## Chapter 3

# Viscosity solutions, boundary values etc.

## 1. Limits of solutions

We start by reformulating the basic Definition 1.3 and we see by Proposition 2.4 that we can rewrite Definition 1.3 as follows:

**Definition 3.1.** *Let $\mathcal{A} \subset \mathbb{R}^d$ and assume that $F : \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ is degenerate elliptic (nonicreasing in its last argument), i.e, it satisfies* (1.1). *A function $u : \mathcal{A} \to \mathbb{R}$ is a subsolution of $F = 0$ in $\mathcal{A}$ if $u \in \mathcal{USC}(\mathcal{A})$ and $F \leq 0$ on $J^{2,+}_{\mathcal{A},u}$.*

*Similarly, a function $u : \mathcal{A} \to \mathbb{R}$ is a supersolution of $F = 0$ in $\mathcal{A}$ if $u \in \mathcal{LSC}(\mathcal{A})$ and $F \geq 0$ on $J^{2,-}_{\mathcal{A},u}$.*

Next we prove a result which shows that it is quite easy to handle limits of subsolutions, (and by symmetry supersolutions).

**Theorem 3.2.** *Let $\mathcal{A} \subset \mathbb{R}^d$ be a locally compact nonempty set and assume that*

(i) *$F_n : \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ is degenerate elliptic (nonicreasing in its last argument), i.e, it satisfies* (1.1), *for each $n \geq 1$;*

(ii) *$u_n : \mathcal{A} \to \mathbb{R}$ is a subsolution of $F_n = 0$ in $\mathcal{A}$ for each $n \geq 1$;*

(iii) *$u_\infty(\mathbf{x}) \in \mathbb{R}$ for every $\mathbf{x} \in \mathcal{A}$ where $u_\infty = \limsup^*{}_{n\to\infty} u_n$;*

*Then $u_\infty$ is a subsolution of $F_\infty = 0$ where $F_\infty = \liminf_{*n\to\infty} F_n$.*

**Proof.** Suppose that $(\mathbf{x}, u_\infty(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u_\infty}$. By Theorem 2.8 there is a subsequence $(n_j)$ and elements $(\mathbf{x}_{n_j}, u_{n_j}(\mathbf{x}_{n_j}), \mathbf{p}_{n_j}, X_{n_j}) \in J^{2,+}_{\mathcal{A},u_{n_j}}$ such that $\mathbf{x}_{n_j} \to \mathbf{x}$,

$u_{n_j}(\mathbf{x}_{n_j}) \to u_\infty(\mathbf{x})$, $\mathbf{p}_{n_j} \to \mathbf{p}$, and $X_{n_j} \to X$ as $j \to \infty$. By definition $F_\infty(\mathbf{x}, u_\infty(\mathbf{x}), \mathbf{p}, X) \le \liminf_{j\to\infty} F_{n_j}(\mathbf{x}_{n_j}, u_{n_j}(\mathbf{x}_{n_j}), p_{n_j}, X_{n_j})$ so we get the desired conclusion since by assumption $F_{n_j}(\mathbf{x}_{n_j}, u_{n_j}(\mathbf{x}_{n_j}), p_{n_j}, X_{n_j}) \le 0$. □

Another version of the same result is the following:

**Proposition 3.3.** *Let $\mathcal{A} \subset \mathbb{R}^d$ be a locally compact nonempty set and assume that*

(i) $F : \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *is degenerate elliptic (nonicreasing in its last argument), i.e, it satisfies* (1.1)*;*

(ii) $\mathcal{F}$ *is a nonempty set of subsolutions of the equation $F = 0$ in $\mathcal{A}$;*

(iii) $u^* < \infty$ *in $\mathcal{A}$ where $u(\mathbf{x}) = \sup\{\, v(\mathbf{x}) : v \in \mathcal{F}\,\}$.*

*Then $u^*$ is a subsolution of $F_* = 0$.*

**Proof.** Let $\mathbf{x} \in \mathcal{A}$ be arbitrary. By definition there are points $\mathbf{x}_n \in \mathcal{A}$ such that $\mathbf{x}_n \to \mathbf{x}$ and $\lim_{n\to\infty} u(\mathbf{x}_n) \to u^*(\mathbf{x})$ and therefore we can find functions $u_n \in \mathcal{F}$ so that $\lim_{n\to\infty} u_n(\mathbf{x}_n)$. If we let $u_\infty = \limsup^*{}_{n\to\infty} u_n$ then we have $u_\infty(\mathbf{y}) \le u^*(\mathbf{y})$ for $\mathbf{y} \in \mathcal{A}$ but $u_\infty(\mathbf{x}) \le u^*(\mathbf{x})$ by our choice of $u_n$. This implies that $J^{2,+}_{\mathcal{A},u^*}(\mathbf{x}) \subset J^{2,+}_{\mathcal{A},u_\infty}(\mathbf{x})$ and we get the claim that $F(\mathbf{x}, u^*(\mathbf{x}), \mathbf{p}, X) \le 0$ if $(\mathbf{x}, u^*(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u^*}(\mathbf{x})$ by using the same argument as in the proof of Theorem 3.2 sand the fact that $F_*$ is lower semicontinuous. □

