Uniqueness of Electromagnetic Inversion by Local Surface Measurements

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Abstract

Assume that a bounded body has been embedded in a homogeneous half space. Our aim is to reconstruct the electromagnetic material parameters, viz. the electric permittivity, conductivity and magnetic permeability, of the body from local field measurements on the surface of the half space. All fields are supposed to be governed by time harmonic Maxwell's equations. We consider two kinds of initial data. The first data set consists of tangential electric and magnetic field vectors of a tangential magnetic dipole for all field and source points in an open subset of the surface. The second alternative is that we know locally on the surface the admittance map which, by definition, associates the tangential magnetic field with the tangential electric field. The purpose of this paper is to show that both of the data sets determine the material parameters uniquely.

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1 Introduction

In general, a wave propagation inverse problem deals with the question whether some material parameters of a body can be determined by boundary measurements without penetration into the interior of the body. In [4] Lassas, Cheney and Uhlmann have studied an inverse problem associated with a scalar Schrödinger operator in a homogeneous half space that contains a bounded perturbation. Lassas et al proved that the perturbation is uniquely determined by the Dirichlet-to-Neumann map

$$\Lambda : u \mapsto \frac{\partial u}{\partial \vec{n}}$$

on an open subset of the boundary of the half space.

The uniqueness question of time harmonic electromagnetic inversion has been completely answered by Ola, Päivärinta and Somersalo in [7] for bounded bodies. They proved that the admittance map

$$Y : \vec{n} \times \vec{E} \mapsto \vec{n} \times \vec{H}$$

(denoted by Λ in the referred work) on the surface of the body determines the material parameters μ, ε and σ uniquely except for some resonance frequencies ω. Later in [8] Ola and Somersalo improved the result by pointing out that it suffices to consider the fields of tangential magnetic dipole sources only.

Our mission is, on one hand, to transfer the result of Lassas et al to electromagnetism. On the other hand, we will study if it is sufficient to restrict ourselves to an open subset of the boundary when executing measurements of Ola-Somersalo type as in [8]. Our method is, unfortunately perhaps, analytic continuation.

2 Fields of Dipole Sources

The time harmonic Maxwell's equations are

$$\begin{aligned} \nabla \times \vec{E}(x) - i\omega\mu(x)\vec{H}(x) &= \vec{M}(x), \\ \nabla \times \vec{H}(x) + i\omega\gamma(x)\vec{E}(x) &= \vec{J}(x), \end{aligned}$$

where $\omega > 0$ is the angular frequency, $\mu(x) > 0$ is the magnetic permeability, and $\gamma(x) = \varepsilon(x) + i\omega\sigma(x)$ is the complex permittivity fabricated of the electric permittivity $\varepsilon(x) > 0$ and the electric conductivity $\sigma(x) \geq 0$. The material parameters μ, ε and σ are supposed to be measurable functions in $\mathbb{R}^3$ and

$$W_{\mu,\gamma} = \operatorname{supp}(\mu - \mu_0) \cup \operatorname{supp}(\gamma - \varepsilon_0)$$

is supposed to be compact, unless explicitly specified. This set is referred to as the *body*, the *scatterer* or the *inhomogeneity*. The corresponding homogeneous

Maxwell's equations are

$$\nabla \times \vec{E}(x) - i\omega\mu(x)\vec{H}(x) = \vec{0}, \quad (1)$$
$$\nabla \times \vec{H}(x) + i\omega\gamma(x)\vec{E}(x) = \vec{0}. \quad (2)$$

If $\mu(x) = \mu_0$ and $\gamma(x) = \varepsilon_0$ are constants or, in other words, the space is homogeneous, the equations (1)–(2) imply the Helmholtz equations

$$\Delta\vec{E}(x) + k^2\vec{E}(x) = \vec{0},$$
$$\Delta\vec{H}(x) + k^2\vec{H}(x) = \vec{0}.$$

The wave number k is chosen such that $k^2 = \omega^2\varepsilon_0\mu_0$ and $\mathrm{Im}k \geq 0$. Note that $0 = \nabla \cdot (\nabla \times \vec{E}) = i\omega\mu_0\nabla \cdot \vec{H}$ implies $\nabla \cdot \vec{H} = 0$. Likewise $\nabla \cdot \vec{E} = 0$.

Every electromagnetic field $(\vec{E}, \vec{H})$ in this treatise satisfies the Silver-Müller radiation conditions (see [2], p. 113)

$$\hat{x} \times \vec{H}(x) + \frac{1}{\eta}\vec{E}(x) = \vec{O}(\frac{1}{r^2}), \quad (3)$$
$$\hat{x} \times \vec{E}(x) - \eta\vec{H}(x) = \vec{O}(\frac{1}{r^2}), \quad (4)$$

as $|x| = r \to \infty$, uniformly for all directions $\hat{x} = x/r$. Here $\eta = (\mu/\varepsilon)^{1/2}$ is the wave impedance.

2.1 Dipole Fields in a Homogeneous Space

Let

$$\phi(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$$

be the fundamental solution of the scalar Helmholtz equation. That is to say

$$\Delta_x\phi(x,y) + k^2\phi(x,y) = -\delta(x-y).$$

The electromagnetic field generated by a magnetic dipole in a homogeneous space has the representation (see [2], p. 112)

$$\overline{\overline{G}}{}^E_H(x,y) \cdot \vec{v} = \nabla_x \times (\phi(x,y)\vec{v}),$$
$$\overline{\overline{G}}{}^H_H(x,y) \cdot \vec{v} = \frac{1}{i\omega\mu_0}\nabla_x \times \nabla_x \times (\phi(x,y)\vec{v}).$$

Here $\vec{v}$ is a constant vector. For brevity, we talk about the dipole $\vec{v}$. The vector $\overline{\overline{G}}{}^E_H(x,y) \cdot \vec{v}$ represents the electric field at the field point $x \in \mathbb{R}^3$ of the magnetic dipole $\vec{v}$ located at the source point $y \in \mathbb{R}^3$. Correspondingly, $\overline{\overline{G}}{}^H_H(x,y) \cdot \vec{v}$ is the magnetic field. The equations

$$\begin{aligned}
\overline{\overline{G}}_E^E(x,y)\cdot\vec{v} &= -\frac{1}{i\omega\varepsilon_0}\nabla_x\times\nabla_x\times(\phi(x,y)\vec{v}),\\
\overline{\overline{G}}_E^H(x,y)\cdot\vec{v} &= \nabla_x\times(\phi(x,y)\vec{v}),
\end{aligned}$$

represent the electromagnetic field generated by an electric dipole $\vec{v}$ in a homogeneous space.

The above dyadics satisfy the Maxwell's equations

$$\begin{aligned}
\nabla\times\overline{\overline{G}}_H^E(\cdot,y)-i\omega\mu_0\overline{\overline{G}}_H^H(\cdot,y) &= \delta(\cdot-y)\overline{\overline{I}},\\
\nabla\times\overline{\overline{G}}_H^H(\cdot,y)+i\omega\varepsilon_0\overline{\overline{G}}_H^E(\cdot,y) &= \overline{\overline{0}},
\end{aligned}$$

and

$$\begin{aligned}
\nabla\times\overline{\overline{G}}_E^E(\cdot,y)-i\omega\mu_0\overline{\overline{G}}_E^H(\cdot,y) &= \overline{\overline{0}},\\
\nabla\times\overline{\overline{G}}_E^H(\cdot,y)+i\omega\varepsilon_0\overline{\overline{G}}_E^E(\cdot,y) &= \delta(\cdot-y)\overline{\overline{I}}.
\end{aligned}$$

Moreover, for

$$(\vec{E},\vec{H})\in\left\{\left(\overline{\overline{G}}_H^E(\cdot,y)\cdot\vec{v},\ \overline{\overline{G}}_H^H(\cdot,y)\cdot\vec{v}\right),\left(\overline{\overline{G}}_E^E(\cdot,y)\cdot\vec{v},\ \overline{\overline{G}}_E^H(\cdot,y)\cdot\vec{v}\right)\right\}$$

the Silver-Müller radiation conditions (3)–(4) are satisfied (see [2], p. 113).

The dyadics $\overline{\overline{G}}_E^E$ and $\overline{\overline{G}}_H^H$ are symmetric while $\overline{\overline{G}}_H^E$ and $\overline{\overline{G}}_E^H$ are antisymmetric with respect to the transposition operator $G\mapsto G^t$. Because $\phi(x,y)$ is a function of the absolute value $|x-y|$ we have, for all $l=1,2,3$, the identity

$$\partial_{y_l}\phi(x,y) \;=\; -\partial_{x_l}\phi(x,y).$$

Hence the reciprocity relations

$$\begin{aligned}
\overline{\overline{G}}_H^E(y,x) &= -\overline{\overline{G}}_H^E(x,y) = -\overline{\overline{G}}_E^H(x,y) = \overline{\overline{G}}_E^H(x,y)^t,\\
\overline{\overline{G}}_E^E(y,x) &= \overline{\overline{G}}_E^E(x,y) = \overline{\overline{G}}_E^E(x,y)^t,\\
\overline{\overline{G}}_H^H(y,x) &= \overline{\overline{G}}_H^H(x,y) = \overline{\overline{G}}_H^H(x,y)^t,
\end{aligned}$$

hold.

