j}}{1 - (\varepsilon_j r)^2} \le m_\infty(r, (1 - z\tilde{J})^{-1}) \le \log^+ \sup_j \frac{1 - (\varepsilon_j r)^{d_j}}{1 - \varepsilon_j r}.$$

So we need estimates for $\sup_j \frac{1-(\varepsilon_j r)^{d_j}}{1-\varepsilon_j r}$.

Let us first assume for simplicity that $\varepsilon_j = 1/j$ and $d_j = j$. We need an estimate for $\sup_{j=1,2,\ldots} f(j)$, where

$$f(j) = \frac{1 - (\frac{r}{j})^j}{1 - \frac{r}{j}}.$$

Proposition 1 *Let* $f(j) = \frac{1-(\frac{r}{j})^j}{1-\frac{r}{j}}$. *Then for* $r > 2\mathrm{e}$

$$\frac{\mathrm{e}^{r/\mathrm{e}} - 1}{\mathrm{e} - 1} \le \sup_{j \ge 1} f(j) \le \frac{\mathrm{e}^{r(1+1/\mathrm{e})} - 1}{\mathrm{e} - 1}.$$

Proof. First of all

$$f'(j) = \frac{(\frac{r}{j})^j(\log\frac{r}{j}-1)(\frac{r}{j}-1)+\frac{r}{j^2}((\frac{r}{j})^j-1)}{(\frac{r}{j}-1)^2}$$

so $f'(j) > 0$ for $j \le \frac{r}{\mathrm{e}}$. So the supremum is achieved with some $j > \frac{r}{\mathrm{e}}$, and

$$\sup_j f(j) \ge f(\frac{r}{\mathrm{e}}) = \frac{\mathrm{e}^{r/\mathrm{e}}-1}{\mathrm{e}-1},$$

which gives the lower limit. Now for $j = cr$, $c > 1$ we have $\frac{r}{j} = \frac{1}{c} < 1$ and

$$f(j) \le \frac{1}{1-\frac{r}{j}} = \frac{c}{c-1}.$$

Now if $r > C\mathrm{e}$ where $C > 1$, then

$$\frac{c}{c-1} < \frac{\mathrm{e}^{r/\mathrm{e}}-1}{\mathrm{e}-1}$$

holds for $c > \frac{\mathrm{e}^{r/\mathrm{e}}-1}{\mathrm{e}^{r/\mathrm{e}}-\mathrm{e}}$, and therefore it holds for $c > \frac{\mathrm{e}^C-1}{\mathrm{e}^C-\mathrm{e}} \ge \frac{\mathrm{e}^{r/\mathrm{e}}-1}{\mathrm{e}^{r/\mathrm{e}}-\mathrm{e}}$. So for $r > 2\mathrm{e}$ we know that for

$$c > \frac{\mathrm{e}^2-1}{\mathrm{e}^2-\mathrm{e}} = 1+\frac{1}{\mathrm{e}}$$

it is true that

$$\frac{c}{c-1} < \frac{\mathrm{e}^{r/\mathrm{e}}-1}{\mathrm{e}-1}.$$

Therefore we know that for $r > 2\mathrm{e}$ the supremum of $f(j)$ is obtained on the interval $j \in [\frac{r}{\mathrm{e}}, r(1+\frac{r}{\mathrm{e}})] =: I_r$. So

$$\sup_{j\ge1} f(j) = \sup_{j\in I_r} f(j) = \sup_{j\in I_r}\sum_{k=0}^{j-1}(\frac{r}{j})^k \le \sum_{k=0}^{r(1+\frac{r}{\mathrm{e}})-1}\mathrm{e}^k = \frac{\mathrm{e}^{r(1+1/\mathrm{e})}-1}{\mathrm{e}-1},$$

which completes the proof. □

Corollary 1 *Let $\tilde{J} = diag(\tilde{J}_j)$, where $\tilde{J}_j$ is given by (7). Assume $\lambda_j = 0$, $\varepsilon_j = 1/j$ and $d_j = j$. Then $m_\infty(r,(1-z\tilde{J})^{-1})$ grows with order $\omega = 1$ and type τ, $\frac{1}{\mathrm{e}} \le \tau \le 1+\frac{1}{\mathrm{e}}$.*

Numerical estimation gives $m_\infty(r) \approx cr$ where $c \approx 0.37 \approx 1/\mathrm{e}$ so the lower limit seems to give the correct behavior.

A similar argumentation is valid also if $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = aj^b$.

Proposition 2 *Let $\lambda_j = 0$, $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = aj^b$. Then for sufficiently large r*

$$\frac{\mathrm{e}^{C_1 r^{b/\delta}}-1}{\mathrm{e}^{\delta/b}-1} \le \sup_{j\ge1} f(j) \le \frac{\mathrm{e}^{C_2 r^{b/\delta}}-1}{\mathrm{e}^{\delta/b}-1}$$

where

$$f(j)=\frac{1-(\frac{\gamma r}{j^\delta})^{d_j}}{1-\frac{\gamma r}{j^\delta}}=\sum_{k=0}^{d_j-1}(\frac{\gamma r}{j^\delta})^k$$

and

$$C_i=\frac{\delta a}{b}c_i^b,\quad i=1,2,\quad c_1=\frac{\gamma^{1/\delta}}{\mathrm{e}^{1/b}},\quad c_2=((1+\frac{1}{\mathrm{e}^{\delta/b}}))^{1/\delta}.$$

Proof. For $j^\delta=cr>\gamma r$ we have

$$f(j)\le\sum_{k=0}^{\infty}(\frac{\gamma r}{j^\delta})^k=\frac{1}{1-\frac{\gamma r}{j^\delta}}=\frac{c}{c-\gamma}.$$

We know that for $r\ge\frac{1}{\gamma}\mathrm{e}^{\delta/b}$

$$\sup_{j\ge1}f(j)\ge f(\frac{(\gamma r)^{1/\delta}}{e^{1/b}})=\frac{\mathrm{e}^{\frac{\delta a\gamma^{b/\delta}}{b\mathrm{e}}r^{b/\delta}}-1}{\mathrm{e}^{\delta/b}-1}.$$

So for $r\ge(\frac{2\mathrm{e}}{a\gamma^{b/\delta}})^{\delta/b}$ we have $\frac{\delta a\gamma^{b/\delta}}{b\mathrm{e}}r^{b/\delta}\ge2\frac{\delta}{b}$ and

$$\frac{\mathrm{e}^{\frac{\delta a\gamma^{b/\delta}}{b\mathrm{e}}r^{b/\delta}}-1}{\mathrm{e}^{\delta/b}-1}>\frac{c}{c-\gamma}$$

holds for

$$c>\frac{\mathrm{e}^{\frac{\delta a\gamma^{b/\delta}}{b\mathrm{e}}r^{b/\delta}}-1}{\mathrm{e}^{\frac{\delta a\gamma^{b/\delta}}{b\mathrm{e}}r^{b/\delta}}-\mathrm{e}^{\delta/b}}$$

and therefore it holds for

$$c>\frac{\mathrm{e}^{2\delta/b}-1}{\mathrm{e}^{\delta/b}(\mathrm{e}^{\delta/b}-1)}=1+\frac{1}{\mathrm{e}^{\delta/b}}\ge\frac{\mathrm{e}^{\frac{\delta a\gamma^{b/\delta}}{b\mathrm{e}}r^{b/\delta}}-1}{\mathrm{e}^{\frac{\delta a\gamma^{b/\delta}}{b\mathrm{e}}r^{b/\delta}}-\mathrm{e}^{\delta/b}}.$$