From these results one sees that if one considers subsolutions of an equation $F = 0$, then it is a natural to require that $F$ is lower semiconitnuous, (and correspondingly upper semi continuous when considering super solutions). Furthermore, when applying the Theorem on Sums it is important to note that provided $F$ is lower semicontinuous and $u$ is a subsolution of $F = 0$ then we have in fact $F \le 0$ on $\overline{J^{2,+}_{\mathcal{A},u}}$. In the case where $F$ is not continuous it may be advantageous to consider generalized viscosity solutions:

**Definition 3.4.** *Let $\mathcal{A} \subset \mathbb{R}^d$ and assume that $F : \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ is degenerate elliptic (nonicreasing in its last argument), i.e, it satisfies* (1.1)*. A function $u : \mathcal{A} \to \mathbb{R}$ is a generalized (viscosity) solution of $F = 0$ in $\mathcal{A}$ if $u$ is a subsolution of $F_* = 0$ and a supersolution of $F^* = 0$ in $\mathcal{A}$.*

## 2. Boundary values

In principle it is clearly possible to add boundary values to an equation by simply demanding that the boundary values are to be satisfied (in some sense that can be made more precise). This may, however, lead to difficulties when one tries to prove existence etc. and another approach is to include the boundary conditions in the formulation of the equation in the following way: Suppose $\Omega$ is an open set and $u$ should satisfy the equation $F(\mathbf{x}, u, Du, D^2) = 0$ in $\Omega$ with boundary values $G(\mathbf{x}, u, Du, D^2u)$ on $\partial\Omega$ (in which case one usually assumes that $G$ does not depend on $D^2u$). Then one defines

$$H(\mathbf{x}, r, \mathbf{p}, X) = \begin{cases} F(\mathbf{x}, u, \mathbf{p}, X), & \mathbf{x} \in \Omega, \\ G(\mathbf{x}, u, \mathbf{p}, X), & \mathbf{x} \in \partial\Omega. \end{cases}$$

In this case $H$ is in general not continuous for $\mathbf{x} \in \partial\Omega$ and thus one is usually forced to consider generalized viscosity solutions and one has

$$H_*(\mathbf{x}, r, \mathbf{p}, X) = \min\{F_*(\mathbf{x}, u, \mathbf{p}, X), G_*(\mathbf{x}, u, \mathbf{p}, X)\}, \quad \mathbf{x} \in \partial\Omega,$$

and

$$H^*(\mathbf{x}, r, \mathbf{p}, X) = \max\{F^*(\mathbf{x}, u, \mathbf{p}, X), G^*(\mathbf{x}, u, \mathbf{p}, X)\}, \quad \mathbf{x} \in \partial\Omega,$$

Then one may use the results in Chapter 5 to get existence, but one should note that one pays the price of not knowing if what sense the boundary conditions are satisfied. Furthermore, in order to get comparison results (i.e., that a subsolution is less than or equal to a supersolution) one often needs to know that the boundary conditions are satisfied in a strict sense (e.g., if $G(\mathbf{x}, r, \mathbf{p}, X) = r - \varphi(\mathbf{x})$ then $u(\mathbf{x}) \leq \varphi(\mathbf{x})$).

**Proposition 3.5.** *Assume that $d \geq 1$ and that*

(i) $\Omega \subset \mathbb{R}^d$ *is open;*

(ii) $\mathbf{x}_\diamond \in \partial\Omega$ *satisfies an exterior ball condition, i.e., there is a vector* $\mathbf{n}_\diamond \in \mathbb{R}^d$ *with* $|\mathbf{n}_\diamond| = 1$ *and numbers* $\rho_\diamond > 0$ *and* $\beta_\diamond > 0$ *such that* $\{\, \mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{x}_\diamond - \rho_\diamond \mathbf{n}_\diamond| \leq \rho_\diamond, |\mathbf{x} - \mathbf{x}_\diamond| \leq \beta_\diamond \,\} \cap \overline{\Omega} = \{\mathbf{x}_\diamond\}$*;*

(iii) $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *is lower semicontinuous and degenerate elliptic, i.e., nonincreasing in its last variable;*

(iv)

$$\liminf_{\substack{\mathbf{x}\to\mathbf{x}_\diamond, \mathbf{x}\in\overline{\Omega} \\ \mathbf{n}\to\mathbf{n}_\diamond, |\mathbf{n}_\diamond|=1 \\ r\to u(\mathbf{x}_\diamond) \\ \lambda\to\infty \\ \eta\to\infty}} F\left(\mathbf{x}, r, \lambda\mathbf{n}, -\eta\lambda^2\, \mathbf{n}\otimes\mathbf{n} + \frac{1}{\rho_\diamond}\,\lambda I\right) > 0;$$