Denote for $x\in\mathbb{R}^n$ and $r>0$

$$\begin{aligned}
S^{n-1}(x,r) &= S(x,r) = \{\,y\in\mathbb{R}^n \mid |y|=r\,\},\\
B^n(x,r) &= B(x,r) = \{\,y\in\mathbb{R}^n \mid |y|<r\,\},\\
\overline{B}^n(x,r) &= \overline{B}(x,r) = B^n(x,r)\cup S^{n-1}(x,r).
\end{aligned}$$

If a point $y \in \mathbb{R}^3$ has a neighbourhood* in which $(\vec{E}, \vec{H})$ satisfies (1)–(2) with constant coefficents $\mu(x) = \mu_0$ and $\gamma(x) = \varepsilon_0$, then for all constant vectors $\vec{v} \in S^2 = S^2(0,1)$ there exists a limit

$$\vec{E}(y) \cdot \vec{v} = \lim_{r \to 0^+} \int_{S(y,r)} \left[\left(\vec{n}(x) \times \vec{H}(x) \right) \cdot \overline{\overline{G}}^E_E(x,y) \cdot \vec{v} + \left(\vec{n}(x) \times \vec{E}(x) \right) \cdot \overline{\overline{G}}^H_E(x,y) \cdot \vec{v} \right] dS(x)$$

uniformly in $\vec{v}$ (see [2], p. 111). Therefore we have the important equality

$$\vec{E}(y) = \lim_{r \to 0^+} \int_{S(y,r)} \left[\left(\vec{n}(x) \times \vec{H}(x) \right) \cdot \overline{\overline{G}}^E_E(x,y) + \left(\vec{n}(x) \times \vec{E}(x) \right) \cdot \overline{\overline{G}}^H_E(x,y) \right] dS(x). \tag{5}$$

Likewise

$$\vec{H}(y) = \lim_{r \to 0^+} \int_{S(y,r)} \left[\left(\vec{n}(x) \times \vec{H}(x) \right) \cdot \overline{\overline{G}}^E_H(x,y) + \left(\vec{n}(x) \times \vec{E}(x) \right) \cdot \overline{\overline{G}}^H_H(x,y) \right] dS(x). \tag{6}$$

2.2 Dipole Fields in an Inhomogeneous Space

We define the dyadic electromagnetic field of a magnetic dipole located at a source point $y \in \mathbb{R}^3 \setminus W_{\mu,\gamma}$ as a pair $(\overline{\overline{E}}, \overline{\overline{H}}) = (\overline{\overline{E}}(\cdot, y), \overline{\overline{H}}(\cdot, y))$ with the following properties:

(i) $\overline{\overline{E}}(\cdot, y)$ and $\overline{\overline{H}}(\cdot, y)$ are C^1-dyadics.

(ii) In the exterior of the scatterer $W_{\mu,\gamma}$ the total field $(\overline{\overline{E}}, \overline{\overline{H}})$ has a decomposition

$$(\overline{\overline{E}}, \overline{\overline{H}}) = (\overline{\overline{E}}_{in}, \overline{\overline{H}}_{in}) + (\overline{\overline{E}}_{sc}, \overline{\overline{H}}_{sc}), \tag{7}$$

where $(\overline{\overline{E}}_{in}, \overline{\overline{H}}_{in}) = (\overline{\overline{G}}^E_H, \overline{\overline{G}}^H_H)$ is the incident field and $(\overline{\overline{E}}_{sc}, \overline{\overline{H}}_{sc})$ is the scattered field.

(iii) The scattered field satisfies, for $x \in \mathbb{R}^3 \setminus W_{\mu,\gamma}$, the homogeneous Maxwell's equations

$$\begin{aligned} \nabla_x \times \overline{\overline{E}}_{sc}(x,y) - i\omega\mu_0 \overline{\overline{H}}_{sc}(x,y) &= \overline{\overline{0}}, \\ \nabla_x \times \overline{\overline{H}}_{sc}(x,y) + i\omega\varepsilon_0 \overline{\overline{E}}_{sc}(x,y) &= \overline{\overline{0}}, \end{aligned}$$

*In our terminology neighbourhoods are open sets.

and radiation conditions

$$\hat{x} \times \overline{\overline{H}}_{sc}(x,y) + \frac{1}{\eta}\overline{\overline{E}}_{sc}(x,y) = \overline{\overline{O}}(\frac{1}{r^2}), \quad (8)$$

$$\hat{x} \times \overline{\overline{E}}_{sc}(x,y) - \eta\overline{\overline{H}}_{sc}(x,y) = \overline{\overline{O}}(\frac{1}{r^2}), \quad (9)$$

as $r \to \infty$.

(iv) The total field satisfies, for $x \in W_{\mu,\gamma}$, the Maxwell's equations

$$\begin{aligned} \nabla_x \times \overline{\overline{E}}(x,y) - i\omega\mu(x)\overline{\overline{H}}(x,y) &= \overline{\overline{0}}, \\ \nabla_x \times \overline{\overline{H}}(x,y) + i\omega\gamma(x)\overline{\overline{E}}(x,y) &= \overline{\overline{0}}. \end{aligned}$$

It is clear from (ii)–(iv) that the total field satisfies

$$\begin{aligned} \nabla_x \times \overline{\overline{E}}(x,y) - i\omega\mu(x)\overline{\overline{H}}(x,y) &= \delta(x-y)\overline{\overline{I}}, \\ \nabla_x \times \overline{\overline{H}}(x,y) + i\omega\gamma(x)\overline{\overline{E}}(x,y) &= \overline{\overline{0}}, \end{aligned} \quad (10)$$

for all $x \in \mathbb{R}^3$, and

$$\begin{aligned} \Delta_x\overline{\overline{E}}(x,y) + k^2\overline{\overline{E}}(x,y) &= \overline{\overline{0}}, \\ \Delta_x\overline{\overline{H}}(x,y) + k^2\overline{\overline{H}}(x,y) &= \overline{\overline{0}}, \end{aligned}$$

for all $x \in \mathbb{R}^3 \setminus (W_{\mu,\gamma} \cup \{y\})$. Actually, $\overline{\overline{E}}(\cdot,y)$ and $\overline{\overline{H}}(\cdot,y)$ are then C^∞-dyadics in $\mathbb{R}^3 \setminus (W_{\mu,\gamma} \cup \{y\})$ (see [9], p. 201). Because both the incident and scattered field satisfy Silver-Müller radiation conditions in dyadic form (cf. (8)–(9)) the same applies to the total field. We denote

$$(\overline{\overline{E}},\overline{\overline{H}}) = (\overline{\overline{G}}^e_h,\ \overline{\overline{G}}^h_h).$$

The electromagnetic field $(\overline{\overline{G}}^e_e,\ \overline{\overline{G}}^h_e)$ of an electric dipole in an inhomogeneous space is defined in the same way. It obeys the equations

$$\begin{aligned} \nabla_x \times \overline{\overline{G}}^e_e(x,y) - i\omega\mu(x)\overline{\overline{G}}^h_e(x,y) &= \overline{\overline{0}}, \\ \nabla_x \times \overline{\overline{G}}^h_e(x,y) + i\omega\gamma(x)\overline{\overline{G}}^e_e(x,y) &= \delta(x-y)\overline{\overline{I}}, \end{aligned}$$

and Silver-Müller radiation conditions. According to [2] (pp. 116–117) every dyadic field

$$(\overline{\overline{E}},\overline{\overline{H}}) \in \{(\overline{\overline{G}}^e_h(\cdot,y),\ \overline{\overline{G}}^h_h(\cdot,y)),\ (\overline{\overline{G}}^e_e(\cdot,y),\ \overline{\overline{G}}^h_e(\cdot,y))\}$$

has the asymptotic form

$$\overline{\overline{E}}(x) = \frac{e^{ikr}}{r}\overline{\overline{F}}(\hat{x}) + \overline{\overline{O}}(\frac{1}{r^2}), \quad (11)$$

$$\overline{\overline{H}}(x) = \frac{e^{ikr}}{\eta r}\hat{x} \times \overline{\overline{F}}(\hat{x}) + \overline{\overline{O}}(\frac{1}{r^2}), \quad (12)$$

where $\overline{\overline{F}} : S^2 \to \mathbb{C}^{3x3}$ is the far-field pattern of $(\overline{\overline{E}}, \overline{\overline{H}})$ with the property

$$\hat{x} \cdot \overline{\overline{F}}(\hat{x}) \;=\; 0.$$

Certainly the same applies for vector fields.

