Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 1998 A401

ON THE NUMERICAL SOLUTION OF INVOLUTIVE ORDINARY DIFFERENTIAL SYSTEMS: BOUNDARY VALUE PROBLEMS

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Jukka Tuomela: On the numerical solution of involutive ordinary differential systems: Boundary value problems; Helsinki University of Technology Institute of Mathematics Research Reports A401 (1998).

Abstract: *We analyse nonlinear two point boundary value problems using differential geometry, in particular jet spaces. This provides a complementary point of view to the more usual approach. We discuss generalized solutions that can arise in our framework, and note that in some situations these may be useful in numerical computations. Then we prove a general existence theorem for a certain class of problems using the concept of linking number.*

AMS subject classifications: primary 34B15, 58F18, secondary 65L10, 58A20.

Keywords: nonlinear boundary value problems, nonlinear boundary conditions, jet spaces, linking number.

ISBN 951-22-4281-8
ISSN 0784-3143
Edita, Espoo, Dec 1998

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1. Introduction

In recent years the boundary value problems which are nonlinear in highest derivatives and/or whose boundary conditions are nonlinear have received some attention, see [2], [3], [5], [7], [8], [9], [10], [11], [14], [15], [17], [18], [19], [20], [21], [22], [24], [27] and references therein. The older results are summarized in the well-known monograph [1] whose bibliography in turn contains numerous references to more classical material. We propose here a geometric approach to these kind of problems and examine certain consequences of this point of view. The tools used are very different from the ones used in the above mentioned references, hence we must spend quite a lot of space to introduce the appropriate framework. In spite of this the treatment cannot possibly be self contained as far as jet geometry is concerned and we refer the reader to [31] which contains an accessible introduction to these matters as well as an extensive list of relevant references. Of course we do not suggest that this geometric point of view replaces or should replace the older ones, rather we view it as an interesting complement to other approaches.

There are several consequences of our approach. First we can prove a general existence theorem for a certain class of boundary conditions. These boundary conditions look perhaps rather peculiar at first sight, but as our example shows such problems can arise in a natural way in variational problems. Another important point is that the generalized solutions obtained in this framework are really smooth curves in high dimensional space, and the singularities encountered in a classical setting are only seen when the curves are projected to appropriate subspaces. So whether these generalized solutions are 'physically reasonable' or not we shall indicate how they are sometimes quite useful in numerical computations. How to actually compute numerically in jet spaces is outside the scope of the present article and we refer to [31] and [30] for information on these matters.

The contents of the article is as follows. In section 2 we introduce the basic differential geometric tools that are needed in the analysis and explain in detail how standard problems are seen in this framework. We also make a few remarks on the use of transversality in the analysis boundary value problems. In section 3 we analyse two examples. In the first one we discuss the relevance of our generalized solutions and conclude with a modified problem where the solution is in fact classical, but using the shooting method to find it, one has generalized solutions in the intermediate stages. Since our generalized solutions are in fact smooth curves, the algorithm does not see the difference between generalized and classical solutions, and consequently the problems with (apparent) singularities are avoided. In the other example we present an elementary proof of an old existence result for a scalar equation, and in fact we can improve the statement by showing that there are at least two solutions. We note in particular that nonlinearity in boundary conditions for this type of problems does not make the problem more difficult than the linear one from the geometric point of view.

In section 4 we finally present the main result. Using the concept of a linking number we prove an existence result for a certain class of boundary conditions. The idea is that if the boundary conditions are linked, then moving the other with the flow one must sooner or later hit the other boundary condition. In particular no asymptotic properties or growth restrictions are required of the relevant vector field.

Acknowledgment All the figures were made with *Mathematica* [32].

2. Geometric formulation of the problem

2.1. **Differential systems in jet spaces.** Here we simply give the basic definitions and refer to [31] for a thorough discussion and motivation of these concepts as well as extensive further references. Basic material on standard differential geometry can be found in [26] and on jet geometry in [25]. All maps are assumed to be smooth, i.e. infinitely differentiable and all manifolds are smooth and without boundary. Let M be a manifold and $p \in M$; TM_p is the tangent space at p and TM is the tangent bundle. A distribution on M is a map which associates to each point $p \in M$ a certain subspace of TM_p. An integral manifold of some distribution is a connected submanifold whose tangent space coincides with the distribution. Let $\pi : \mathcal{E} \to \mathcal{B}$ be a bundle and let $J_q(\mathcal{E})$ be the bundle of q-jets of $\mathcal{E}$.

Definition 2.1. *A (partial) differential system (or equation) of order q on $\mathcal{E}$ is a submanifold $\mathcal{R}_q$ of $J_q(\mathcal{E})$.*

Let $\mathcal{E} = \mathbb{R} \times \mathbb{R}^n$ and let us denote the coordinates of $J_q(\mathcal{E})$ by $(x, y^1, \dots, y^n, y^1_1, \dots, y^n_q)$. Let us define the one forms

$$\alpha^i_j = dy^i_{j-1} - y^i_j dx \qquad i = 1, \dots, n \qquad j = 1, \dots, q \tag{2.1}$$

Let $p \in J_q(\mathcal{E})$ and $v_p \in (TJ_q(\mathcal{E}))_p$ and let us further set

$$\begin{aligned} \mathcal{C}_p &= \big\{ v_p \in (TJ_q(\mathcal{E}))_p \, \big| \, \alpha^i_j(v_p) = 0 \big\} \\ \mathcal{D}_p &= (T\mathcal{R}_q)_p \cap C_p \end{aligned} \tag{2.2}$$

$\mathcal{C}$ is called the Cartan distribution and $\dim(\mathcal{C}_p) = n + 1$. Suppose that we are given a system of k q'th order ordinary differential equations

$$f(x, y, y_1, \dots, y_q) = 0 \tag{2.3}$$

We interprete f as a morphism of bundles $J_q(\mathcal{E})$ and $\mathbb{R} \times \mathbb{R}^k$ which in terms of coordinates can be taken to be a map $\mathbb{R}^{(n+1)q+1} \to \mathbb{R}^k$. If $k = n$ we have the ordinary situation of n equations and n unknowns, and if $k > n$ we have an overdetermined system, or a DAE system. Recall that geometrically there is no difference between ODEs and DAEs, see [31] for more details.

The equation (2.3) defines a certain submanifold of $J_q(\mathcal{E})$ which we denote by $\mathcal{R}_q$ and in terms of coordinates $\mathcal{R}_q$ is simply given by $f^{-1}(0)$. Now we can define the solutions of our equations as follows.

Definition 2.2. *Let $\mathcal{R}_q \subset J_q(\mathcal{E})$ be involutive and suppose that the distribution $\mathcal{D}$ defined in* (2.2) *is one-dimensional. A solution of $\mathcal{R}_q$ is an integral manifold of $\mathcal{D}$.*

Recall that one-dimensional distributions always have integral manifolds, so solutions always exist as far as initial value problems are concerned. We cannot discuss the important notion of involution here and refer to [31] for ample explanations. Intuitively one might say that the system is involutive if it contains all of its differential consequences up to order q. This concept is not really needed below, but for completeness we stated the definition in appropriate generality.