So the supremum is obtained on the interval $j\in[1,((1+\frac{1}{\mathrm{e}^{\delta/b}})r)^{1/\delta}]$. Moreover one can show that $f'(j)>0$ for $j<\frac{1}{\mathrm{e}^{1/b}}(\gamma r)^{1/\delta}$ so in fact the supremum is obtained on the interval

$$j\in[c_1r^{1/\delta},c_2r^{1/\delta}]=:I_r,\quad c_1=\frac{\gamma^{1/\delta}}{\mathrm{e}^{1/b}},\quad c_2=((1+\frac{1}{\mathrm{e}^{\delta/b}}))^{1/\delta}.$$

Therefore

$$\sup_{j\ge1}f(j)=\sup_{j\in I_r}f(j)=\sup_j\sum_{k=0}^{aj^b-1}(\frac{\gamma r}{j^\delta})^k\le\sum_{k=0}^{ac_2^br^{b/\delta}-1}(\frac{\gamma r}{c_1^\delta r})^k=\frac{\mathrm{e}^{\frac{\delta}{b}ac_2^br^{b/\delta}}-1}{\mathrm{e}^{\delta/b}-1}.$$

□

Corollary 2 *Let $\tilde J=diag(\tilde J_j)$, where $\tilde J_j$ is given by (7). Assume $\lambda_j=0$, $\varepsilon_j=\gamma j^{-\delta}$ and $d_j=aj^b$. Then $m_\infty(r,(1-z\tilde J)^{-1})$ grows with order $\omega=\frac{b}{\delta}$ and type τ, $C_1\le\tau\le C_2$, where C_1 and C_2 are as in Proposition 2.*

When $\lambda_j \neq 0$, things get slightly more complicated. First of all, we need the following lemmas.

Lemma 3 *Let $\lambda_j \in \mathbb{R}$ and $z = r\mathrm{e}^{\mathrm{i}\varphi}$. Then*

$$\sup_j \frac{1}{|1-\lambda_j z|} \leq \frac{1}{|\sin\varphi|}.$$

Proof.

$$|1-\lambda_j z| = |1-\lambda_j r\mathrm{e}^{\mathrm{i}\varphi}| = |\mathrm{e}^{-\mathrm{i}\varphi} - \lambda_j r| = \sqrt{(\cos\varphi - \lambda_j r)^2 + \sin^2\varphi} \geq |\sin\varphi|.$$

□

Lemma 4 *Let $\lambda_j = \alpha j^{-\beta}$, $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = aj^b$. Assume furthermore that $\delta \geq \beta$. Fix $M > 1$ and let $j > \max\{1, ((M\gamma+\alpha)r)^{1/\beta}.\}$ Then*

$$\frac{1}{|1-\lambda_j z|} \frac{1 - |\frac{\varepsilon_j z}{1-z\lambda_j}|^{d_j}}{1 - |\frac{\varepsilon_j z}{1-z\lambda_j}|} < (\frac{M}{M-1}) \frac{1}{|\sin\varphi|}.$$

Proof. That $j > ((M\gamma+\alpha)r)^{1/\beta}$ implies $\frac{M\gamma}{\alpha} + 1 \leq \frac{j^\beta}{\alpha r}$ so

$$\begin{aligned} \frac{M\gamma}{\alpha}\frac{1}{j^{\delta-\beta}} &< \frac{M\gamma}{\alpha} < \frac{j^\beta}{\alpha r} - 1 \leq \frac{j^\beta}{\alpha r} - \cos\varphi = \sqrt{(\frac{j^\beta}{\alpha r} - \cos\varphi)^2} \\ &\leq \sqrt{(\frac{j^\beta}{\alpha r} - \cos\varphi)^2 + \sin^2\varphi}. \end{aligned}$$

Therefore

$$M|\varepsilon_j z| = M\gamma j^{-\delta} r < \alpha j^{-\beta} r \sqrt{(\frac{j^\beta}{\alpha r} - \cos\varphi)^2 + \sin^2\varphi} = |1 - \alpha j^{-\beta} r\mathrm{e}^{\mathrm{i}\varphi}| = |1 - z\lambda_j|,$$

and furthermore

$$\frac{|\varepsilon_j z|}{|1-z\lambda_j|} < \frac{1}{M} < 1$$

so finally we have

$$\begin{aligned} \frac{1}{|1-\lambda_j z|} \frac{1 - |\frac{\varepsilon_j z}{1-z\lambda_j}|^{d_j}}{1 - |\frac{\varepsilon_j z}{1-z\lambda_j}|} &\leq \frac{1}{|1-\lambda_j z|} \frac{1}{1 - |\frac{\varepsilon_j z}{1-z\lambda_j}|} \\ &\leq \frac{1}{|1-\lambda_j z|} \frac{1}{1 - \frac{1}{M}} < (\frac{M}{M-1}) \frac{1}{|\sin\varphi|}, \end{aligned}$$

where the last inequality follows from Lemma 3. □

Lemma 5 *Let $\lambda_j = \alpha j^{-\beta}$, $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = aj^b$. Assume furthermore that $\delta < \beta$. Fix $M > 1$ and let $j > \max\{1, ((M\gamma+\alpha)r)^{1/\delta}.\}$ Then*

$$\frac{1}{|1-\lambda_j z|} \frac{1 - |\frac{\varepsilon_j z}{1-z\lambda_j}|^{d_j}}{1 - |\frac{\varepsilon_j z}{1-z\lambda_j}|} < (\frac{M}{M-1}) \frac{1}{|\sin\varphi|}.$$

Proof. That $j > ((M\gamma+\alpha)r)^{1/\delta}$ implies $j^\delta > \alpha r(\frac{\gamma M}{\alpha}+1)$ so (as $j>1$)

$$\frac{\gamma M}{\alpha} + j^{-(\beta-\delta)} < \frac{\gamma M}{\alpha} + 1 < \frac{1}{\alpha r} j^\delta$$

and furthermore

$$\frac{\gamma M}{\alpha} j^{(\beta-\delta)} < \frac{1}{\alpha r} j^\beta - 1 < \frac{1}{\alpha r} j^\beta - \cos\varphi < \sqrt{(\frac{1}{\alpha r} j^\beta - \cos\varphi)^2 + \sin^2\varphi},$$

and the rest follows as in Lemma 4.