2.3 Maxwell Duality

Denote by D a region in $\mathbb{R}^3$, i.e., D is a nonempty bounded connected open subset of $\mathbb{R}^3$ whose boundary is piecewise smooth†. Assume that μ and γ are C^1-functions in a neigbourhood of $\overline{D}$. Let us define two formal operators, $\mathcal{M} = \mathcal{M}_{\mu,\gamma}$ and $\mathcal{M}^* = \mathcal{M}^*_{\mu,\gamma}$, by setting

$$\mathcal{M}\begin{pmatrix} \mathsf{E}_1 \\ \mathsf{H}_1 \end{pmatrix} = \begin{pmatrix} \nabla \times \mathsf{E}_1 - i\omega\mu\mathsf{H}_1 \\ \nabla \times \mathsf{H}_1 + i\omega\gamma\mathsf{E}_1 \end{pmatrix},$$

$$\mathcal{M}^*\begin{pmatrix} \mathsf{H}_2 \\ \mathsf{E}_2 \end{pmatrix} = \begin{pmatrix} \nabla \times \mathsf{H}_2 + i\omega\gamma\mathsf{E}_2 \\ \nabla \times \mathsf{E}_2 - i\omega\mu\mathsf{H}_2 \end{pmatrix},$$

and a formal bilinear form

$$\left\langle \begin{pmatrix} \mathsf{A} \\ \mathsf{B} \end{pmatrix}, \begin{pmatrix} \mathsf{C} \\ \mathsf{D} \end{pmatrix} \right\rangle_D = \int_D \left(\mathsf{A}^t \cdot \mathsf{C} + \mathsf{B}^t \cdot \mathsf{D}\right) dV,$$

where E_1, H_1, E_2, H_2, A, B, C and D are either vectors‡ or dyadics. We call the integral equation of the following lemma *Maxwell duality*:

Lemma 2.1 (Maxwell duality) *If, in a neighbourhood of $\overline{D}$, $(\mathsf{E}_1, \mathsf{H}_1)$ is either a C^1-vector field or C^1-dyadic field and, independently, $(\mathsf{E}_2, \mathsf{H}_2)$ is either a C^1-vector field or C^1-dyadic field, then*

$$\left\langle \mathcal{M}\begin{pmatrix} \mathsf{E}_1 \\ \mathsf{H}_1 \end{pmatrix}, \begin{pmatrix} \mathsf{H}_2 \\ \mathsf{E}_2 \end{pmatrix} \right\rangle_D = \tag{13}$$

$$\left\langle \begin{pmatrix} \mathsf{E}_1 \\ \mathsf{H}_1 \end{pmatrix}, \mathcal{M}^*\begin{pmatrix} \mathsf{H}_2 \\ \mathsf{E}_2 \end{pmatrix} \right\rangle_D + \int_{\partial D} \left((\vec{n} \times \mathsf{E}_1)^t \cdot \mathsf{H}_2 + (\vec{n} \times \mathsf{H}_1)^t \cdot \mathsf{E}_2\right) dS.$$

Proof: The formula follows immediately from the divergence theorem by applying well known identities from vector and dyadic analysis (see e.g. [1], pp. 487–491 and 506–509). □

†We don't specify exactly what we mean with a piecewise smooth boundary. The only purpose to introduce this smoothness requirement is that we should be able to apply the divergence theorem to D.

‡For vectors $\vec{a} = \sum_{l=1}^{n} a_l \vec{e}_l$ (which we usually write as $\vec{a} = a_l \vec{e}_l$ using the Einstein's summation convention) the transposition is regarded as identity operator.

If $(\vec{E}_1, \vec{H}_1)$, $(\vec{E}_2, \vec{H}_2)$ satisfy the radiation conditions (3)–(4), then the far-field patterns satisfy

$$\vec{F}_1(\hat{x}) \times (\hat{x} \times \vec{F}_2(\hat{x})) \; = \; \hat{x}(\vec{F}_1(\hat{x}) \cdot \vec{F}_2(\hat{x})) - \vec{F}_2(\hat{x})(\vec{F}_1(\hat{x}) \cdot \hat{x}) \; = \; \hat{x}(\vec{F}_1(\hat{x}) \cdot \vec{F}_2(\hat{x}))$$

and in the same manner

$$\vec{F}_2(\hat{x}) \times (\hat{x} \times \vec{F}_1(\hat{x})) \; = \; \hat{x}(\vec{F}_2(\hat{x}) \cdot \vec{F}_1(\hat{x})).$$

Thus, as $r \to \infty$,

$$\begin{aligned} \vec{E}_1 \times \vec{H}_2 + \vec{H}_1 \times \vec{E}_2 \;&=\; \vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1 \\ &=\; \frac{e^{2ikr}}{\eta r^2} \vec{F}_1(\hat{x}) \times \Big(\hat{x} \times \vec{F}_2(\hat{x})\Big) + \vec{O}(\frac{1}{r^3}) \\ &\qquad - \frac{e^{2ikr}}{\eta r^2} \vec{F}_2(\hat{x}) \times \Big(\hat{x} \times \vec{F}_1(\hat{x})\Big) - \vec{O}(\frac{1}{r^3}) \\ &=\; \vec{O}(\frac{1}{r^3}). \end{aligned}$$

In the integral equation (13) for vectors then

$$\vec{n} \times \vec{E}_1 \cdot \vec{H}_2 + \vec{n} \times \vec{H}_1 \cdot \vec{E}_2 \; = \; \vec{n} \cdot (\vec{E}_1 \times \vec{H}_2 + \vec{H}_1 \times \vec{E}_2) \; = \; \vec{O}(\frac{1}{r^3}).$$

For a measurable subset T of S^2 the Lebesgue measure of rT is $O(r^2)$; so we obtain

$$\lim_{r \to \infty} \int_{rT} \Big(\vec{n} \times \vec{E}_1 \cdot \vec{H}_2 + \vec{n} \times \vec{H}_1 \cdot \vec{E}_2\Big) \, dS \; = \; 0. \tag{14}$$

Suppose T is a two dimensional region on S^2 which implies, by definition, that the boundary curve ∂T is piecewise smooth. We shall use the notation

$$\mathcal{C}T \; = \; \bigcup_{r>0} rT.$$

From (14) and the Maxwell duality (13) it follows for the cone $\mathcal{C}T$ that

$$\begin{aligned} &\left\langle \mathcal{M} \begin{pmatrix} \vec{E}_1 \\ \vec{H}_1 \end{pmatrix}, \begin{pmatrix} \vec{H}_2 \\ \vec{E}_2 \end{pmatrix} \right\rangle_{\mathcal{C}T} = \qquad (15) \\ &\left\langle \begin{pmatrix} \vec{E}_1 \\ \vec{H}_1 \end{pmatrix}, \mathcal{M}^* \begin{pmatrix} \vec{H}_2 \\ \vec{E}_2 \end{pmatrix} \right\rangle_{\mathcal{C}T} + \int_{\partial \mathcal{C}T} \Big(\vec{n} \times \vec{E}_1 \cdot \vec{H}_2 + \vec{n} \times \vec{H}_1 \cdot \vec{E}_2\Big) \, dS. \end{aligned}$$

2.4 Boundary Integral Equation

By denoting $(\vec{E}_1, \vec{H}_1) = (\vec{E}, \vec{H})$ in (14) and setting

$$(\vec{E}_2, \vec{H}_2) \; = \; (\overline{\overline{G}}{}^e_e(\cdot, y) \cdot \vec{v}, \; \overline{\overline{G}}{}^h_e(\cdot, y) \cdot \vec{v})$$

the equation (14) becomes

$$\lim_{r\to\infty}\int\limits_{rT}\left[\left(\vec{n}\times\vec{H}(x)\right)\cdot\overline{\overline{G}}_e^e(x,y)\cdot\vec{v}+\left(\vec{n}\times\vec{E}(x)\right)\cdot\overline{\overline{G}}_e^h(x,y)\cdot\vec{v}\right]dS(x)\;=\;0,$$

which leads to

$$\lim_{r\to\infty}\int\limits_{rT}\left[\left(\vec{n}\times\vec{H}(x)\right)\cdot\overline{\overline{G}}_e^e(x,y)+\left(\vec{n}\times\vec{E}(x)\right)\cdot\overline{\overline{G}}_e^h(x,y)\right]dS(x)\;=\;\vec{0}.\tag{16}$$

The corresponding formula for magnetic dipoles is

$$\lim_{r\to\infty}\int\limits_{rT}\left[\left(\vec{n}\times\vec{H}(x)\right)\cdot\overline{\overline{G}}_h^e(x,y)+\left(\vec{n}\times\vec{E}(x)\right)\cdot\overline{\overline{G}}_h^h(x,y)\right]dS(x)\;=\;\vec{0}.\tag{17}$$

Note the complementary nature of (5)–(6) and (16)–(17). In the next lemma we prove an analogue of the Stratton-Chu formula for conical unbounded surfaces:

Lemma 2.2 *Let T be a two dimensional region on S^2 and y an interior point of the cone $\mathcal{C}T$. In addition we assume the following:*

- *(i) The inclusion $W_{\mu,\gamma}\subset B(0,R)\setminus\overline{B}(y,r)$ holds for some radii $r>0$ and $R>0$.*
- *(ii) The C^1-field $(\vec{E},\vec{H})$ satisfies the homogeneous Maxwell's equations (1)–(2) and Silver-Müller radiation conditions (3)–(4) in a neighbourhood of $\mathcal{C}T$.*