2.2. **A class of boundary value problems.** We have seen that the geometric framework outlined above leads us to consider manifolds, one-dimensional distributions on them and the corresponding integral manifolds. Hence it is rather natural to formulate the concept of boundary condition directly with these terms.

Definition 2.3. *Let M be any smooth manifold and let $\mathcal{D}$ be a one-dimensional distribution on M. A submanifold $\mathcal{B}$ of M is a* boundary condition, *if $T\mathcal{B}_p \cap \mathcal{D}_p = \{0\}$ for all $p \in \mathcal{B}$.*

The conditions on $\mathcal{B}$ could be relaxed somewhat, but this definition is convenient for the purposes of the present paper. Intuitively one may express the content of the definition by saying that boundary conditions are differential equations which have no solutions. Perhaps this sounds a bit strange, but there is an analoguous situation in hyperbolic PDEs: namely, the Cauchy data should be prescribed on a manifold which does not contain any characteristics. Compared to the more usual ways to express the boundary conditions, our definition does not cover the cases where initial and final states are related, but is more general than the usual ones in other cases; for example the boundary condition may involve also the highest order derivatives.

We are going to study the following type of boundary value problems.

Definition 2.4. *A* two point boundary value problem *is the following collection of data:*

- *a manifold M of dimension m,*
- *a one-dimensional distribution $\mathcal{D}$ on M and*
- *two disjoint boundary conditions $\mathcal{B}_1$ and $\mathcal{B}_2$ such that* $\dim(\mathcal{B}_1) = k$ *and* $\dim(\mathcal{B}_2) = m - k - 1$.

A solution *of the two point boundary value problem is an integral manifold $\mathcal{I}$ of $\mathcal{D}$ such that $\mathcal{I} \cap \mathcal{B}_1 \neq \emptyset$ and $\mathcal{I} \cap \mathcal{B}_2 \neq \emptyset$.*

2.3. **Transversality.** We expect that in a 'regular' situation there are only a finite number of solutions to a given boundary value problem, or at least the solutions should be isolated. But is this nice situation typical among 'all' problems? The notion of transversality allows us to conclude that perturbing a little the given problem we can always recover the regular problem, with certain reservations to be discussed below.

Let us recall that if M and N are submanifolds of Q, then M and N intersect transversely, if for all $p \in M \cap N$, $TM_p + TN_p = TQ_p$. Moreover if the intersection is transversal, then $M \cap N$ is a submanifold of Q and $\dim(M \cap N) = \dim(M) + \dim(N) - \dim(Q)$. The following basic theorem says that in a 'typical' situation the intersection is transverse [13].

Theorem 2.1. *Let M and N be submanifolds of Q. Every neighborhood of the inclusion $i_N : N \to Q$ contains an embedding which is transverse to M.*

In a similar fashion it is seen that the condition $T\mathcal{B}_p \cap \mathcal{D}_p = \{0\}$ in Definition 2.3 can always by achieved (at least locally) by perturbing a little either the boundary condition $\mathcal{B}$ or the distribution $\mathcal{D}$ (or both).

In order to use transversality to study the behaviour of the solution set, it is convenient to introduce the following terms [4].

Definition 2.5. *Let p, $z \in M$ and let $\mathcal{D}$ be a distribution on M; $p \sim z$ if there is an integral manifold $\mathcal{I}$ of $\mathcal{D}$ such that p, $z \in \mathcal{I}$. Let $A \subset M$; the* saturation *of A is the following set*

$$\mathrm{sat}(A) = \big\{ p \in M \mid \exists\ z \in A \text{ such that } p \sim z \big\}$$

The existence of the solutions for the problem in Definition 2.4 can thus be expressed as follows:

> *The boundary value problem in Definition 2.4 has a solution if and only if*
>
> $$\mathrm{sat}(\mathcal{B}_1) \cap \mathcal{B}_2 \neq \emptyset \quad (\text{ equivalently } \quad \mathrm{sat}(\mathcal{B}_2) \cap \mathcal{B}_1 \neq \emptyset \text{ }).$$

Now the solution set in a nice situation behaves as follows.

Proposition 2.1. *Suppose that $\mathrm{sat}(\mathcal{B}_1)$ is a closed submanifold, $\mathcal{B}_2$ is closed and that the intersection of $\mathrm{sat}(\mathcal{B}_1)$ and $\mathcal{B}_2$ is transverse. Then $\mathrm{sat}(\mathcal{B}_1) \cap \mathcal{B}_2$ does not have any accumulation points. If in addition $\mathcal{B}_2$ is compact, then the intersection is a finite set.*

In other words, when the intersection is transverse the solutions are isolated and if one of the boundary conditions is compact there is only a finite number of solutions.

Proof. By counting the dimensions it is seen that the intersection must be zero dimensional. Let $p \in \mathrm{sat}(\mathcal{B}_1) \cap \mathcal{B}_2$. Then there is a neighborhood of $p \in U \subset M$ and the coordinate map such that $\mathrm{sat}(\mathcal{B}_1) \cap U$ is represented by the first coordinates and $\mathcal{B}_2 \cap U$ by the last coordinates [16]. Hence there are no other points of intersection in U. The finiteness property follows because on a compact manifold there cannot be an infinite number of isolated points. ☐

Note that the condition that $\mathcal{B}_2$ be a closed submanifold is not restrictive at all in practice because usually boundary conditions are given as zero sets and thus by Sard's theorem they usually satisfy the required condition. As an immediate corollary of the above proposition and Theorem 2.1 we get

Corollary 2.1. *Consider the problem in Definition 2.4 and suppose that* $\operatorname{sat}(\mathcal{B}_1)$ *is a closed submanifold. Then by perturbing* $\mathcal{B}_2$ *arbitrarily little the intersection of* $\operatorname{sat}(\mathcal{B}_1)$ *and* $\mathcal{B}_2$ *becomes transverse.*

Hence if the saturation is nice then any problem can be approximated by a nice problem. Of course in general the saturation can be a very complicated set, for instance if the system is chaotic. Another indication of complexities that may arise is given by the fact that the space $M/\!\sim$ may be non-Hausdorff, see [4] for an example. However, if the saturation as a whole is not a closed submanifold we may still get a meaningful problem by restricting our attention to an appropriate subset.