□

Proposition 3 *Assume $\lambda_j = \alpha j^{-\beta}$, $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = a j^b$. Fix $M > 1$. Then*

$$\sup_{j\geq 1} f(j) \leq \frac{a(cr)^{b/\rho}}{|\sin\varphi|}\Big(1 + (\frac{\gamma r}{|\sin\varphi|})^{a(cr)^{b/\rho}-1}\Big) \tag{11}$$

where $\rho := \min\{\beta,\delta\}$ and $c := M\gamma+\alpha$.

Proof. Choose r such that $a(cr)^{b/\rho} > \frac{M}{M-1}$ and $cr > 1$. Then by Lemmas 4 and 5 we have

$$f(j) < \frac{M}{M-1}\frac{1}{|\sin\varphi|} \quad \text{for} \quad j > (cr)^{1/\rho}. \tag{12}$$

Consider now $j \in I_r$, where $I_r := [1,(cr)^{1/\rho}]$. We have

$$\frac{|\varepsilon_j z|}{|1-\lambda_j z|} = \frac{\gamma j^{-\delta} r}{|1-\alpha j^{-\beta} r e^{i\varphi}|} = \frac{\gamma j^{-\delta} r}{|e^{-i\varphi} - \alpha j^{-\beta} r|} \leq \frac{\gamma r}{|\sin\varphi|}$$

so

$$\begin{aligned} f(j) &= \frac{1}{|1-\lambda_j z|}\sum_{k=0}^{d_j-1} |\frac{\varepsilon_j z}{1-\lambda_j z}|^k \leq \frac{1}{|\sin\varphi|}\sum_{k=0}^{a(cr)^{b/\rho}-1} (\frac{\gamma r}{|\sin\varphi|})^k \\ &\leq \frac{a(cr)^{b/\rho}}{|\sin\varphi|}\max\{1, (\frac{\gamma r}{|\sin\varphi|})^{a(cr)^{b/\rho}-1}\} \\ &\leq \frac{a(cr)^{b/\rho}}{|\sin\varphi|}(1 + (\frac{\gamma r}{|\sin\varphi|})^{a(cr)^{b/\rho}-1}) \end{aligned}$$

As $a(cr)^{b/\rho} > \frac{M}{M-1}$, (12) guarantees that $\sup_{j\in I_r} f(j) = \sup_{j\geq 1} f(j)$, so the above yields (11).

□

Corollary 3 *Let $\tilde J = diag(\tilde J_j)$, where $\tilde J_j$ is given by (7). Assume $\lambda_j = \alpha j^{-\beta}$, $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = a j^b$. Then the order ω_m of $m_\infty(r,(1-z\tilde J)^{-1})$ satisfies*

$$\omega_m \leq \frac{b}{\rho},$$

where $\rho := \min\{\beta,\delta\}$.

Corollary 4 *Let $\tilde{J} = diag(\tilde{J}_j)$, where $\tilde{J}_j$ is given by (7). Assume $\lambda_j = \alpha j^{-\beta}$, $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = a j^b$. Then for $\beta > \delta$ $m_\infty(r, (1 - z\tilde{J})^{-1})$ grows with order*

$$\omega_m = \frac{b}{\delta}.$$

Proof. That $\omega_m \leq \frac{b}{\delta}$ follows from Corollary 3. Choose $\delta < \mu < \beta$ and choose r such that $(cr)^{1/\mu} \geq 1$. Let $J := (cr)^{1/\mu}$. Now

$$\sup_{j \geq 1} f(j) \geq f(J) \geq \frac{1}{\gamma r} J^\delta \left| \frac{\gamma J^{-\delta} r}{1 - \alpha J^{-\beta} r e^{i\varphi}} \right|^{aJ^b}$$

and

$$\left| \frac{\gamma J^{-\delta} r}{1 - \alpha J^{-\beta} r e^{i\varphi}} \right| \geq \left| \frac{\gamma J^{-\delta} r}{1 + \alpha J^{-\beta} r} \right| = \left| \frac{\hat{\gamma} r^{\frac{\mu - \delta}{\mu}}}{1 + \hat{\alpha} r^{\frac{\mu - \beta}{\mu}}} \right|$$

where $\hat{\gamma} = \gamma c^{-\gamma/\mu}$ and $\hat{\alpha} = \alpha c^{-\beta/\mu}$. So

$$\sup_{j \geq 1} f(j) \geq \frac{1}{\hat{\gamma}} r^{\frac{\delta - \mu}{\mu}} \left| \frac{\hat{\gamma} r^{\frac{\mu - \delta}{\mu}}}{1 + \hat{\alpha} r^{\frac{\mu - \beta}{\mu}}} \right|^{a(cr)^{b/\mu}}$$

and

$$a(cr)^{b/\mu} \log^+ \left| \frac{\hat{\gamma} r^{\frac{\mu - \delta}{\mu}}}{1 + \hat{\alpha} r^{\frac{\mu - \beta}{\mu}}} \right| \leq \log^+ \left| \hat{\gamma} r^{\frac{\mu - \delta}{\mu}} \sup_{j \geq 1} f(j) \right|$$

Finally

$$m_\infty(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |\sup f(j)| \mathrm{d}\varphi \geq a(cr)^{b/\mu} \log^+ \left| \frac{\hat{\gamma} r^{\frac{\mu - \delta}{\mu}}}{1 + \hat{\alpha} r^{\frac{\mu - \beta}{\mu}}} \right| - \log^+ \hat{\gamma} r^{\frac{\mu - \delta}{\mu}}$$

holds for all μ such that $\delta < \mu < \beta$, which shows that the order ω_m of $m_\infty(r)$ satisfies $\omega_m \geq \frac{b}{\delta}$. □

5.2.3 The main result

Let us summarize what we've learned in the following theorem.

Theorem 7 *Let $\tilde{J} = diag(\tilde{J}_j)$, where $\tilde{J}_j$ is given by (7). Let $\lambda_j = \alpha j^{-\beta}$, $\varepsilon_j = \gamma j^{-\delta}$ and $d_j = a j^b$. Then $N_\infty(r, (1 - z\tilde{J})^{-1})$ grows with order ω_N and type τ_N, and $m_\infty(r, (1 - z\tilde{J})^{-1})$ grows with order ω_m and type τ_m, where*

$$\omega_N = \frac{b+1}{\beta}, \qquad \begin{cases} \omega_m \leq \frac{b}{\beta} & \text{for } \beta \leq \delta, \\ \omega_m = \frac{b}{\delta} & \text{for } \beta > \delta. \end{cases}$$

So for $\delta \geq \beta \frac{b}{b+1}$

$$\omega_m \leq \omega_N$$

while for $\delta < \beta \frac{b}{b+1}$

$$\omega_m > \omega_N.$$

6 A nonnormal example

Transforming A into Jordan form is not a good idea if the condition number $\kappa(S)$ of the transformation matrix S is large, for in such a case (4) does not give any information. In such a case it is useful to look at the resolvent itself. Consider the following example.