Then

$$\vec{E}(y)\;=\;\int\limits_{\partial\mathcal{C}T}\left[\left(\vec{n}\times\vec{H}(x)\right)\cdot\overline{\overline{G}}_e^e(x,y)+\left(\vec{n}\times\vec{E}(x)\right)\cdot\overline{\overline{G}}_e^h(x,y)\right]dS(x),\tag{18}$$

$$\vec{H}(y)\;=\;\int\limits_{\partial\mathcal{C}T}\left[\left(\vec{n}\times\vec{H}(x)\right)\cdot\overline{\overline{G}}_h^e(x,y)+\left(\vec{n}\times\vec{E}(x)\right)\cdot\overline{\overline{G}}_h^h(x,y)\right]dS(x).\tag{19}$$

Proof: One can assume, without loss of generality, that $\overline{B}(y,r)\subset\mathcal{C}T\cap B(0,R)$. We apply Maxwell duality (13) to the fields $(\mathsf{E}_1,\mathsf{H}_1)=(\vec{E},\vec{H})$ and $(\mathsf{E}_2,\mathsf{H}_2)=(\overline{\overline{G}}_e^e(\cdot,y),\;\overline{\overline{G}}_e^h(\cdot,y))$ in the region $D=D_{R,r}=\mathcal{C}T\cap B(0,R)\setminus\overline{B}(y,r)$. Because both of the fields satisfy the homogeneous Maxwell's equations in D the duality (13) implies

$$\vec{0}=\int\limits_{\partial D_{R,r}}\left[\left(\vec{n}\times\vec{H}(x)\right)\cdot\overline{\overline{G}}_e^e(x,y)+\left(\vec{n}\times\vec{E}(x)\right)\cdot\overline{\overline{G}}_e^h(x,y)\right]dS(x)\tag{20}$$

By decomposing $\partial D_{R,r}$ to a disjoint union

$$\partial D_{R,r} \;=\; S(y,r) \cup (S(0,R) \cap \mathcal{C}T) \cup \left(\overline{B}(0,R) \cap \partial \mathcal{C}T\right)$$

the boundary integral becomes a sum of three integrals. According to (5) the first of them tends to $-\vec{E}(y)$ as $r \to 0^+$. This is an implication of the decomposition (7) and the fact that the normal vectors $\vec{n}$ in (5) and here point to opposite directions. According to (16) the second integral tends to 0 as $R \to \infty$. The third boundary integral tends to the right hand side of (18) as $r \to 0^+$ and $R \to \infty$. Thus we get (18) from (20) via a limit process. The latter formula (19) for magnetic dipoles is proved mutatis mutandis. $\square$

The formula (18) has a straightforward formal derivation from the Maxwell duality[§] (13) by the replacements

$$(\mathsf{E}_1, \mathsf{H}_1) = (\vec{E}, \vec{H}), \quad (\mathsf{E}_2, \mathsf{H}_2) = (\overline{\overline{G}}{}^e_e(\cdot, y),\, \overline{\overline{G}}{}^h_e(\cdot, y)).$$

This is a consequence of the Maxwell's equations

$$\mathcal{M}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}, \quad \mathcal{M}^*\begin{pmatrix} \overline{\overline{G}}{}^h_e(\cdot, y) \\ \overline{\overline{G}}{}^e_e(\cdot, y) \end{pmatrix} = \begin{pmatrix} \delta(\cdot - y)\overline{\overline{I}} \\ \overline{\overline{0}} \end{pmatrix},$$

and the radiation conditions that kill the boundary integral like in (16). The same kind of a formal derivation applies, of course, for (19).

2.5 Reciprocity

We are going to express the field $\overline{\overline{G}}(y,z)$ at y generated by a dipole located at z by means of the field $\overline{\overline{G}}(z,y)$ at z generated by a dipole located at y.

Lemma 2.3 *Let $W_{\mu,\gamma} \subset B(0,R) \setminus (\overline{B}(y,r) \cup \overline{B}(z,r))$ for some radii $r > 0$ and $R > 0$. Then we have the following reciprocity relations:*

$$\overline{\overline{G}}{}^h_h(y,z) = \overline{\overline{G}}{}^h_h(z,y)^t, \tag{21}$$

$$\overline{\overline{G}}{}^e_e(y,z) = \overline{\overline{G}}{}^e_e(z,y)^t, \tag{22}$$

$$\overline{\overline{G}}{}^h_e(y,z) = \overline{\overline{G}}{}^e_h(z,y)^t. \tag{23}$$

Proof: A rigorous proof could be accomplished (as in Lemma 2.2) by applying the Maxwell duality (13) in the region $D = B(0,R) \setminus \left(\overline{B}(y,r) \cup \overline{B}(z,r)\right)$ and letting $r \to 0^+$ and $R \to \infty$. Formally (21) is proved by replacing in (13)

$$\mathcal{M}\begin{pmatrix} \mathsf{E}_1 \\ \mathsf{H}_1 \end{pmatrix} = \begin{pmatrix} \delta(\cdot - y)\overline{\overline{I}} \\ \overline{\overline{0}} \end{pmatrix}, \quad \mathcal{M}^*\begin{pmatrix} \mathsf{H}_2 \\ \mathsf{E}_2 \end{pmatrix} = \begin{pmatrix} \overline{\overline{0}} \\ \delta(\cdot - z)\overline{\overline{I}} \end{pmatrix},$$

[§] Note that the field of a dipole located at y has a singularity at y. Therefore the field is not C^1.

(22) is proved by replacing in (13)

$$\mathcal{M}\left(\begin{array}{c} \mathsf{E}_1 \\ \mathsf{H}_1 \end{array}\right) = \left(\begin{array}{c} \overline{\overline{0}} \\ \delta(\cdot - y)\overline{\overline{I}} \end{array}\right), \quad \mathcal{M}^*\left(\begin{array}{c} \mathsf{H}_2 \\ \mathsf{E}_2 \end{array}\right) = \left(\begin{array}{c} \delta(\cdot - z)\overline{\overline{I}} \\ \overline{\overline{0}} \end{array}\right),$$

and (23) is proved by replacing in (13)

$$\mathcal{M}\left(\begin{array}{c} \mathsf{E}_1 \\ \mathsf{H}_1 \end{array}\right) = \left(\begin{array}{c} \delta(\cdot - y)\overline{\overline{I}} \\ \overline{\overline{0}} \end{array}\right), \quad \mathcal{M}^*\left(\begin{array}{c} \mathsf{H}_2 \\ \mathsf{E}_2 \end{array}\right) = \left(\begin{array}{c} \delta(\cdot - z)\overline{\overline{I}} \\ \overline{\overline{0}} \end{array}\right).$$

□

3 Determination of μ, ε and σ by Local Measurements

In this section we prove the first of our main results. Suppose we know approximately where the inhomogeneous body is located. In other words we are given a compact subset W of the lower half space outside of which the space is known to be homogeneous. Then we measure the tangential component of the electromagnetic field of a magnetic dipole $\vec{v}$ for all tangential orientations of $\vec{v}$ (actually two linearly independent tangential orientations will do for linearity), for all field points $x \in U$, and for all source points $y \in U$ where $U \neq \emptyset$ is an arbitrary open subset of the planar surface of the half space. We claim that within W there is at most one possible material distribution which is compatible with our measurement results or, strictly speaking, our data determine the functions $\mu\big|_W$, $\varepsilon\big|_W$ and $\sigma\big|_W$ uniquely.

This result is, in a sense, seemingly stronger than which was proved by Ola and Somersalo in [8] where the body is totally surrounded by a closed surface $\partial\Omega$ of source points without gaps. For every source point a measurement has to be made at every point of $\partial\Omega$. In addition there are some magnetic resonance frequencies ω for which the uniqueness in [8] possibly doesn't hold. In case of our measurement arrangement no such frequencies appear. Nevertheless, our localized and frequency independent version is just a seeming improvement because it rests on the theorem of Ola and Somersalo (Theorem B on p. 1131 in [8]).