2.4. **Examples.** Let us then see how the standard two point boundary value problem looks like in this framework. Consider the following scalar problem.

$$\begin{cases} y'' - f(x,y,y') = 0 \\ y(a) = c \\ y(b) = d \end{cases}$$

Hence $\mathcal{R}_2 \subset J_2(\mathbb{R}\times\mathbb{R}) \simeq \mathbb{R}^4$ is defined by $y_2 - f(x,y,y_1) = 0$ and the boundary conditions by

$$\mathcal{B}_1 \ : \ \begin{cases} y_2 - f(x,y,y_1) = 0 \\ x - a = 0 \\ y - c = 0 \end{cases} \qquad\qquad \mathcal{B}_2 \ : \ \begin{cases} y_2 - f(x,y,y_1) = 0 \\ x - b = 0 \\ y - d = 0 \end{cases}$$

Clearly $\dim(\mathcal{B}_1) = \dim(\mathcal{B}_2) = 1$ and $\dim(\mathcal{R}_2) = 3$, so the dimensions of the problem are 'right'. Since y_2 is given explicitly $\mathcal{R}_2$ and the corresponding dirstribution can be diffeomorphically projected to $J_1(\mathbb{R}\times\mathbb{R})$. Doing this we can directly see the problem in three dimensional space. The distribution is given by the nullspace of the following matrix.

$$A = \begin{pmatrix} -y_1 & 1 & 0 & 0 \\ -y_2 & 0 & 1 & 0 \\ -\partial f/\partial x & -\partial f/\partial y & -\partial f/\partial y_1 & 1 \end{pmatrix}$$

So using the projection $\pi_1^2 \, : \, (x,y,y_1,y_2) \mapsto (x,y,y_1)$ we get the following problem in $J_1(\mathbb{R}\times\mathbb{R}) \simeq \mathbb{R}^3$.

$$\begin{cases} \mathcal{D} = \operatorname{span}\big(1, y_1, f(x,y,y_1)\big) \\[2ex] \mathcal{B}_1 \ : \ \begin{cases} x - a = 0 \\ y - c = 0 \end{cases} \qquad\qquad \mathcal{B}_2 \ : \ \begin{cases} x - b = 0 \\ y - d = 0 \end{cases} \end{cases}$$

Hence the boundary conditions are vertical lines and the distribution is defined and one-dimensional in all of $\mathbb{R}^3$. Similarly if we have the problem

$$\begin{cases} \mathcal{R}_q \ : \ y_q - f(x,y,\dots,y_{q-1}) = 0 \\ \mathcal{B}_1 \ : \ \begin{cases} y_q - f(x,y,\dots,y_{q-1}) = 0 \\ g_1(x,y,\dots,y_{q-1}) = 0 \end{cases} \qquad \mathcal{B}_2 \ : \ \begin{cases} y_q - f(x,y,\dots,y_{q-1}) = 0 \\ g_2(x,y,\dots,y_{q-1}) = 0 \end{cases} \end{cases}$$

then this is equivalent to the problem

$$\begin{cases} \mathcal{D} = \mathrm{span}\big(1, y_1, \dots, y_{q-1}, f(x,y,\dots,y_{q-1})\big) \\ \mathcal{B}_1 \ : \ g_1(x,y,\dots,y_{q-1}) = 0 \quad \mathcal{B}_2 \ : \ g_2(x,y,\dots,y_{q-1}) = 0 \end{cases} \tag{2.4}$$

In standard problems of course one can get rid of the highest derivatives in the boundary conditions using the differential equation. In the general case, however, this is not possible, and thus we can write the general form of the problem as follows.

$$\begin{cases} \mathcal{R}_q \ : \ f(x,y,\dots,y_q) = 0 \\ \mathcal{B}_1 \ : \ \begin{cases} f(x,y,\dots,y_q) = 0 \\ g_1(x,y,\dots,y_q) = 0 \end{cases} \quad \mathcal{B}_2 \ : \ \begin{cases} f(x,y,\dots,y_q) = 0 \\ g_2(x,y,\dots,y_q) = 0 \end{cases} \end{cases} \tag{2.5}$$

Writing in this way the statement that boundary conditions are just certain kind of differential equations becomes quite natural. Let us stress that geometrically the presence (resp. absence) of y_q in g_i does not a priori make the problem more difficult (resp. easier).

3. New solutions and proofs

In this section we analyse two examples to show what kind of benefit our geometric formulation can have in the study of two point boundary value problems. In the first one the main point is that our new generalized solutions are in fact smooth curves in a higher dimensional space, so the singularities encountered in the classical setting are avoided. Whether these generalized solutions are in fact 'physically reasonable' depends of course on the particular application, but we show that in any case they are useful in numerical computations.

The other example is about a new and elementary proof of an old result about scalar equation. We show that in fact we can get a stronger result in a particular case and that the same type of reasoning extends in a straightforward way to the case where one of the boundary conditions is one dimensional.

3.1. **Periodic solution.** Let us consider an example taken from [10]

$$\mathcal{R}_1 \ : f(x,y,y_1) = 2(y_1-y)^3 - 9(y_1-y)^2 + 12(y_1-y) - x = 0 \tag{3.1}$$

with the boundary condition $y(0) = y(9)$.[1] In fact this is *not* the type of a boundary condition which is covered by Definition 2.3. However, we take it up because this problem illustrates quite well how the solutions in our sense can be different from the classical solutions as well as various generalized solutions, and what are the consequences this approach from the point of view of applications.

Let $\mathcal{R}_1 = f^{-1}(0) \subset J_1(\mathbb{R} \times \mathbb{R}) \simeq \mathbb{R}^3$. Obviously $\mathcal{R}_1$ is a smooth manifold and the distribution $\mathcal{D}$ is given by the nullspace of

$$A = \begin{pmatrix} -y_1 & 1 & 0 \\ -1 & -b & b \end{pmatrix} \quad \text{where} \quad b = 6(y_1 - y)^2 - 18(y_1 - y) + 12$$

Evidently $\mathcal{D}$ is one-dimensional in the whole of $\mathcal{R}_1$, and it is spanned by the vector field $V = (b, by_1, 1 + by_1)$. Hence we can consider the following system of ODEs

$$\begin{cases} x' = 6(y_1 - y)^2 - 18(y_1 - y) + 12 \\ y' = \big(6(y_1 - y)^2 - 18(y_1 - y) + 12\big)\, y_1 \\ y_1' = 1 + \big(6(y_1 - y)^2 - 18(y_1 - y) + 12\big)\, y_1 \end{cases} \tag{3.2}$$

Denote the auxiliary independent variable by s, let $z = (x, y, y_1)$ and let

$$\mathcal{B}_1 = \big\{(x, y, y_1) \in \mathcal{R}_1 \,|\, x = 0\big\} \quad \text{and} \quad \mathcal{B}_2 = \big\{(x, y, y_1) \in \mathcal{R}_1 \,|\, x = 9\big\}$$

Then the problem can be formulated as follows: find a solution z of (3.2) such that $z(0) \in \mathcal{B}_1$, $z(s^*) \in \mathcal{B}_2$ and $y(0) = y(s^*)$. Using this formulation we shall prove