Let $\mathbb{D}$ be the unit disk in the complex plane and let $A(\mathbb{D})$ denote the set of analytic functions in $\mathbb{D}$. Let H^2 be the Hilbert space

$$H^2 := \{f \in A(\mathbb{D}) \mid \sum_{k=0}^{\infty} a_k z^k = f(z), \quad \sum |a_k|^2 < \infty\},$$

equipped with the inner product

$$(f, g) = (\sum_{k=0}^{\infty} a_k z^k, \sum_{k=0}^{\infty} b_k z^k) = \sum_k a_k \overline{b_k} = \frac{1}{2\pi} \int_{\partial\mathbb{D}} f(w)\overline{g}(w)\, dw$$

and the norm induced by the inner product. Consider the integration operator

$$V : f(z) \to \int_0^z f(x) dx,$$

that is

$$V : \sum_{k=0}^{\infty} a_k z^k \to \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}.$$

So integration in H^2 corresponds to operating with W in ℓ_2, where the operator W is defined by

$$W = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \frac{1}{2} & 0 & \\ & & \frac{1}{3} & \ddots \\ & & & \ddots \end{pmatrix}.$$

The resolvent of W is

$$(1 - zW)^{-1} = \begin{pmatrix} 1 & & & & \\ z & 1 & & & \\ \frac{z^2}{2!} & \frac{z}{2} & 1 & & \\ \frac{z^3}{3!} & & \frac{z}{3} & 1 & \\ \vdots & & & \ddots & \ddots \\ \frac{z^k}{k!} & & & & \\ \vdots & & & & \end{pmatrix}.$$

where the i^{th} subdiagonal contains the terms

$$\frac{z^i (k-i)!}{k!}, \quad k = i, i+1, \ldots$$

It is entire in z. The resolvent has no poles, so $N_\infty(r) = 0$ and $T_\infty(r) = m_\infty(r)$.

Theorem 8 *The resolvent of W is entire of order $\omega = 1$ and of type $\tau = 1$.*

Proof. First take the absolute value of each term of the resolvent matrix; notice that the elements of the matrix thus obtained decrease along each subdiagonal:

$$|z|^i \frac{(k-i)!}{k!} = |z|^i \frac{1}{i!} \frac{(k-i)!i!}{k!} = |z|^i \frac{1}{i!} \frac{1}{\binom{k}{i}} \leq |z|^i \frac{1}{i!} \quad \text{for} \quad k \geq i.$$

Denote by T the corresponding Toeplitz matrix

$$T = \begin{pmatrix} 1 & & & & \\ |z| & 1 & & & \\ \frac{|z|^2}{2} & |z| & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{|z|^k}{k!} & & & & \\ \vdots & \ddots & & & \end{pmatrix}.$$

Now it is known that the norm of a Toeplitz matrix is less than the sum of the absolute values of it's first column. We are interested in the norm of the resolvent. But clearly

$$\|(1 - zW)^{-1}\|_2 \leq \|T\|_2 \leq \sum_k \frac{|z|^k}{k!} = \mathrm{e}^{|z|}.$$

So the order and the type of the resolvent satisfy

$$\omega \leq 1, \quad \tau \leq 1.$$

On the other hand, choose $v = (1, 0, 0, 0, \ldots)^T$ to get

$$\|(1 - zW)^{-1}\|_2^2 \geq \|(1 - zW)^{-1}v\|_2^2 = \sum_k \frac{|z|^{2k}}{(k!)^2} = \sum_k a_k |z|^k,$$

where for even k (use Stirling)

$$a_k = \frac{1}{((k/2)!)^2} \sim \frac{1}{((\frac{k/2}{\mathrm{e}})^{k/2})^2} = (\frac{2\mathrm{e}}{k})^k$$

and as by Theorem 2 the order and the type can be read from these coefficients:

$$a_k \sim (\frac{\tau \mathrm{e} \omega}{k})^{k/\omega}$$

we have for the type and the order of the resolvent squared

$$\omega' \geq 1, \quad \tau' \geq 2.$$

But $f^2(r) \sim e^{\tau' r^{\omega'}}$ implies that $f(r) \sim e^{\frac{\tau'}{2} r^{\omega'}}$, so the order and the type of the resolvent satisfy

$$\omega \geq 1, \quad \tau \geq 1.$$

□

Theorem 9 *$T_\infty(r, (1-zW)^{-1})$ grows with the same order and the same type as $M_\infty(r, (1-zW)^{-1})$:*

$$\omega_{mer} = \omega, \quad \tau_{mer} = \tau.$$

This follows from Lemma 1 in Section 2.

7 Numerical experiments

In the following we shall present some numerical examples, which illustrate the behavior of $T_\infty(r, (1-zA)^{-1})$, $N_\infty(r, (1-zA)^{-1})$ and $m_\infty(r, (1-zA)^{-1})$ for different A. Here A belongs to one of the model classes discussed in Section 3, so it is block diagonal, where the blocks are of form Λ_j, J_j or $\tilde{J}_j$, where

$$\Lambda_j = \begin{pmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{pmatrix}, \quad \tilde{J}_j = \begin{pmatrix} \lambda_j & \varepsilon_j & & \\ & \ddots & \ddots & \\ & & & \varepsilon_j \\ & & & \lambda_j \end{pmatrix}, \tag{13}$$

and $J_j = \tilde{J}_j$ with $\varepsilon_j = 1$ for all j. The dimension of the blocks are d_j in each case. Note in particular that the condition number of the transformation matrix $\kappa(S)$ does not appear in these examples. The numerical computations are actually done for submatrices of the infinite matrix A.

In the following we shall write $T_\infty(r)$, $N_\infty(r)$ and $m_\infty(r)$ for $T_\infty(r, (1-zA)^{-1})$, $N_\infty(r, (1-zA)^{-1})$ and $m_\infty(r, (1-zA)^{-1})$ respectively. We always plot $\log T_\infty(r)$, $\log m_\infty(r)$ and $\log N_\infty(r)$ as functions of $\log r$. If $T_\infty(r)$ grows with order ω and type τ, then $T_\infty(r) \sim \tau r^\omega$ and $\log T_\infty(r) \sim \log \tau + \omega \log r$, so the order ω equals the slope of $\log T_\infty(r)$. We use solid, dashed and dash-dotted lines for $\log T_\infty(r)$, $\log m_\infty(r)$ and $\log N_\infty(r)$ respectively. Instead of plotting the eigenvalues λ_j of A in the complex plane we frequently plot the poles $z_j = 1/\lambda_j$ of the resolvent $(1-zA)^{-1}$.