3.1 Field Continuation Outside a Surface Element

Let us denote

$$\begin{array}{rcl} \mathbb{R}^3_+ &=& \{\, x \in \mathbb{R}^3 \mid x_3 > 0 \,\}, \\ \mathbb{R}^3_- &=& \{\, x \in \mathbb{R}^3 \mid x_3 < 0 \,\}, \\ \mathbb{R}^2 &=& \{\, x \in \mathbb{R}^3 \mid x_3 = 0 \,\}. \end{array}$$

Thus $\mathbb{R}^3_+$ is the upper half space, $\mathbb{R}^3_-$ is the lower half space and $\mathbb{R}^2$ is the surface or boundary of the lower (or as well of the upper) half space. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\vec{v} = v_l \vec{e}_l \in \mathbb{R}^3$ we denote

$$\begin{aligned} \tilde{x} &= (x_1, x_2, -x_3), \\ \tilde{\vec{v}} &= v_1\vec{e}_1 + v_2\vec{e}_2 - v_3\vec{e}_3. \end{aligned}$$

Lemma 3.1 *Suppose U is an open nonempty subset of $\mathbb{R}^2$, V is an open connected subset of $\overline{\mathbb{R}}^3_+ = \mathbb{R}^3_+ \cup \mathbb{R}^2$, and $U \subset V$. If $(\vec{E}, \vec{H})$ is a C^1-field in V, such that for every $x \in V$*

$$\nabla \times \vec{E}(x) - i\omega\mu_0\vec{H}(x) = \vec{0}, \tag{24}$$

$$\nabla \times \vec{H}(x) + i\omega\varepsilon_0\vec{E}(x) = \vec{0}, \tag{25}$$

then the tangential field $(\vec{n} \times \vec{E}\big|_U, \vec{n} \times \vec{H}\big|_U)$ determines $(\vec{E}\big|_V, \vec{H}\big|_V)$ uniquely. Here $\vec{n} = \vec{e}_3$.*

Proof: Let us write (24) as

$$\partial_2 E_3 - \partial_3 E_2 - i\omega\mu_0 H_1 = 0, \tag{26}$$

$$\partial_3 E_1 - \partial_1 E_3 - i\omega\mu_0 H_2 = 0, \tag{27}$$

$$\partial_1 E_2 - \partial_2 E_1 - i\omega\mu_0 H_3 = 0. \tag{28}$$

From the knowledge of $(\vec{n} \times \vec{E}\big|_U, \vec{n} \times \vec{H}\big|_U)$ we obtain $E_s\big|_U$, $H_s\big|_U$, for $s = 1, 2$, and therefore also $\partial_r E_s\big|_U$, $\partial_r H_s\big|_U$, for $r, s = 1, 2$. Thus we can solve $H_3\big|_U$ from (28) and thereafter we obtain $\partial_r H_3\big|_U$, $r = 1, 2$. In the same manner we find out $E_3\big|_U$ and $\partial_r E_3\big|_U$, $r = 1, 2$, by writing (25) componentwise. After that we solve $\partial_3 E_s\big|_U$, $s = 1, 2$, from (26) and (27), and likewise $\partial_3 H_s\big|_U$, $s = 1, 2$, from the first two components of (25). Finally we solve the divergence equations $\nabla \cdot \vec{E} = 0$ and $\nabla \cdot \vec{H} = 0$ to obtain $\partial_3 E_3\big|_U$ and $\partial_3 H_3\big|_U$. Especially, we observe that the Cauchy data

$$\vec{E}\big|_U, \quad \partial_3\vec{E}\big|_U, \quad \vec{H}\big|_U, \quad \partial_3\vec{H}\big|_U,$$

are uniquely determined. Since $\vec{E}$ and $\vec{H}$ both satisfy Helmholtz equation† in V, the unique continuation of $(\vec{E}\big|_U, \vec{H}\big|_U)$ to V follows from the Holmgren's Uniqueness Theorem (see [2], p. 194). □

For future needs we prove a generalization of the preceding lemma for curved surface elements in Appendix.

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*In other words: if we have a single tangential field $(\vec{n} \times \vec{E}\big|_U, \vec{n} \times \vec{H}\big|_U)$ on U for some material distributions (μ, γ) and (μ', γ') outside V, then we also have a single field $(\vec{E}\big|_V, \vec{H}\big|_V)$ in V for both of the distributions, provided that (24)–(25) are satisfied within V in both cases. This is exactly what we mean when talking about unique determination in the sequel.

†The vector Helmholtz $\Delta\vec{A} + k^2\vec{A} = \vec{0}$ consists of three scalar Helmholtz equations $\Delta A_l + k^2 A_l = 0$, l=1,2,3.

3.2 Admittance Map Y and Impedance Map Z

We are going to express the tangential component $\vec{n} \times \vec{H}\big|_{\mathbb{R}^2}$ of a magnetic field as a function of the tangential component $\vec{n} \times \vec{E}\big|_{\mathbb{R}^2}$ of the electric field and vice versa. Here $\vec{n} = \vec{e}_3$. To this end we define, for a function $f : \mathbb{R}^3 \to \mathbb{C}$, the symmetrized functions f_+ and f_- by setting

$$f_+(x) = \begin{cases} f(x), & x_3 \geq 0, \\ f(\tilde{x}), & x_3 < 0, \end{cases} \qquad f_-(x) = \begin{cases} f(x), & x_3 \leq 0, \\ f(\tilde{x}), & x_3 > 0. \end{cases}$$

Let $(\overline{\overline{G}}_h^{e+}, \overline{\overline{G}}_h^{h+})$, $(\overline{\overline{G}}_e^{e+}, \overline{\overline{G}}_e^{h+})$ denote dipole fields associated with the material distribution μ_+, γ_+ for a compact $W_{\mu,\gamma} \subset \mathbb{R}^3_+$, and analogously, let $(\overline{\overline{G}}_h^{e-}, \overline{\overline{G}}_h^{h-})$, $(\overline{\overline{G}}_e^{e-}, \overline{\overline{G}}_e^{h-})$ represent dipole fields associated with μ_-, γ_- for a compact $W_{\mu,\gamma} \subset \mathbb{R}^3_-$.

Definition 3.1 *The* magnetically symmetric Green's dyadics $\overline{\overline{\mathcal{G}}}_h^{e-}$, $\overline{\overline{\mathcal{G}}}_h^{h-}$ *for the lower half space in the symmetric material distribution* μ_-, γ_- *are defined by*

$$\begin{aligned} \overline{\overline{\mathcal{G}}}_h^{e-}(x,y) \cdot \vec{v} &= \overline{\overline{G}}_h^{e-}(x,y) \cdot \vec{v} + \overline{\overline{G}}_h^{e-}(x,\tilde{y}) \cdot \tilde{\vec{v}}, \\ \overline{\overline{\mathcal{G}}}_h^{h-}(x,y) \cdot \vec{v} &= \overline{\overline{G}}_h^{h-}(x,y) \cdot \vec{v} + \overline{\overline{G}}_h^{h-}(x,\tilde{y}) \cdot \tilde{\vec{v}}. \end{aligned}$$

The electrically symmetric Green's dyadics $\overline{\overline{\mathcal{G}}}_e^{e-}$, $\overline{\overline{\mathcal{G}}}_e^{h-}$ *for the lower half space in the symmetric material distribution* μ_-, γ_- *are defined by*

$$\begin{aligned} \overline{\overline{\mathcal{G}}}_e^{e-}(x,y) \cdot \vec{v} &= \overline{\overline{G}}_e^{e-}(x,y) \cdot \vec{v} + \overline{\overline{G}}_e^{e-}(x,\tilde{y}) \cdot \tilde{\vec{v}}, \\ \overline{\overline{\mathcal{G}}}_e^{h-}(x,y) \cdot \vec{v} &= \overline{\overline{G}}_e^{h-}(x,y) \cdot \vec{v} + \overline{\overline{G}}_e^{h-}(x,\tilde{y}) \cdot \tilde{\vec{v}}. \end{aligned}$$

Here $\vec{v} = v_l \vec{e}_l \in \mathbb{R}^3$ *is an arbitrary directional vector. The corresponding dyadics* $\overline{\overline{\mathcal{G}}}_h^{e+}$, $\overline{\overline{\mathcal{G}}}_h^{h+}$, $\overline{\overline{\mathcal{G}}}_e^{e+}$ *and* $\overline{\overline{\mathcal{G}}}_e^{h+}$ *for the upper half space are defined analogously.*

The field $(\overline{\overline{\mathcal{G}}}_h^{e\pm}, \overline{\overline{\mathcal{G}}}_h^{h\pm})$ satisfies the PEC (perfect electric conductor) boundary condition on the plane $\mathbb{R}^2$ of symmetry (cf. [6], pp. 117–122); that is to say

$$\begin{aligned} \vec{u} \cdot \overline{\overline{\mathcal{G}}}_h^{e\pm}(\tilde{x},y) \cdot \vec{v} &= -\tilde{\vec{u}} \cdot \overline{\overline{\mathcal{G}}}_h^{e\pm}(x,y) \cdot \vec{v}, \\ \vec{u} \cdot \overline{\overline{\mathcal{G}}}_h^{h\pm}(\tilde{x},y) \cdot \vec{v} &= \tilde{\vec{u}} \cdot \overline{\overline{\mathcal{G}}}_h^{h\pm}(x,y) \cdot \vec{v}. \end{aligned} \tag{29}$$

The field $(\overline{\overline{\mathcal{G}}}_e^{e\pm}, \overline{\overline{\mathcal{G}}}_e^{h\pm})$ satisfies the PMC (perfect magnetic conductor) boundary condition on $\mathbb{R}^2$. Accordingly

$$\begin{aligned} \vec{u} \cdot \overline{\overline{\mathcal{G}}}_e^{e\pm}(\tilde{x},y) \cdot \vec{v} &= \tilde{\vec{u}} \cdot \overline{\overline{\mathcal{G}}}_e^{e\pm}(x,y) \cdot \vec{v}, \\ \vec{u} \cdot \overline{\overline{\mathcal{G}}}_e^{h\pm}(\tilde{x},y) \cdot \vec{v} &= -\tilde{\vec{u}} \cdot \overline{\overline{\mathcal{G}}}_e^{h\pm}(x,y) \cdot \vec{v}. \end{aligned} \tag{30}$$