Proposition 3.1. *The problem 3.1 with condition $y(0) = y(9)$ has at least one solution.*

Proof. We see immediately that $y_1(s) - y(s) = s$ and thus $x(s) = 2s^3 - 9s^2 + 12s$ which in turn implies that $s^* = 3$. This proves also that all solutions starting at $\mathcal{B}_1$ eventually reach $\mathcal{B}_2$ which is not a priori clear. Then one checks that if the initial point $p = (0, y, y_1) \in \mathcal{B}_1$ is taken such that y is sufficiently big then the corresponding solution satisfies $y(9) > y(0)$ and if y is sufficiently small the corresponding solution satisfies $y(9) < y(0)$. Hence by continuity there is (at least) one $p \in \mathcal{B}_1$ such that the corresponding solution satisfies $y(0) = y(9)$. ☐

Note that 'time' x flows backwards when $1 < s < 2$ which explains the remark in [10] about the inexistence of the classical solution on a larger time interval than $[0, 5]$ because $x(1) = 5$. In our framework we have smooth solutions on $\mathcal{R}_1$ independent of the time interval. The singularities are only seen when the solution is projected to (x, y) – plane using the standard projection $\pi : (x, y, y_1) \mapsto (x, y)$. In figure 3.1 on the left there is $\mathcal{R}_1$ and in figure 3.2 there is a solution computed by the shooting method. In the present case the system is very stable backwards in s, hence the convergence is very fast.

[1] The interval in the original problem was $[0, 6]$, but changing it to $[0, 9]$ does not alter the nature of the problem.

Obviously the generalized solution obtained here is different from the one obtained in [10] and it depends on the application which one if any is the relevant one.[2] Of course it is also possible that at $x = 5$ the model simply ceases to be physically relevant and there is no point of trying to go beyond that point. However, from the numerical point of view our framework would still be useful. Namely, numerically one usually encounters difficulties when approaching a singularity and hence in a classical setting the solution near $x = 5$ would perhaps be inaccurate. In jet context the points at $x = 5$ are simply regular points, so the numerical solution should be as accurate as everywhere else. Similar situation arises when analysing impasse points, see [28] and references therein for more information on impasse points. Jets are also useful in the resolution of other types of singularities, see [29].

Let us give another example of the numerical usefulness of the jet point of view. Consider the following modified problem

(3.3)

$$f(x, y, y_1) = 2(y_1 - y)^3 - 9\big(1 - \varphi(y + y_1)\big)(y_1 - y)^2 + 12(y_1 - y) - x = 0$$

where

$$\varphi(z) = \begin{cases} \exp\left(\dfrac{-5z^2}{81 - 9z^2}\right) & , \ |z| < 3 \\ 0 & , \ |z| \geq 3 \end{cases}$$

The manifold is shown in figure 3.1 on the right and one may expect that there is a classical solution. However, in the initial guess of the shooting method the solution does not stay in the corridor and hence the projection is singular. The next iteration, however, already gives a classical solution, see figure 3.2 on the right. So these generalized solutions might appear in the intermediate stages in the numerical solution even though the the actual solution is smooth in the classical sense. Because these generalized solutions are smooth in the relevant manifold, they are indistinguishable from the classical solutions for the algorithm and therefore the (apparent) singularities do not make the problem numerically harder.

Finally note that the numerical computations could be performed without explicit knowledge of the vector field in (3.2). This is explained in [31] to which we refer for further details.

3.2. **Problems with fixed initial and final 'time'.** A lot of problems arising in practice are of the following type.

$$(3.4) \qquad \begin{cases} \mathcal{D} = \operatorname{span}\big(1, y_1, \dots, y_{q-1}, f(x, y, \dots, y_{q-1})\big) \\ \mathcal{B}_1 : \begin{cases} x - a = 0 \\ g_1(y, \dots, y_{q-1}) = 0 \end{cases} \qquad \mathcal{B}_2 : \begin{cases} x - b = 0 \\ g_2(y, \dots, y_{q-1}) = 0 \end{cases} \end{cases}$$

[2]In [10] the authors remark that equation (3.1) admits smooth solution curves implicitly given by an equation of the form $F(x, y) = c$. However, this seems to be incorrect, see figure 3.2.

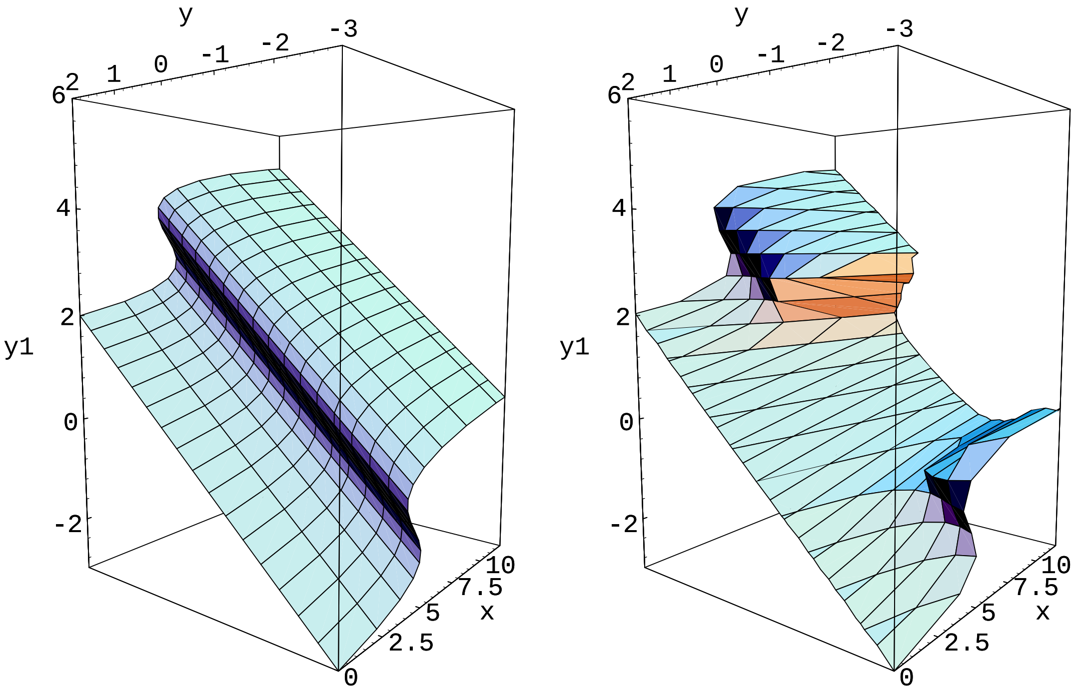

FIGURE 3.1. Differential equations (3.1) (on the left) and (3.3) (on the right).