Case 7.1 Figure 1 illustrates the effect of the length of the Jordan blocks. Here $A = diag(J_j)$ and the eigenvalues of the matrix A are $\lambda_j = (\frac{\tilde{\tau}\tilde{\omega}}{j})^{1/\tilde{\omega}}$, where $\tilde{\tau} = 1$, $\tilde{\omega} = \frac{1}{2}$. In each subfigure the dimension d_j of the Jordan blocks J_j is a different constant: $d_j = d = 1, 2, 4, 5$ in figures (a), (b), (c) and (d) respectively. The longer the Jordan block, the longer it takes for $N_\infty(r)$ to exceed $m_\infty(r)$, but eventually this will happen and for large enough r it is true that $T_\infty(r) \approx N_\infty(r)$.

Case 7.2 Figure 2 shows that it is not essential that the eigenvalues of A be on a finite number of lines for $m_\infty(r)$ to be bounded (see Theorem 6). Here $A = diag(\lambda_j)$ and the eigenvalues of the matrix A satisfy

$$\lambda_{4j-k} = (\frac{1}{2j})^2 e^{i\varphi_{k,j}}, \quad k = 0, \ldots, 3, \quad j = 1, 2, \ldots$$

In the first case $\varphi_{k,j} = k\frac{\pi}{2}$, so $\varphi_{k,j} = \varphi_{k,n}$. In the second case $\varphi_{k,j} = k\frac{\pi}{2} + \varepsilon_j$, where $\varphi_{k,j} = \varphi_{m,n}$ only if $k = m$ and $j = n$. On the left hand side you see the poles $z_j = 1/\lambda_j$ of the resolvent $(1 - zA)^{-1}$ of A and on the right hand side the growth of $\log T_\infty(r)$, $\log m_\infty(r)$ and $\log N_\infty(r)$ as functions of $\log r$. In the first case the eigenvalues are on the real and the imaginary axes. In the second case for each j the eigenvalues are rotated slightly, so that all eigenvalues have different phase angles. So in the first case the 2. requirement in Theorem 6 is satisfied, in the second case it is not. Yet there is no difference in the behaviors of $T_\infty(r)$, $m_\infty(r)$ and $N_\infty(r)$. So clearly the second requirement is not the 'right' one, but only necessary for this particular proof.

Case 7.3 In Figure 3 we compare four cases, where in each the eigenvalues of the matrix A are $\lambda_j = (\frac{\tilde{\tau}\tilde{\omega}}{j})^{1/\tilde{\omega}}$, where $\tilde{\tau} = 1$, $\tilde{\omega} = \frac{1}{2}$. Let Λ_j and J_j be as defined by (13). The figures (a)–(d) show the functions $\log T_\infty(r)$, $\log m_\infty(r)$ and $\log N_\infty(r)$, where A in the different subfigures is

$$\begin{array}{lll} (a) & A = diag(\Lambda_j), & \dim\Lambda_j = 2, \\ (b) & A = diag(J_j), & \dim J_j = 2, \\ (c) & A = diag(\Lambda_j, J_j), & \dim\Lambda_j = \dim J_j = 2, \\ (d) & A = diag(J_j), & \dim J_j = 4. \end{array}$$

In the first two cases all eigenvalues are of multiplicity 2, and in the latter two cases all eigenvalues are of multiplicity 4. The poles $z_j = 1/\lambda_j$ of the resolvent of A are of order 1 in (a), 2 in (b) and (c) and 4 in (d). The higher the multiplicity of the pole, the longer it takes for $N_\infty(r)$ to exceed $m_\infty(r)$. The multiplicity of the eigenvalue is not decisive.

Case 7.4 Though in Case 7.2 it is true that there are a fixed number of eigenvalues with $|\lambda| = (\frac{\tilde{\tau}\tilde{\omega}}{k})^{1/\tilde{\omega}}$ this is not essential. In Figure 4 the number of eigenvalues (all simple) with $|\lambda| = (\frac{\tilde{\tau}\tilde{\omega}}{k})^{1/\tilde{\omega}}$ increases with k, and still $T_\infty(r) \sim N_\infty(r)$.

Case 7.5 It is essential that $d_{max} < \infty$. To see this consider Figure 5. Here the eigenvalues $\lambda_k = (\frac{1}{2k})^2$ are all of multiplicity k. In 5 (a) we have $A = diag(J_j)$, where $\dim J_j = j$, which tends to infinity as $j \to \infty$. Here indeed $T_\infty(r) \sim m_\infty(r)$. A word of caution: Figure 5(a) is not very reliable, as A is close to singular and for large $|z|$ so is $(1 - zA)$ which has to be inverted when calculating m_∞. In 5(b) we have $A = diag(\Lambda_j)$, where again $\dim\Lambda_j = j$, so though the eigenvalue λ_j is of multiplicity j, the pole $z_j = 1/\lambda_j$ of the resolvent of A is simple. Again as it should $T_\infty(r) \sim N_\infty(r)$.

Case 7.6 Finally let $A = diag(\tilde{J}_j)$, where $\lambda_j = \alpha j^{-\beta}$ and $\varepsilon_j = \gamma j^{-\delta}$. In Section 5.2 we concluded that when $\delta < \beta\frac{b}{b+1}$, then $m_\infty(r)$ grows faster than $N_\infty(r)$, whereas if $\delta > \beta\frac{b}{b+1}$, then $N_\infty(r)$ will grow faster. Figure 6 illustrates this phenomenon. Here $\lambda_j = (\frac{1}{2j})^2$ and $d_j = j$, which means that $\beta = 2$ and $b = 1$, so $\beta\frac{b}{b+1} = 1$. Moreover $\varepsilon_j = \lambda_j^k$, so $\delta = k\beta$, where the values for k and δ can be found in the table below.

subplot	k	δ
a	2	4
b	1	2
c	$\frac{3}{4}$	1.5
d	0.6	1.2
e	$\frac{1}{2}$	1
f	0.4	0.8

In the subplots (a)–(d) we have $\delta > \beta\frac{b}{b+1}$, and $N_\infty(r)$ grows faster than $m_\infty(r)$, as it should. In subplot (e) we have $\delta = \beta\frac{b}{b+1}$. Here $m_\infty(r)$ and $N_\infty(r)$ grow with the same speed. Finally, in subplot (f) $\delta < \beta\frac{b}{b+1}$, and $m_\infty(r)$ grows faster than $N_\infty(r)$. The slowdown of the growth of $m_\infty(r)$ in subplot (e) is due to numerical errors which occur for large r in the case of resolvents of nearly singular matrices [3].