From (29) and (30) we conclude that, for all $x \in \mathbb{R}^2$, $y, \vec{v} \in \mathbb{R}^3$,

$$\begin{aligned} \vec{n} \times \overline{\overline{\mathcal{G}}}_h^{e\pm}(x,y) \cdot \vec{v} &= \vec{0}, \quad \vec{n} \cdot \overline{\overline{\mathcal{G}}}_h^{h\pm}(x,y) \cdot \vec{v} &= 0, \\ \vec{n} \times \overline{\overline{\mathcal{G}}}_e^{h\pm}(x,y) \cdot \vec{v} &= \vec{0}, \quad \vec{n} \cdot \overline{\overline{\mathcal{G}}}_e^{e\pm}(x,y) \cdot \vec{v} &= 0. \end{aligned} \tag{31}$$

If we set $T = S^2 \cap \mathbb{R}^3_-$ in (19)[‡], then we obtain for all $y \in \mathbb{R}^3_- \setminus W$

$$\vec{H}(y) = \int_{\mathbb{R}^2} \left[\left(\vec{n} \times \vec{H}(x)\right) \cdot \overline{\overline{\mathcal{G}}}_h^{e-}(x,y) + \left(\vec{n} \times \vec{E}(x)\right) \cdot \overline{\overline{\mathcal{G}}}_h^{h-}(x,y) \right] dS(x),$$

whenever $(\vec{E}, \vec{H})$ satisfies the homogeneous Maxwell's equations (1)–(2) and Silver-Müller radiation conditions (3)–(4) in a neighbourhood of $\overline{\mathbb{R}}^3_-$. Since, for $x \in \mathbb{R}^2$,

$$\left(\vec{n} \times \vec{H}(x)\right) \cdot \overline{\overline{\mathcal{G}}}_h^{e-}(x,y) = -\vec{H}(x) \cdot \left(\vec{n} \times \overline{\overline{\mathcal{G}}}_h^{e-}(x,y)\right) = -\vec{H}(x) \cdot \overline{\overline{0}} = \vec{0},$$

we have

$$\vec{H}(y) = \int_{\mathbb{R}^2} \left(\vec{n} \times \vec{E}(x)\right) \cdot \overline{\overline{\mathcal{G}}}_h^{h-}(x,y) dS(x),$$

and consequently

$$\vec{n} \times \vec{H}(y) = -\int_{\mathbb{R}^2} \left(\vec{n} \times \vec{E}(x)\right) \cdot \left(\overline{\overline{\mathcal{G}}}_h^{h-}(x,y) \times \vec{n}\right) dS(x).$$

The left side has a limit as $y_3 \to 0^-$. Therefore the right side also has a limit as $y_3 \to 0^-$ and

$$\vec{n} \times \vec{H}(y_{12}) = -\lim_{y_3 \to 0^-} \int_{\mathbb{R}^2} \left(\vec{n} \times \vec{E}(x)\right) \cdot \left(\overline{\overline{\mathcal{G}}}_h^{h-}(x,y) \times \vec{n}\right) dS(x),$$

where we have denoted $y_{12} = (y_1, y_2, 0)$. Because of a strong singularity in $\overline{\overline{\mathcal{G}}}_h^{h-}(x,y)$ at $x = y$ the limit cannot be moved under the integral sign. By imitating the preceding procedure we obtain the corresponding expression for $\vec{n} \times \vec{E}(y_{12})$. The two integral equations can also be derived for the upper half space. In the following proposition we put these results together:

Proposition 3.2 *Let $W_{\mu,\gamma} \cap \overline{\mathbb{R}}^3_\pm$ be a compact subset of $\mathbb{R}^3_\pm$, and suppose $(\vec{E}, \vec{H})$ is a C^1-field in a neighbourhood of $\overline{\mathbb{R}}^3_\pm$. If $(\vec{E}, \vec{H})$ satisfies the homogeneous*

[‡]It is clear that (19) also applies to a sum of two dipole fields although we derived the formula for a single dipole only.

Maxwell's equations (1)–(2) and Silver-Müller radiation conditions (3)–(4) in a neighbourhood of $\overline{\mathbb{R}}^3_\pm$*, then*

$$\vec{n} \times \vec{H}(y_{12}) = - \lim_{y_3 \to 0\pm} \int_{\mathbb{R}^2} \left(\vec{n} \times \vec{E}(x)\right) \cdot \left(\overline{\overline{\mathcal{G}}}_h^{h\pm}(x,y) \times \vec{n}\right) dS(x), \quad (32)$$

$$\vec{n} \times \vec{E}(y_{12}) = - \lim_{y_3 \to 0\pm} \int_{\mathbb{R}^2} \left(\vec{n} \times \vec{H}(x)\right) \cdot \left(\overline{\overline{\mathcal{G}}}_e^{e\pm}(x,y) \times \vec{n}\right) dS(x), \quad (33)$$

for $\vec{n} = \mp\vec{e}_3$.

At this point we are ready to define concepts analogous to the Dirichlet-to-Neumann map Λ and its inverse Λ^{-1}:

Definition 3.2 *The equation (32) defines the* admittance map

$$Y_\pm : \vec{n} \times \vec{E}\big|_{\mathbb{R}^2} \mapsto \vec{n} \times \vec{H}\big|_{\mathbb{R}^2}$$

and (33) defines the impedance map

$$Z_\pm : \vec{n} \times \vec{H}\big|_{\mathbb{R}^2} \mapsto \vec{n} \times \vec{E}\big|_{\mathbb{R}^2}$$

for the half space $\mathbb{R}^3_\pm$.

As a straightforward consequence of the definitions we see that $Y_\pm$ and its inverse $Z_\pm$ are independent of the material distribution in $\mathbb{R}^3_\mp$. A note-worthy feature of the half space geometry, when compared with the interior of a closed surface as in [7] or [8], is that these maps exist for all frequencies $\omega > 0$. This is, of course, imposed by the radiation condition.

3.3 Unique Continuation of Local Measurements

Assume that W is a compact subset of $\mathbb{R}^3 \setminus \mathbb{R}^2$, $\mathbb{R}^3 \setminus W$ is connected and $W_{\mu,\gamma} \subset W$. We consider the $\vec{u}$-component

$$(\vec{u} \cdot \overline{\overline{G}}_h^e(x,y) \cdot \vec{v},\ \vec{u} \cdot \overline{\overline{G}}_h^h(x,y) \cdot \vec{v}) \quad (34)$$

of an electromagnetic field at x generated by a magnetic dipole $\vec{v}$ located at y. By a *local measurement* we mean an arrangement which results in the knowledge of (34) for

$$(x,y) \in U \dot{\times} U,\ \vec{u} \in \mathbb{R}^2,\ \vec{v} \in \mathbb{R}^2,$$

where $U \neq \emptyset$ is a fixed open subset of $\mathbb{R}^2$ and $A \dot{\times} B$ is defined for sets A and B by

$$A \dot{\times} B = \{ (x,y) \in A \times B \mid x \neq y \}.$$

A local measurement extends, as a consequence of Lemma 3.1, to the knowledge of (34) for all

$$(x,y) \in (\mathbb{R}^3 \setminus W)\dot{\times}U,\ \vec{u} \in \mathbb{R}^3,\ \vec{v} \in \mathbb{R}^2, \tag{35}$$

and further, by reciprocity (21), to the knowledge of

$$\vec{v} \cdot \overline{\overline{G}}_h^h(y,x) \cdot \vec{u} \;=\; \vec{u} \cdot \overline{\overline{G}}_h^h(x,y) \cdot \vec{v}$$

on the conditions (35). Especially, we know the perpendicular tangential components $\vec{e}_l \cdot \overline{\overline{G}}_h^h(\cdot,x) \cdot \vec{u}$, $l=1,2$, of the field $\overline{\overline{G}}_h^h(\cdot,x) \cdot \vec{u}$. Since these components satisfy the Helmholtz equation on $\mathbb{R}^2 \setminus \{x\}$ the Unique Continuation Principle for elliptic partial differential equations (see [3], p. 212, or [5], pp. 64–69) implies the existence of a unique extension of $\vec{e}_l \cdot \overline{\overline{G}}_h^h(\cdot,x) \cdot \vec{u}$ from $U \setminus \{x\}$ to $\mathbb{R}^2 \setminus \{x\}$, $l=1,2$. If we know the restrictions $\mu\big|_{\mathbb{R}^3_+}$, $\gamma\big|_{\mathbb{R}^3_+}$, then for $x \in \mathbb{R}^3_- \setminus W$ and $\vec{u} \in \mathbb{R}^3$ the impedance map Z_+ gives a unique

$$\vec{n} \times \overline{\overline{G}}_h^e(\cdot,x)\big|_{\mathbb{R}^2} \cdot \vec{u} \;=\; Z_+(\vec{n} \times \overline{\overline{G}}_h^h(\cdot,x)\big|_{\mathbb{R}^2} \cdot \vec{u}),$$

and from Lemma 3.1 we obtain a unique field

$$(\overline{\overline{G}}_h^e(\cdot,x)\big|_{\mathbb{R}^3\setminus W} \cdot \vec{u},\ \overline{\overline{G}}_h^h(\cdot,x)\big|_{\mathbb{R}^3\setminus W} \cdot \vec{u}).$$