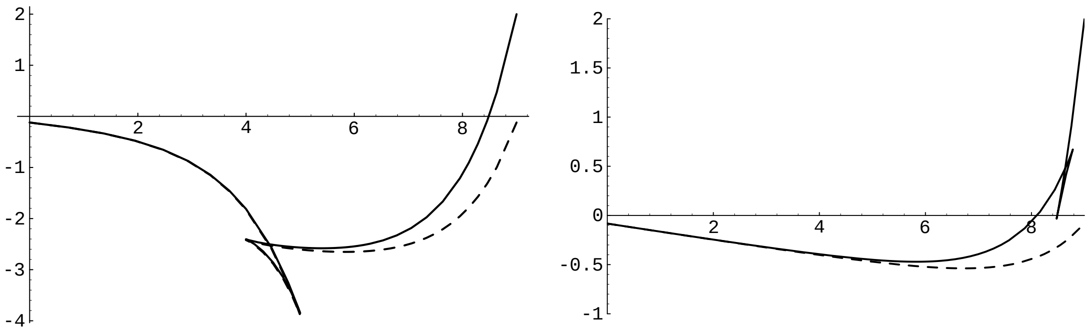

FIGURE 3.2. Solutions of the equations (3.1) (on the left) and (3.3) (on the right) by the shooting method. The solid lines are the first guesses and the dashed lines the second iterations whose accuracy is already quite satisfactory.

To analyse these problems in more detail let us first introduce some notations. The (vector field spanning the) distribution naturally defines a flow $\mathbb{R} \times J_{q-1}(\mathbb{R} \times \mathbb{R}^n) \to J_{q-1}(\mathbb{R} \times \mathbb{R}^n)$, but for the problems of this type another flow is more approriate. Recall that $J_{q-1}(\mathbb{R} \times \mathbb{R}^n) \simeq \mathbb{R} \times \mathbb{R}^{nq}$ and define $\psi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{nq} \to \mathbb{R}^{nq}$ as follows. Let z_p be a parametrization of an integral manifold of the distribution in (3.4) with parameter x which satisfies the initial condition $z_p(s) = p$. Then we define $\psi(x, s, p) = z_p(x)$. Further if we fix the parameters we can define $\psi_a^b(p) = \psi(b,a,p)$. Note that ψ and ψ_a^b may not be defined for all values of the arguments, but for simplicity of notation this is not indicated. Let us further set

$$T_a = \{p \in J_{q-1}(\mathbb{R}\times\mathbb{R}^n) \,|\, x = a\} \simeq \mathbb{R}^{nq}$$

Hence $\psi_a^b : T_a \to T_b$. Now $\mathcal{B}_1 \subset T_a$ and $\mathcal{B}_2 \subset T_b$ and they can be identified with submanifolds M_1 and M_2 defined as

$$M_i = \{p \in \mathbb{R}^{nq} \,|\, g_i(p) = 0\}$$

Then we finally can formulate the existence of the solutions of problem (3.4) with these terms.

> *Problem* (3.4) *has a solution if and only if there is $p \in M_1$ such that $\psi_a^b(p) \in M_2$.*

Let us consider the scalar second order case.

$$(3.5)\qquad \begin{cases} \mathcal{D} = \operatorname{span}\big(1, y_1, f(x,y,y_1)\big) \\ \\ \mathcal{B}_1 \ : \ \begin{cases} x - a = 0 \\ g_1(y,y_1) = 0 \end{cases} \qquad \mathcal{B}_2 \ : \ \begin{cases} x - b = 0 \\ g_2(y,y_1) = 0 \end{cases} \end{cases}$$

Let us further define the following sets.

$$\Omega_+ = \big\{p \in \mathbb{R}^2 \,|\, g_2(p) > 0\big\}$$
$$\Omega_- = \big\{p \in \mathbb{R}^2 \,|\, g_2(p) < 0\big\}$$

Then we can formulate

Proposition 3.2. *Suppose that M_i in the problem* (3.5) *are connected and ψ_a^b is defined for all $p \in \mathbb{R}^2$. If there is $p_+ \in M_1 \cap \Omega_+$ such that $\psi_a^b(p_+) \in \Omega_+$ and $p_- \in M_1 \cap \Omega_-$ such that $\psi_a^b(p_-) \in \Omega_-$, then the problem* (3.5) *has at least one solution.*

Proof. Since M_1 is connected, then by continuity there must be $p \in M_1$ such that $\psi_a^b(p) \in M_2$. □

Of course this proposition admits an easy generalization to the general case if one of the boundary conditions is one-dimensional. Note also that the connectedness is no real restriction because we could examine each component seprately as in the example below. In spite of the elementary character of the above proposition we can in fact use it to improve a result from [11]. Consider the following problem.

$$(3.6)\qquad \begin{cases} \mathcal{D} = \operatorname{span}\big(1, y_1, f(x,y,y_1)\big) \\ \\ \mathcal{B}_1 \ : \ \begin{cases} x = 0 \\ y_1^2 - y^2 + c = 0 \end{cases} \qquad \mathcal{B}_2 \ : \ \begin{cases} x = 1 \\ 2y_1^2 - 2yy_1 + d = 0 \end{cases} \end{cases}$$

Here c and d are arbitrary real parameters and the vector field $V = \big(1, y_1, f(x,y,y_1)\big)$ is supposed to satisfy the condition $|f(x,y,y_1)| \le w$ for $(x,y,y_1) \in [0,1]\times\mathbb{R}^2$ for some $w > 0$. In [11] it was proved using degree theory that the problem has at least one solution. We shall prove that

Proposition 3.3. *If $c > 0$ and $d < 0$, then the problem* (3.6) *has at least two solutions.*

Proof. First let us note that because f is bounded ψ_0^1 is defined for all $p \in \mathbb{R}^2$. Let us further define the sets

$$\begin{aligned} M_1^c =&\{(y, y_1) \in \mathbb{R}^2 \mid y_1^2 - y^2 + c = 0\} \\ M_2^d =&\{(y, y_1) \in \mathbb{R}^2 \mid 2y_1^2 - 2yy_1 + d = 0\} \end{aligned}$$

Suppose that $c > 0$ and $d < 0$, see figure 3.3. Let us denote the lower (resp. upper) half plane branch of M_2^d by M_{2-}^d (resp. M_{2+}^d) and the left (resp. right) half plane branch of M_1^c by M_{1-}^c (resp. M_{1+}^c). Let us define the sets

$$\begin{aligned} \Omega_+ &= \left\{(y, y_1) \in \mathbb{R}^2 \mid y_1 > \tfrac{1}{2}\, y + \tfrac{1}{2}\, \sqrt{y^2 - 2d}\right\} \\ \Omega_- &= \left\{(y, y_1) \in \mathbb{R}^2 \mid y_1 < \tfrac{1}{2}\, y + \tfrac{1}{2}\, \sqrt{y^2 - 2d}\right\} \end{aligned}$$

Let $a > 0$ and put $\tilde{p} = (-a, \sqrt{a^2 - c}) \in M_{1-}^c$. Then using this initial point and taking a sufficiently big we have

$$y_1(x) \geq y_1(0) - \int_0^x |f\big(t, y(t), y_1(t)\big)|dt \geq \tfrac{1}{2}\, a - wx$$

Similarly we get $y_1(x) \leq a + wx$. Hence

$$y(x) = -a + \int_0^x y_1(t)dt \leq -a + ax + \tfrac{1}{2}\, wx^2$$

Thus $y(1) \leq \frac{1}{2}\, w$ and $y_1(1) \geq \frac{1}{2}\, a - w$ which implies that $\psi_0^1(\tilde{p}) \in \Omega_+$ for sufficiently big a.