8 Summary

In this paper we have studied the growth of $T_\infty(r, (1 - zA)^{-1})$ for various a bounded linear operators A in a separable Hilbert space H, in which case A admits a matrix representation. We have mainly considered operators A for which there exists a bounded S with a bounded inverse S^{-1} such that

$$A = S^{-1}\tilde{J}S,$$

where $\tilde{J} = diag(\tilde{J}_j)$, $\dim\tilde{J}_j = d_j$ and

$$\tilde{J}_j = \begin{pmatrix} \lambda_j & \varepsilon_j & & \\ & \ddots & \ddots & \\ & & & \varepsilon_j \\ & & & \lambda_j \end{pmatrix}.$$

Furthermore we have assumed that

$$\lambda_j = \alpha j^{-\beta}, \quad \varepsilon_j = \gamma j^{-\delta}, \quad d_j = aj^b \tag{14}$$

where α, β, γ, δ, a and b are nonnegative real numbers. In Section 4 we show what in this case $N_\infty(r, (1 - zA)^{-1})$ grows with order ω_N and type τ_N, where

$$\omega_N = \frac{b+1}{\beta}, \quad \tau_N = \frac{a\beta}{(b+1)^2}\alpha^{\frac{b+1}{\beta}}. \tag{15}$$

Figure 1: The eigenvalues of the matrix A are $\lambda_j = (\frac{1}{2j})^2$. The lengths of the Jordan blocks corresponding to each eigenvalue are 1 in (a), 2 in (b), 4 in (c) and 5 in (d). The longer the Jordan block, the longer it takes for $N_\infty(r)$ to exceed $m_\infty(r)$. $\log T_\infty(r)$ (solid), $\log m_\infty(r)$ (dashed) and $\log N_\infty(r)$ (dash-dotted) are plotted as functions of $\log r$.

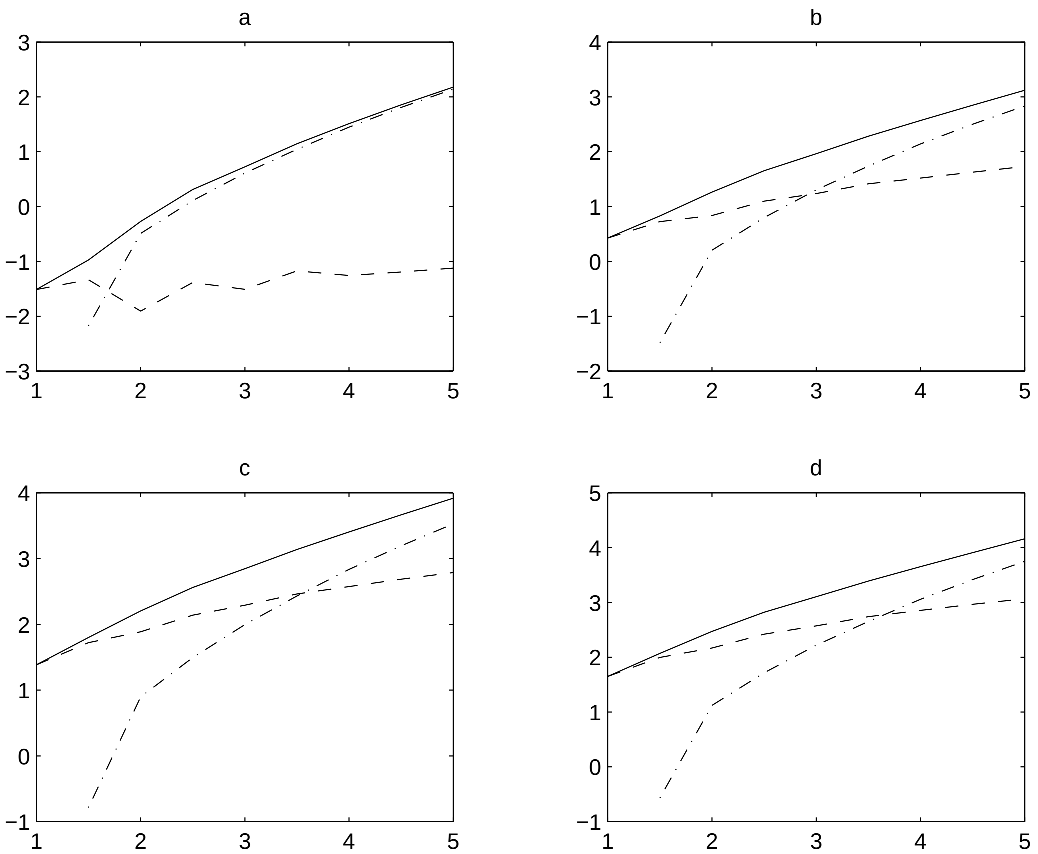


Figure 2: The eigenvalues of the matrix A satisfy $|\lambda_{4j-k}| = (\frac{1}{2j})^2$, $k = 0, \ldots, 3$, and each eigenvalue is simple. On the left hand side are the poles of the resolvent of A and on the right hand side the growth of $\log T_\infty(r)$ (solid), $\log m_\infty(r)$ (dashed) and $\log N_\infty(r)$ (dash-dotted) as functions of $\log r$. In the first case the eigenvalues are on the real and the imaginary axes. In the second case for each j the eigenvalues are rotated slightly, so that all eigenvalues have different phase angles. This does not seem to affect the behaviors of $T_\infty(r)$, $m_\infty(r)$ and $N_\infty(r)$.