Thus we have proved:

Lemma 3.3 *Let W be a compact subset of $\mathbb{R}^3 \setminus \mathbb{R}^2$, such that $\mathbb{R}^3 \setminus W$ is connected and $W_{\mu,\gamma} \subset W$. We suppose further that $\mu\big|_{\mathbb{R}^3_+}$ and $\gamma\big|_{\mathbb{R}^3_+}$ are known. If $U \neq \emptyset$ is an open subset of $\mathbb{R}^2$, then the restriction*

$$\begin{aligned} (U\dot{\times}U) \times (\mathbb{R}^2 \times \mathbb{R}^2) \quad &\to \quad \mathbb{C} \times \mathbb{C}, \\ ((x,y),(\vec{u},\vec{v})) \quad &\mapsto \quad (\vec{u} \cdot \overline{\overline{G}}_h^e(x,y) \cdot \vec{v},\ \vec{u} \cdot \overline{\overline{G}}_h^h(x,y) \cdot \vec{v}), \end{aligned}$$

determines the map

$$\begin{aligned} ((\mathbb{R}^3_- \setminus W)\dot{\times}(\mathbb{R}^3_- \setminus W)) \times (\mathbb{R}^3 \times \mathbb{R}^3) \quad &\to \quad \mathbb{C} \times \mathbb{C}, \\ ((x,y),(\vec{u},\vec{v})) \quad &\mapsto \quad (\vec{u} \cdot \overline{\overline{G}}_h^e(x,y) \cdot \vec{v},\ \vec{u} \cdot \overline{\overline{G}}_h^h(x,y) \cdot \vec{v}), \end{aligned}$$

uniquely.

3.4 From Local Measurements to Material Parameters

In what follows W is a fixed compact subset of $\mathbb{R}^3_-$ and $W_{\mu,\gamma} \subset W$. Because of the formulation of the THEOREM B in [8] we also require that ε, μ and σ are

C^∞-functions and there are strictly positive constants ε_m, ε_M, μ_m, μ_M and σ_M for which

$$\varepsilon_m \le \varepsilon(x) \le \varepsilon_M, \quad \mu_m \le \mu(x) \le \mu_M, \quad 0 \le \sigma(x) \le \sigma_M.$$

In [8] the fields of magnetic dipoles are defined as distributions that satisfy Maxwell's equations of type (10) and radiation conditions. For our purposes it is crucial that these distributions actually turn out to be C^1-fields (as a matter of fact they are C^∞) which can be seen by representing them by the layer potential operators (see [8], LEMMA 3.5 on p. 1138 and (23) on p. 1139). This guarantees that the conditions (i)–(iv) in Section 2.2 are satisfied. Now we are ready to prove the first of our main results:

Theorem 3.4 *Assume that W, ε, μ and σ fulfill the foregoing requirements and $U \neq \emptyset$ is an open subset of $\mathbb{R}^2$. The local measurement*

$$\begin{array}{rcl} (U \dot{\times} U) \times (\mathbb{R}^2 \times \mathbb{R}^2) & \to & \mathbb{C} \times \mathbb{C}, \\ ((x,y),(\vec{u},\vec{v})) & \mapsto & (\vec{u} \cdot \overline{\overline{G}}{}^e_h(x,y) \cdot \vec{v},\ \vec{u} \cdot \overline{\overline{G}}{}^h_h(x,y) \cdot \vec{v}), \end{array} \tag{36}$$

on U determines the material parameters ε, μ and σ in W uniquely.

Proof: Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^3$ with a smooth boundary, connected exterior $\mathbb{R}^3 \setminus \overline{\Omega}$, and the property that $W \subset \Omega \subset \overline{\Omega} \subset \mathbb{R}^3_-$. THEOREM B in [8] states that the boundary measurement

$$\begin{array}{l} (\vec{u}_x \cdot \overline{\overline{G}}{}^e_h(x,y) \cdot \vec{v}_y,\ \vec{u}_x \cdot \overline{\overline{G}}{}^h_h(x,y) \cdot \vec{v}_y), \\ \vec{u}_x \cdot \vec{n}(x) = 0, \quad \vec{v}_y \cdot \vec{n}(y) = 0, \quad (x,y) \in \partial\Omega \dot{\times} \partial\Omega, \end{array} \tag{37}$$

uniquely determines the material distribution within Ω except for a discrete set $F_\Omega \subset \mathbb{R}$ of magnetic resonance frequencies (these frequencies only appear when $\sigma = 0$). We may assume, without loss of generality, that $\omega \notin F_\Omega$ (otherwise we take another Ω). On the other hand, from Lemma 3.3 we know that the local measurement (36) determines the boundary measurement (37) uniquely. □

4 Determination of μ, ε and σ from Local Admittance Kernel

In this section the values of material parameters in the upper half space $\mathbb{R}^3_+$ are of no importance. We shall derive a counterpart to the uniqueness theorem of Lassas et al (see [4], Theorem 2.1 on p. 681) for $\mathbb{R}^3_-$. It states that for the scalar Schrödinger equation $(\Delta + q(x))u(x) = 0$, where $\operatorname{Im} q(x) > 0$ and $\operatorname{supp}(q - q_0) \subset W \subset \mathbb{R}^3_-$ for a fixed compact W, the inhomogeneity $q\big|_W$ is uniquely determined by knowledge of the Dirichlet-to-Neumann map

$$\Lambda : u\big|_{\mathbb{R}^2} \mapsto \frac{\partial u}{\partial \vec{n}}\Big|_{\mathbb{R}^2}$$

on an open subset $U \neq \emptyset$ of the boundary of the half space*. The map Λ is formally an integral operator whose kernel $K_\Lambda(.,.)$ has a restriction to $U \dot{\times} U$. This makes it meaningful to talk about Λ on U.

We shall prove the corresponding theorem in electromagnetism: $\varepsilon\big|_W$, $\mu\big|_W$ and $\sigma\big|_W$ are uniquely determined by knowledge of the admittance map Y_- on U provided that we a priori know the inhomogeneity to be contained in W. Like Λ, according to Definition 3.2, the map Y_- is formally an integral operator with the kernel $K_{Y_-}(\cdot,\cdot) = \overline{\overline{\mathcal{G}}}_h^{h-}(\cdot,\cdot) \times \vec{n}$.

4.1 Connection between the Kernels of Y and Z

We assume that W is a compact subset of $\mathbb{R}^3_-$ and $W_{\mu,\gamma} \cap \overline{\mathbb{R}}^3_- \subset W$. Let $U \neq \emptyset$ be an open subset of $\mathbb{R}^2$. It is a straightforward consequence of the definition of $\overline{\overline{\mathcal{G}}}_h^{h-}$ and the reciprocity formula (21) that

$$\overline{\overline{\mathcal{G}}}_h^{h-}(y,x) \;=\; \overline{\overline{\mathcal{G}}}_h^{h-}(x,y)^t. \tag{38}$$

If $U \neq \emptyset$ is an open subset of $\mathbb{R}^2$, the restriction $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} U}$ is uniquely determined by the restriction $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} U} \times \vec{n}$ of the formal kernel $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{\mathbb{R}^2 \dot{\times} \mathbb{R}^2} \times \vec{n}$ of the admittance map Y_-. This is easily verified by componentwise checking and using the fact that, by definition, $\overline{\overline{\mathcal{G}}}_h^{h-}(\cdot,y) \cdot \vec{n} = \vec{0}$ for all $y \in \mathbb{R}^2$.

Denote

$$\ddot{W} \;=\; \{\, x \in \mathbb{R}^3 \mid x \in W \ \text{ or } \ \tilde{x} \in W \,\}$$

and assume that $\mathbb{R}^3 \setminus \ddot{W}$ is connected (or equivalently: $\mathbb{R}^3_- \setminus W$ is connected). For a fixed $y \in U$ and $\vec{v} \in \mathbb{R}^3$ we see from (31) that $\vec{n} \times \overline{\overline{\mathcal{G}}}_h^{e-}(\cdot,y)\big|_{U \setminus \{y\}} \cdot \vec{v} = \vec{0}$. On the other hand, Lemma 3.1 implies that

$$(\vec{n} \times \overline{\overline{\mathcal{G}}}_h^{e-}(\cdot,y) \cdot \vec{v},\; \vec{n} \times \overline{\overline{\mathcal{G}}}_h^{h-}(\cdot,y) \cdot \vec{v})\big|_{U \setminus \{y\}}$$

determines the field

$$(\overline{\overline{\mathcal{G}}}_h^{e-}(\cdot,y) \cdot \vec{v},\; \overline{\overline{\mathcal{G}}}_h^{h-}(\cdot,y) \cdot \vec{v})\big|_{\mathbb{R}^3 \setminus (\{y\} \cup \ddot{W})}$$

uniquely. We conclude that $(\overline{\overline{\mathcal{G}}}_h^{e-}, \overline{\overline{\mathcal{G}}}_h^{h-})\big|_{(\mathbb{R}^3 \setminus \ddot{W}) \dot{\times} U}$ is uniquely determined by $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} U}$. From (38) it follows that especially $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} (\mathbb{R}^3 \setminus \ddot{W})}$ and thus

*For the sake of coherence we use a notation different from that of the referred article.