Then taking still $a > 0$ and using $\hat{p} = (-\sqrt{a^2 + c}, -a) \in M_{1-}^c$ as an initial point we easily deduce that

$$y_1(x) = -a + \int_0^x f\big(t, y(t), y_1(t)\big)dt \leq -a + wx$$

Clearly $y_1(1) < 0$ for sufficiently big a and hence $\psi_0^1(\hat{p}) \in \Omega_-$. By the previous proposition we conclude that there is a $p \in M_{1-}^c$ such that $\psi_0^1(p) \in M_{2+}^d$. Reasoning similarly with sets M_{2-}^d, M_{1+}^c and

$$\begin{aligned} D_+ &= \left\{(y, y_1) \in \mathbb{R}^2 \mid y_1 > \tfrac{1}{2}\, y - \tfrac{1}{2}\, \sqrt{y^2 - 2d}\right\} \\ D_- &= \left\{(y, y_1) \in \mathbb{R}^2 \mid y_1 < \tfrac{1}{2}\, y - \tfrac{1}{2}\, \sqrt{y^2 - 2d}\right\} \end{aligned}$$

produces another solution. □

The same conclusion holds also in the limiting cases $c = 0$ or $d = 0$. For other values of the parameters the above simple argument does not work because of the way the asymptotes of the curves happen to coincide. This, however, is naturally not a generic situation, in other words perturbing the coefficients of the equations slightly makes the method of proof applicable. Recall that in [11] the degree of a map associated to this problem was two, hence in a sense it is 'natural' that there are (at least) two solutions.

Note that we needed only information about asymptotic properties of the curves and hence the fact that the boundary conditions were

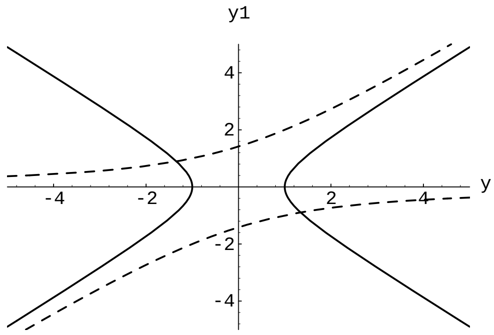

FIGURE 3.3. The curves M_1^1 (solid line) and M_2^{-2} (dashed).

nonlinear was rather irrelevant. Also nonlinearities need not be polynomial. For example consider the problem

$$(3.7)\quad \begin{cases} \mathcal{D} = \operatorname{span}\big(1, y_1, f(x,y,y_1)\big) \\ \mathcal{B}_1 \; : \; \begin{cases} x = 0 \\ y^2 - e^{y_1} = 0 \end{cases} \qquad \mathcal{B}_2 \; : \; \begin{cases} x = 1 \\ 2y_1^2 + 2yy_1 - 1 = 0 \end{cases} \end{cases}$$

Then the same argument as above easily yields that the problem (3.7) has at least *three* solutions.

The whole process can be interpreted as follows: in the beginning some manifolds $M_1 = \psi_a^a(M_1)$ and M_2 intersect and we hope to find 'natural' conditions on the flow such that $\psi_a^b(M_1)$ and M_2 also intersect for some values $b \neq a$. Hence it is seen that with different boundary conditions we might require rather different conditions on the flow as far as the direction is concerned. The conditions on the size are more 'uniform' because they are required to guarantee that the flow is well-defined for the relevant values of the parameters.

As noted above the method of proof extends to the case when one of the boundary conditions is one dimensional, because then the other boundary condition separates the relevant space into different components. In general one needs tools from algebraic topology to study the intersections; this is beyond the scope of the present article.

Since the proof relies on asymptotic properties of the boundary conditions, we cannot get any upper bound on the number of solutions. Also the proof cannot be applied to the cases where at least one of the boundary conditions is compact. In the next section we study one type of problems where both boundary conditions are compact and hence asymptotic properties do not play any role.

4. Existence theorem for a certain class of problems

4.1. **Linking number.** We review briefly the basic definitions and refer to [26, vol. 1] and [6] for more details. All manifolds in this section are assumed to be connected, oriented and compact. Let M and N be two m–dimensional manifolds and let $f : M \to N$. This induces a map between tangent spaces: $df_p : TM_p \to TN_{f(p)}$.

Definition 4.1. *Let z be a regular value of f. The degree of f is*

$$\deg(f) = \sum_{p_i \in f^{-1}(z)} \operatorname{sign}\big(\det(df_{p_i})\big)$$

Note that regular values exist, by Sard's theorem, that the sum above is finite and that the result does not depend on the choice of the regular value. Recall also that points not in the image of f are also called regular values, and in that case $\deg(f) = 0$.

Definition 4.2. *Maps $f,\ g : M \to N$ are homotopic, if there is a map $h : M \times [0,1] \to N$ such that $h(p,0) = f(p)$ and $h(p,1) = g(p)$. Two submanifolds are said to be homotopic if their inclusion maps are homotopic.*

We will need the following basic property.

Theorem 4.1. *If f and g are homotopic, then $\deg(f) = \deg(g)$.*

Let M and N be manifolds of dimensions k and $m - k - 1$, let $f : M \to \mathbb{R}^m$, $g : N \to \mathbb{R}^m$ and $f(M) \cap g(N) = \emptyset$. Let us further define a map

$$\alpha : M \times N \to S^{m-1} \quad , \quad \alpha(p,z) = \frac{g(z) - f(p)}{|g(z) - f(p)|}$$

where S^{m-1} is the unit sphere of dimension $m - 1$.

Definition 4.3. *The linking number of f and g is $\operatorname{link}(f,g) = \deg(\alpha)$. If M and N are disjoint submanifolds of $\mathbb{R}^m$ we define $\operatorname{link}(M,N) = \operatorname{link}(i_1(M), i_2(N))$ where i_1 and i_2 are the inclusion maps.*

Let h_1 (resp. h_2) define a homotopy between f and $\bar{f}$ (resp. g and $\bar{g}$). Then from Theorem 4.1 we deduce

Theorem 4.2. *If for all $t \in [0,1]$*

$$\{h_1(p,t) \,|\, p \in M\} \cap \{h_2(z,t) \,|\, z \in N\} = \emptyset$$

then $\operatorname{link}(f,g) = \operatorname{link}(\bar{f},\bar{g})$.