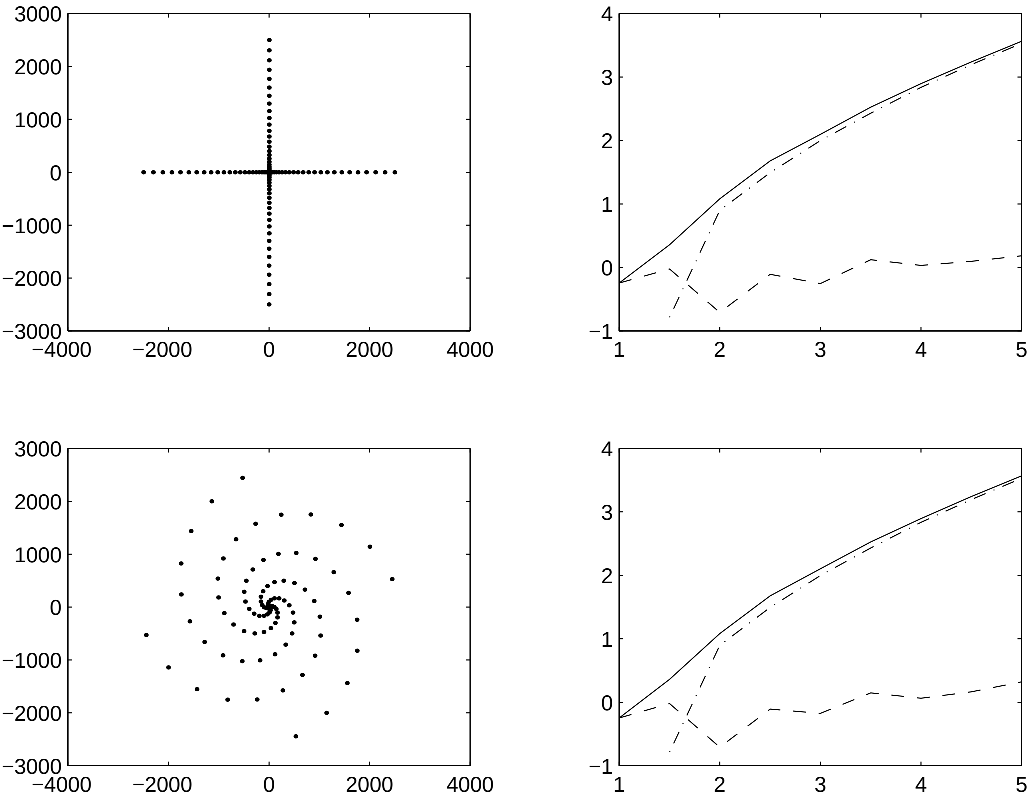


Figure 3: Here we compare four cases, where in each the eigenvalues of the matrix A are $\lambda_j = (\frac{1}{2j})^2$. In the first two cases all eigenvalues are of multiplicity 2, and the Jordan blocks associated with each eigenvalue are of length one in (a) and two in (b). In the other two cases all eigenvalues are of multiplicity 4. The Jordan blocks associated with each eigenvalue are of lengths 1,1 and 2 in (c) and 4 in (d). This means that the poles of the resolvent of A are of order 1 in (a), 2 in (b) and (c) and 4 in (d). $\log T_\infty(r)$ (solid), $\log m_\infty(r)$ (dashed) and $\log N_\infty(r)$ (dash-dotted) are plotted as functions of $\log r$.

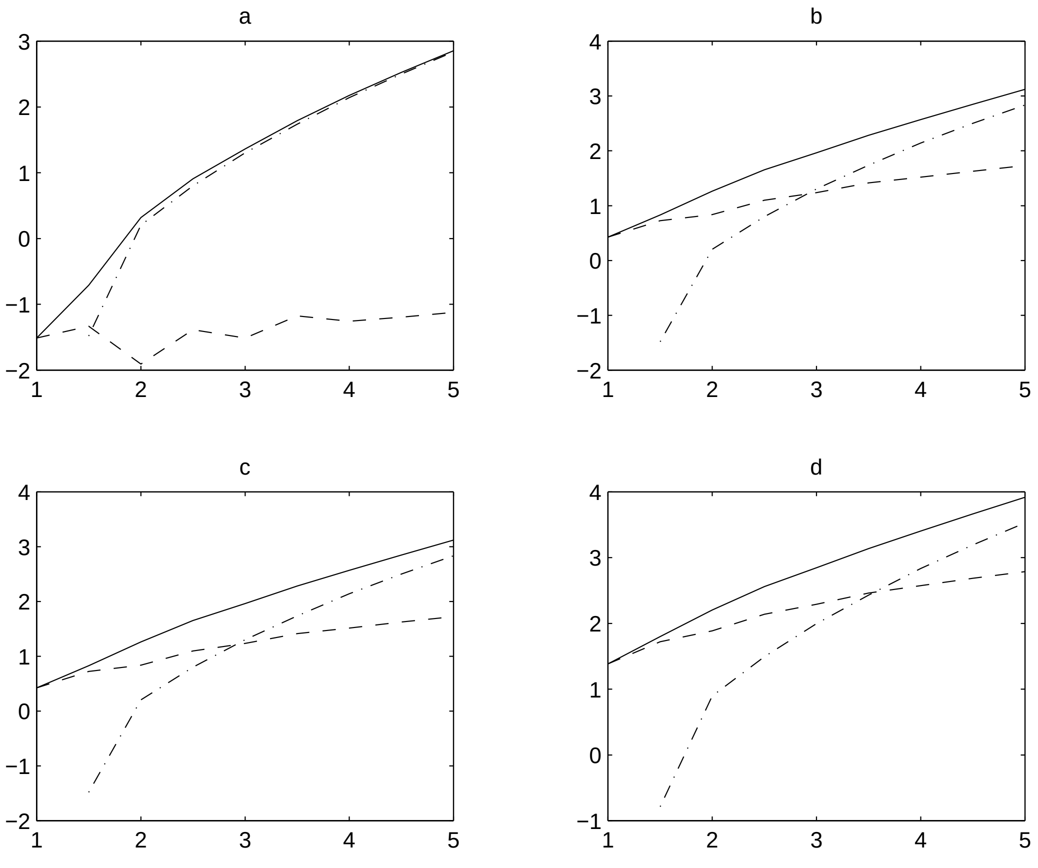


Figure 4: The eigenvalues of A are all simple, and the number of eigenvalues with $|\lambda| = (\frac{1}{2k})^2$ increases with k. On the left the poles of the resolvent of A. On the right $\log T_\infty(r)$ (solid), $\log m_\infty(r)$ (dashed) and $\log N_\infty(r)$ (dash-dotted) as functions of $\log r$.

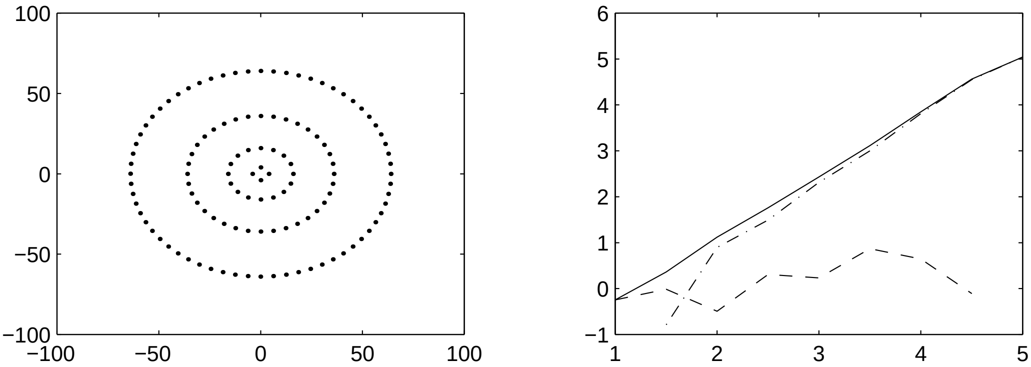


Figure 5: Here the eigenvalues $\lambda_k = (\frac{1}{2k})^2$ are all of multiplicity k. In (a) the Jordan block corresponding to the eigenvalue λ_k is of length k, In 5 (b) the A is diagonal with multiple eigenvalues. $\log T_\infty(r)$ (solid), $\log m_\infty(r)$ (dashed) and $\log N_\infty(r)$ (dash-dotted) are plotted as functions of $\log r$.