$\vec{n} \times \overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} (\mathbb{R}^3 \setminus \ddot{W})}$ are uniquely determined by $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} U}$. According to (31) $\vec{n} \times \overline{\overline{\mathcal{G}}}_h^{e-}\big|_{U \dot{\times} (\mathbb{R}^3 \setminus \ddot{W})} = \overline{\overline{0}}$. Therefore Lemma 3.1 implies that

$$(\overline{\overline{\mathcal{G}}}_h^{e-}, \overline{\overline{\mathcal{G}}}_h^{h-})\big|_{(\mathbb{R}^3 \setminus \ddot{W}) \dot{\times} (\mathbb{R}^3 \setminus \ddot{W})} \tag{39}$$

is uniquely determined by $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} U}$.

Because of the reciprocity (23), the first component of (39) determines the second component of

$$(\overline{\overline{\mathcal{G}}}_e^{e-}, \overline{\overline{\mathcal{G}}}_e^{h-})\big|_{(\mathbb{R}^3 \setminus \ddot{W}) \dot{\times} (\mathbb{R}^3 \setminus \ddot{W})} \tag{40}$$

via transposition. The first component of (40) is then uniquely determined by Maxwell's equations. Hence we have proved:

Proposition 4.1 *Let W be a compact subset of $\mathbb{R}^3_-$, such that $\mathbb{R}^3_- \setminus W$ is connected, and let $U \neq \emptyset$ be an open subset of $\mathbb{R}^2$. If $W_{\mu,\gamma} \cap \overline{\mathbb{R}}^3_- \subset W$, then the pair*

$$(\overline{\overline{\mathcal{G}}}_h^{e-}, \overline{\overline{\mathcal{G}}}_h^{h-})\big|_{(\mathbb{R}^3 \setminus \ddot{W}) \dot{\times} (\mathbb{R}^3 \setminus \ddot{W})}, \quad (\overline{\overline{\mathcal{G}}}_e^{e-}, \overline{\overline{\mathcal{G}}}_e^{h-})\big|_{(\mathbb{R}^3 \setminus \ddot{W}) \dot{\times} (\mathbb{R}^3 \setminus \ddot{W})} \tag{41}$$

is uniquely determined by the restriction $\overline{\overline{\mathcal{G}}}_h^{h-}\big|_{U \dot{\times} U} \times \vec{n}$ of the formal kernel of the lower admittance map Y_-.

4.2 From Local Kernel to Material Parameters

According to Definition 3.1

$$\begin{aligned} \overline{\overline{G}}_h^{h-}(\cdot, y) \cdot \vec{v} &= \frac{1}{2} \overline{\overline{\mathcal{G}}}_h^{h-}(\cdot, y) \cdot \vec{v}, \\ \overline{\overline{G}}_e^{h-}(\cdot, y) \cdot \vec{v} &= \frac{1}{2} \overline{\overline{\mathcal{G}}}_e^{h-}(\cdot, y) \cdot \vec{v}, \end{aligned}$$

for all $y \in \mathbb{R}^2$ and $\vec{v} \in \mathbb{R}^2$. From these equations and the reciprocity rules (21) and (23) it follows that the tangential components on $\mathbb{R}^2$ for all magnetic dipoles located in $\mathbb{R}^3 \setminus \ddot{W}$ are uniquely determined by (41). Lemma 3.1 then implies that (41) determines the total fields of the above dipoles everywhere in $\mathbb{R}^3 \setminus \ddot{W}$. Now we can convince us of the uniqueness of the material distribution in $\ddot{W}$ just like we did in Theorem 3.4. In other words, we have proved the second of the two main theorems:

Theorem 4.2 *Suppose ε, μ and σ are C^∞-functions in $\mathbb{R}^3_-$ and there are strictly positive constants ε_m, ε_M, μ_m, μ_M and σ_M for which*

$$\varepsilon_m \leq \varepsilon(x) \leq \varepsilon_M, \quad \mu_m \leq \mu(x) \leq \mu_M, \quad 0 \leq \sigma(x) \leq \sigma_M,$$

whenever $x \in \mathbb{R}^3_-$. *Let* W *be a compact subset of* $\mathbb{R}^3_-$, *such that* $W_{\mu,\gamma} \cap \overline{\mathbb{R}}^3_- \subset W$. *If* $U \neq \emptyset$ *is an open subset of* $\mathbb{R}^2$, *then the values of* ε, μ *and* σ *in* $W \cap \mathbb{R}^3_-$ *are uniquely determined by the restriction* $\overline{\overline{\mathcal{G}}}_h^{h-}|_{U \dot{\times} U} \times \vec{n}$ *of the formal kernel of the lower admittance map* Y_-.

Note that in Theorem 3.4 it is required that the inhomogeneity lies totally in the lower half space $\mathbb{R}^3_-$ but in case of Theorem 4.2 this assumption is unnecessary. In the proof of the former result we used the upper impedance map Z_+ while the latter is based on Y_- and Z_-.

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Appendix. Field Continuation Outside a Surface Element

In this appendix we will generalize Lemma 3.1 for curved surfaces. By a *surface element* T we mean a 2-manifold in $\mathbb{R}^3$ with the property that there is an open set $D \subset \mathbb{R}^3$ and a C^2-diffeomorphism $\varphi : \mathbb{R}^3 \to D$ for which $\varphi(\mathbb{R}^2) = T$. The surface element T corresponds to U in Lemma 3.1.

Lemma 3.1' *Suppose* T *and* φ *are as above. Denote* $V = \varphi(\overline{\mathbb{R}}^3_+)$ *and let* $(\vec{E}, \vec{H})$ *be a* C^1*-field, such that for every* $x \in \varphi(\mathbb{R}^3_+)$

$$\begin{aligned} \nabla \times \vec{E}(x) - i\omega\mu_0 \vec{H}(x) &= \vec{0}, \\ \nabla \times \vec{H}(x) + i\omega\varepsilon_0 \vec{E}(x) &= \vec{0}. \end{aligned} \tag{42}$$

Then the tangential field $(\vec{n} \times \vec{E}|_T, \vec{n} \times \vec{H}|_T)$ *determines* $(\vec{E}|_V, \vec{H}|_V)$ *uniquely.*

Proof: Because $(\vec{E}, \vec{H})$ is a C^1-field (42) also holds for $x \in T$. We utilize the generic formula (see [3], p. 162)

$$\mathrm{Div}(\vec{n} \times \vec{E}) \;=\; -\vec{n} \cdot \nabla \times \vec{E}$$

where Div means surface divergence. Applied to $\vec{E}$ and $\vec{H}$ in (42) it yields

$$\mathrm{Div}(\vec{n} \times \vec{E}) \;=\; -i\omega\mu_0 \vec{n} \cdot \vec{H}, \quad \mathrm{Div}(\vec{n} \times \vec{H}) \;=\; i\omega\varepsilon_0 \vec{n} \cdot \vec{E}.$$

Thus if we know the tangential components of $\vec{E}$ and $\vec{H}$ on T, then we also know their normal components. In the sequel we assume that the total field $(\vec{E}\big|_T, \vec{H}\big|_T)$ is known.

Let us keep $u \in \mathbb{R}^2$ fixed and let $x = \varphi(u) \in T$. In case of $\vec{E}$ the chain rule gives

$$D(\vec{E} \circ \varphi)(u) \;=\; D\vec{E}(x) D\varphi(u) \;\in\; \mathbb{R}^{3\times 3}, \tag{43}$$

where the left side is known for except for the last column. The matrix element r, s on the right side is

$$\nabla E_r(x) \cdot \partial_s \varphi(u). \tag{44}$$

Since $(\partial_1\varphi(u), \partial_2\varphi(u))$ is a basis for the tangent space of T at x we obtain any tangential derivative of E_r, $r = 1, 2, 3$, as a linear combination of the known scalars (44), $s = 1, 2$. Likewise we find out the tangential derivatives of H_r, $r = 1, 2, 3$.

Because the curl operator is invariant under rotations and translations, so are the Maxwell's equations (42). Thefore we can treat them in any orthonormal right-handed coordinate system. Let $(\vec{e}_1, \vec{e}_2)$ be a basis for the tangent space at x and $\vec{e}_3 = \vec{n}(x)$. By solving the six scalar equations equivalent to (42) together with the divergence equations $\nabla \cdot \vec{E}(x) = 0$ and $\nabla \cdot \vec{H}(x) = 0$ we obtain the normal derivatives of $\vec{E}$ and $\vec{H}$ at x.

At this stage we have the Cauchy data at every point x of the surface T. The rest of the proof goes as we did when proving Lemma 3.1 by Holmgren's theorem. □

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