We shall need the following simple consequence of this Theorem.

Lemma 4.1. *Let M and N be submanifolds of $\mathbb{R}^m$ and suppose that there exist $r > 0$ such that $|p| < r$ for all $p \in M$ and $|z| > r$ for all $z \in N$. Then $\operatorname{link}(M,N) = 0$.*

Proof. We can shrink M to a point with a homotopy taking the inclusion to a constant map without violating the condition in Theorem 4.2. But if f or g is constant then α cannot be onto. ☐

Evidently an analoguous statement is valid in the case M and N are separated by a hyperplane. Note finally that two manifolds can be nontrivially linked although their linking number is zero, see [6, p. 293] for an example.

4.2. **Main theorem.** Let us consider the problem (2.4) where the boundary conditions are compact, oriented and connected.

Theorem 4.3. *If* $\operatorname{link}(\mathcal{B}_1, \mathcal{B}_2) \neq 0$*, then the problem* (2.4) *has at least two solutions.*

Proof. Let V be the unit vector field such that

$$V_p \in \mathcal{D}_p = \operatorname{span}\big(1, y_1, \dots, y_{q-1}, f(x, y, \dots, y_{q-1})\big)$$

Hence V is a vector field on $J_{q-1}(\mathbb{R}\times\mathbb{R}^n) \simeq \mathbb{R}^m$ where $m = nq+1$. Since V is globally Lipschitz we have the flow $\psi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and the corresponding one parameter family of diffeomorphisms $\psi^t : \mathbb{R}^m \to \mathbb{R}^m$ which are defined for all $t \in \mathbb{R}$.

Take $r > 0$ such that $|z| < r$ for all $z \in \mathcal{B}_1 \cup \mathcal{B}_2$. Now choosing $b = (1, 0, \dots, 0)$ or $b = (-1, 0, \dots, 0)$ we have $\langle V_p, b\rangle > 0$ for all $p \in \mathbb{R}^m$ and hence by Lemmas 4.2 and 4.3 below for any $p \in \mathcal{B}_1$ there is t_p^* and an open neighborhood U_p of p such that $|\psi^{t_p^*}(z)| > r$ for all $t \geq t_p^*$ and for all $z \in U_p$. By compactness there is a finite subcover

$$\mathcal{B}_1 \subset \bigcup_{i=1}^{j} U_{p_i}$$

Defining $t^* = \max_i t_{p_i}^*$ we have $|\psi^{t^*}(p)| > r$ for all $t \geq t^*$ and for all $p \in \mathcal{B}_1$. Now $\psi : [0, t^*] \times \mathbb{R}^m \to \mathbb{R}^m$ defines a homotopy between $\mathcal{B}_1$ and $\psi^{t^*}(\mathcal{B}_1)$. By Lemma 4.1 $\operatorname{link}(\psi^{t^*}(\mathcal{B}_1), \mathcal{B}_2) = 0$ which implies that the condition in Theorem 4.2 is violated. Thus there is $p \in \mathcal{B}_1$ and $\tilde{t} \in [0, t^*]$ such that $\psi(\tilde{t}, p) \in \mathcal{B}_2$. Reasoning similarly with $-V$ produces another solution. ☐

Lemma 4.2. *Let* V *be a unit vector field on* $\mathbb{R}^m$ *and suppose that there exist a unit vector* b *such that* $\langle V, b\rangle > 0$*. Then for any* $r > 0$ *and* $p \in \mathbb{R}^m$ *there exists* t_p *such that* $|\psi^{t_p}(p)| > r$ *for all* $t \geq t_p$*.*

Proof. Suppose on the contrary that there exist a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $|\psi^{t_n}(p)| \leq r$. Passing to a subsequence if necessary there is thus z such that $\lim_{n\to\infty} \psi^{t_n}(p) = z$. Let us prove that in fact $\lim_{t\to\infty} \psi^t(p) = z$. Let $p_n = \psi^{t_n}(p)$, consider the interval $[t_n, t_{n+1}]$ and let $0 \leq h \leq t_{n+1} - t_n$. Now $|\psi^h(p_n) - p_n| \leq |p_{n+1} - p_n|$ because

$$|\psi^h(p_n) - p_n| \geq |\langle \psi^h(p_n) - p_n, b\rangle| = \int_0^h \langle V\big(\psi^s(p_n)\big), b\rangle ds > 0$$

and hence $|\psi^h(p_n) - p_n|$ is a monotonically increasing function of h. This implies that

$$|\psi^h(p_n) - z| \leq |p_{n+1} - p_n| + |p_n - z|$$

Hence for any $\varepsilon > 0$ there is n such $|\psi^t(p) - z| < \varepsilon$ for $t \geq t_n$ or in other words $\lim_{t\to\infty} \psi^t(p) = z$. Now taking any fixed $h > 0$ we have

$$z = \lim_{t\to\infty} \psi^{t+h}(p) = \lim_{t\to\infty} \psi^h \circ \psi^t(p) = \psi^h\Big(\lim_{t\to\infty} \psi^t(p)\Big) = \psi^h(z)$$

But this is impossible because $V(z) \neq 0$ and hence $\psi^h(z) \neq z$. ☐

Lemma 4.3. *Let V satisfy the hypothesis of the previous lemma, let $p \in \mathbb{R}^n$ and $|p| \leq r$. Then there is an open neighborhood U_p of p and t_p^* such that $|\psi^t(z)| > r$ for all $z \in U_p$ and $t \geq t_p^*$.*

Proof. By the previous lemma there is t_p^* such that $|\psi^{t_p^*}(p)| \geq 6r$. Then given ε and taking U_p sufficiently small we deduce by the continuity of $\psi^{t_p^*}$ that $|\psi^{t_p^*}(z)| \geq 6r - \varepsilon$ for all $z \in U_p$. Shrinking U_p further if necessary we also have $|z - p| < \varepsilon$. Recall that $|\psi^t(z) - z|$ is a monotonically increasing function of t. Hence taking any $c \in \mathbb{R}^n$ with $|c| \leq r$ and $t \geq t_p^*$ we get the following estimates

$$\begin{aligned} |\psi^t(z) - c| &\geq |\psi^t(z) - z| - |z - c| \geq |\psi^{t_p^*}(z) - z| - |p - c| - \varepsilon \\ &\geq |\psi^{t_p^*}(z)| - |z| - |p - c| - \varepsilon \geq 6r - |p| - |p - c| - 3\varepsilon \\ &\geq 3r - 3\varepsilon \end{aligned}$$

Hence $|\psi^t(z) - c| > 2r$ which implies that $|\psi^t(z)| > r$. □

Note in particular that there are no growth restrictions on f. Intuitively the reason is that the solution is found before the integral manifolds can escape to infinity, hence the asymptotic properties of the integral manifolds or f do not play any role.