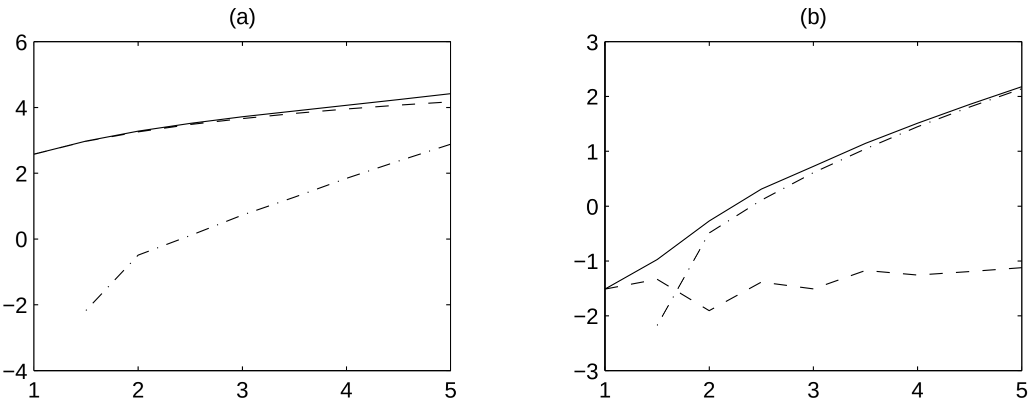
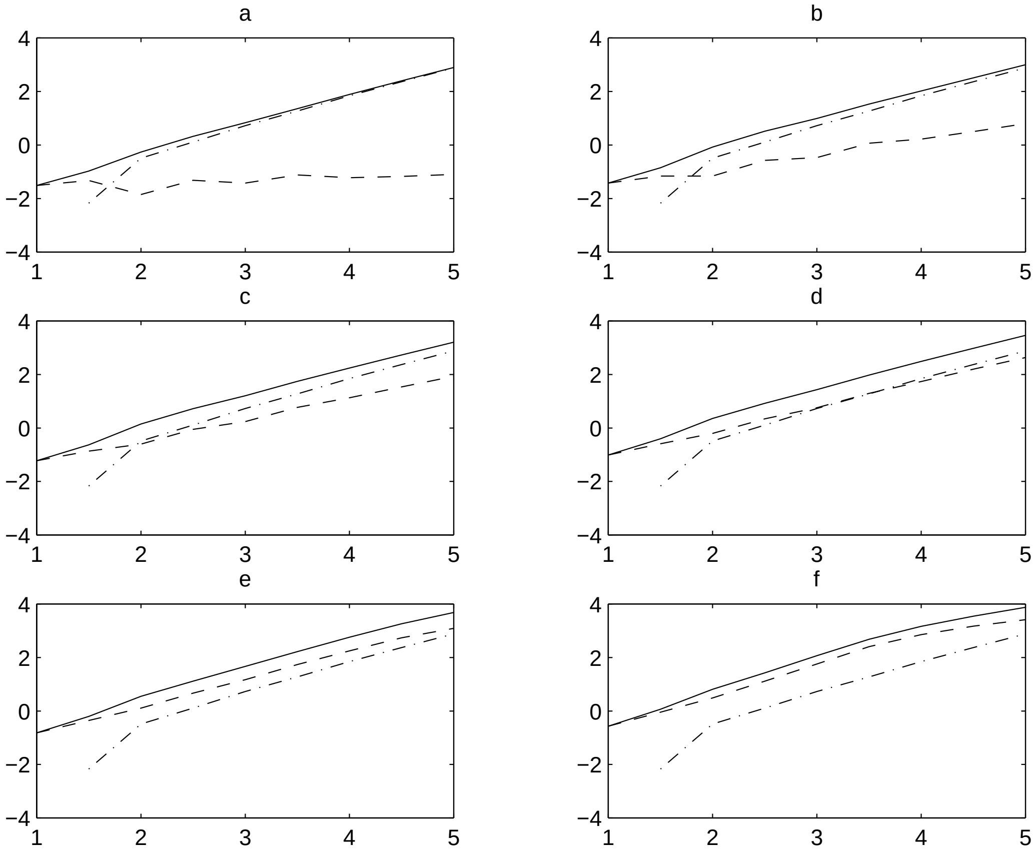


Figure 6: Here the eigenvalues $\lambda_j = (\frac{1}{2j})^2$ and the Jordan block corresponding to the eigenvalue λ_j is of length j. The off-diagonal elements of each block are ε_j instead of 1, where $\varepsilon_j = \lambda_j^k$, and $k = 2$, 1, 0.75, 0.6, 0.5 and 0.4 in the different subplots. In the first four subplots $N_\infty(r)$ grows faster than $m_\infty(r)$, in the last subplot the situation is reversed. $\log T_\infty(r)$ (solid), $\log m_\infty(r)$ (dashed) and $\log N_\infty(r)$ (dash-dotted) are plotted as functions of $\log r$.



Note, that these are independent of ε_j. Furthermore, putting together Theorems 3 and 7 we know that $m_\infty(r,(1-zA)^{-1})$ grows with order ω_m and type τ_m, where where

$$\begin{cases} \omega_m \le \frac{b}{\beta} & \text{for } \beta \le \delta, \\ \omega_m = \frac{b}{\delta} & \text{for } \beta > \delta. \end{cases}$$

We mention two special cases. First of all, if the size of the blocks $\tilde{J}_j$ is constant, then $b = 0$ and $\omega_N = \frac{1}{\beta}$, $\omega_m = 0$. This follows also from Theorems 4 and 6, from which we see that it actually holds as well in the more general case when the size of the blocks is bounded by a constant. Secondly, if $\lambda_j = 0$ then $\omega_m = \frac{b}{\delta}$, $\omega_N = 0$, and furthermore τ_m is, by Corollary 2, bounded from above and below by constants, which depend on γ, δ, a and b.

So for $\delta \ge \beta\frac{b}{b+1}$ we have $\omega_m \le \omega_N$, in which case the $T_\infty(r,(1-zA)^{-1})$ grows with $\omega = \omega_N$ and type $\tau = \tau_N$, where ω_N and τ_N are as in (15). Moreover, for $\delta < \beta\frac{b}{b+1}$ we have

$$\omega_m > \omega_N,$$

so $T_\infty(r,(1-zA)^{-1})$ grows with order $\omega = \omega_m = \frac{b}{\delta}$. In this case determining τ is difficult.

The results have been formulated for λ_j, ε_j and d_j satisfying (14). They can, of course, easily be adapted to give bounds for the growth of $N_\infty(r,(1-zA)^{-1})$ and $m_\infty(r,(1-zA)^{-1})$ in cases such that λ_j, ε_j and d_j have bounds of the form (14).

We do not need to have the canonical form $\tilde{J}$ available in order to be able to determine the growth of $T_\infty(r,(1-zA)^{-1})$. In Section 6 we look at the weighted shift operator W, and show that the resolvent of W is entire, so

$$N_\infty(r,(1-zW)^{-1}) = 0, \quad T_\infty(r,(1-zW)^{-1}) = m_\infty(r,(1-zW)^{-1}),$$

and that $T_\infty(r,(1-zW)^{-1})$ grows with order $\omega = 1$ and type $\tau = 1$.

Numerical calculations presented in Section 7 show that the bounds obtained hold in practice quite well, also in cases when the spectrum is not on the real axis.

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