In the problems we have in mind the manifolds will be defined as zero sets of some maps. For these manifolds the orientability is no restriction because of the following result [6].

Theorem 4.4. *Let $k < n$, let zero be a regular value of $f : \mathbb{R}^n \to \mathbb{R}^k$ and let $M = f^{-1}(0)$ be nonempty. Then M is an orientable submanifold of $\mathbb{R}^n$.*

Of course the connectedness is not really a restriction either because one could simply examine each component of the boundary conditions separately.

As an example of the application of the theorem consider the following problem.

$$\left\{ \begin{array}{l} \mathcal{D} = \operatorname{span}\left(1, y_1, \frac{39}{100}\, y + \frac{31}{50}\, x^2 - \frac{43}{16}\right) \\ \mathcal{B}_1 \ : \ \left\{ \begin{array}{l} 2y + x^2 = 0 \\ y_1^2 + 2xy_1 + \frac{1}{16}\, x^4 + x^2 - 2 = 0 \end{array} \right. \\ \mathcal{B}_2 \ : \ \left\{ \begin{array}{l} y - 2x^2 + 1 = 0 \\ y_1^2 - 8xy_1 + 8x^4 + 16x^2 + 8x - 8 = 0 \end{array} \right. \end{array} \right. \tag{4.1}$$

This problem is the Euler equation for the variational problem

$$J(y) = \int_a^b L(x, y, y_1)dx$$

where L is given by

$$\begin{aligned} L(x, y, y_1) =& \tfrac{1}{2}\, y_1^2 - \tfrac{1}{5}\Big(x^3 + 2xy + \tfrac{127}{80}\, x + 4\Big)y_1 - \\ & \tfrac{1}{50}\, x^4 + \tfrac{1}{50}\, x^2 y - \tfrac{58}{25}\, x^2 - \tfrac{1}{200}\, y^2 - \tfrac{4}{5}\, x - \tfrac{601}{200}\, y + 1 \end{aligned}$$

and $(a, y(a))$ (resp. $(b, y(b))$) is constrained to lie on the curve $2y+x^2 = 0$ (resp. $y - 2x^2 + 1 = 0$). The other equations in the boundary conditions are the transversality conditions [12]. In this case the Euler equations turned out to be linear, but it would be easy to construct similar examples with nonlinear Euler equations.

Here boundary conditions are one-dimensional closed curves, see figure 4.1. In other words they are knots (or to be more precise they are unknots!) and the link is the simplest possible nontrivial link, called Hopf link. In this case there are various other ways to define the linking number than the one given above. In [23] there are 7 alternative definitions which are all equivalent to Definition 4.3 (at least up to sign), but of course in a specific situation some might be much more convenient than the others to work with. Anyway not all of these definitions extend to the many dimensional case which mainly intrests us here. The discussion of how to actually compute the linking number in various situations would lead us into the realm of algebraic topology, and is beyond the scope of the present article.

Theorem 4.3 admits the following straightforward generalization.

Corollary 4.1. *Suppose that* $\pi^q_{q-1} : \mathcal{R}_q \to J_{q-1}(\mathcal{E})$ *is a diffeomorphism and let* $M_i = \pi^q_{q-1}(\mathcal{B}_i)$. *Suppose further that* g_i *in* (2.5) *do not depend on the highest derivatives. If* $\operatorname{link}(M_1, M_2) \neq 0$, *then the problem* (2.5) *has at least two solutions.*

Proof. Now $d\pi^q_{q-1} : (T\mathcal{R}_q)_p \to (TJ_{q-1}(\mathcal{E}))_{\pi^q_{q-1}(p)}$ is an isomorphism, hence there is a bijective correspondence between the integral manifolds of $\mathcal{D}$ and $d\pi^q_{q-1}(\mathcal{D})$. Further from the hypothesis it follows that $\mathcal{B}_i$ (resp. M_i) satisfy the conditions of Definition 2.3 for the original (resp. projected) problem. Hence by previous Theorem the projected problem has at least two solutions. Lifting these integral manifolds to $\mathcal{R}_q$ by $(\pi^q_{q-1})^{-1}$ provides then (at least) two solutions to the original problem. □

For example the following scalar equations verify the hypothesis on the projection.

$$y_q^3 + y_q + f(x, y, \dots, y_{q-1}) = 0$$

$$\exp(y_q) + f(x, y, \dots, y_{q-1}) = 0$$

Note that in the latter case the map $\pi^q_{q-1} : \mathcal{R}_q \to J_{q-1}(\mathcal{E})$ is not surjective.

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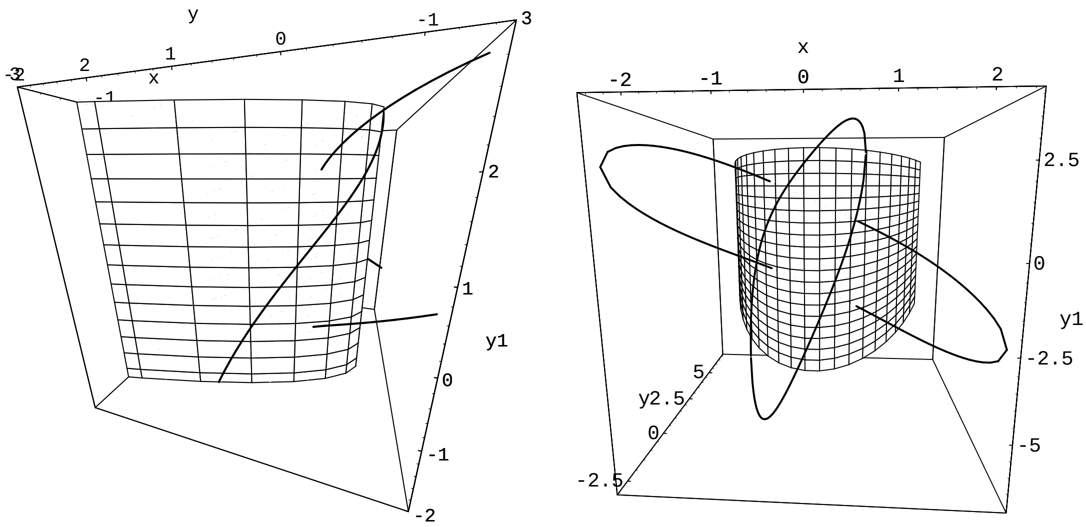

FIGURE 4.1. Boundary conditions of the problem (4.1) and part of the surface $y - 2x^2 + 1 = 0$. On the left is a detail of the picture on the right.

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ISBN 951-22-4281-8
ISSN 0784-3143
Edita, Espoo, Dec 1998