(v) $G : \partial\Omega \times \mathbb{R} \to [-\infty, \infty]$ *is lower semicontinuous;*

(vi) $u : \overline{\Omega} \to \mathbb{R}$ *is a subsolution of* $H_* = 0$ *in* $\overline{\Omega}$ *where*

$$H(\mathbf{x}, r, \mathbf{p}, X) = \begin{cases} F(\mathbf{x}, r, \mathbf{p}, X), & \mathbf{x} \in \Omega, \\ G(\mathbf{x}, r), & \mathbf{x} \in \partial\Omega. \end{cases}$$

*Then* $G(\mathbf{x}_\diamond, u(\mathbf{x}_\diamond)) \leq 0$.

**Proof.** Suppose to the contrary that $G(\mathbf{x}_\diamond, u(\mathbf{x}_\diamond)) > 0$. Let $\mathbf{n}_\diamond$ and $\rho_\diamond$ be the unit vector and number in assumption (ii), let $\eta > 1$ be such that (possible by (iv))

$$\liminf_{\substack{\mathbf{x}\to\mathbf{x}_\diamond,\mathbf{x}\in\overline{\Omega}\\ \mathbf{n}\to\mathbf{n}_\diamond,|\mathbf{n}_\diamond|=1\\ r\to u(\mathbf{x}_\diamond)\\ \lambda\to\infty}} F\left(\mathbf{x}, r, \lambda\mathbf{n}, -\eta\lambda^2\,\mathbf{n}\otimes\mathbf{n} + \frac{1}{\rho_\diamond}\,\lambda I\right) > 0.$$

Furthermore, let $\mathbf{y}_\diamond = x_\diamond + \rho_\diamond\mathbf{n}_\diamond$, and define the function $\Psi$ by

$$\Psi(\alpha,\mathbf{x}) = \psi(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond)), \qquad \alpha \geq 1, \quad \mathbf{x}\in\overline{\Omega},$$

where $\psi$ is some twice continuously differentiable function with $\psi'(t) \geq \frac{1}{2}$, $t \geq 0$, $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi''(0) = -2\eta$. (Take for example $\psi(t) = \frac{t}{2} + \frac{1}{8\eta-2}\left(1 - \frac{1}{(1+t)^{4\eta-1}}\right)$ when $t > -\frac{1}{2}$.) Let $\mathcal{A} = \overline{\Omega}\cap\{\,\mathbf{x} : |\mathbf{x}-\mathbf{x}_\diamond| \leq \beta_\diamond\,\}$ and observe that the only point $\mathbf{x}\in\mathcal{A}$ where $\Psi(\alpha,\mathbf{x}) = 0$ is $\mathbf{x}_\diamond$. Since $u$ is upper semicontinuous in the compact set $\mathcal{A}$ it is bounded from above there and for each $\alpha \geq 1$ there is a point $\mathbf{x}_\alpha \in \mathcal{A}$ such that

$$u(\mathbf{x}_\alpha) - \Psi(\alpha, x_\alpha) = \sup_{\mathbf{x}\in\mathcal{A}}\left(u(\mathbf{x}) - \Psi(\alpha,\mathbf{x})\right).$$

It follows from Lemma 4.10 that $\lim_{\alpha\to\infty}\Psi(\alpha,\mathbf{x}_\alpha) = 0$ and that $\lim_{\alpha\to\infty}\mathbf{x}_\alpha = \mathbf{x}_\diamond$. Thus we see that if $\alpha$ is sufficiently large, then $\mathbf{x}_\alpha$ is a local maximum point of $u(\mathbf{x}) - \Psi(\alpha,\mathbf{x})$ in $\overline{\Omega}$. Clearly we have $u(\mathbf{x}_\alpha) \geq u(\mathbf{x}_\diamond)$ and since $u$ is upper semicontinuous we conclude that $\lim_{\alpha\to\infty} u(\mathbf{x}_\alpha) = u(\mathbf{x}_\diamond)$. Since $G$ is lower semicontinuous this implies that if $\alpha$ is sufficiently large and $\mathbf{x}_\alpha\in\partial\Omega$ then $G(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha) > 0$. The assumption that $u$ is a subsolution of $H_* = 0$ then implies that

$$F(\mathbf{x}_\alpha, u(\mathbf{x}_\alpha), D_{\mathbf{x}}\Psi(\alpha,\mathbf{x}_\alpha), D^2_{\mathbf{x}}\Psi(\alpha,\mathbf{x}_\alpha)) \leq 0. \tag{3.1}$$

Now a calculation shows that

$$D_{\mathbf{x}}\Psi(\alpha,\mathbf{x}) = \alpha\frac{\psi'(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|}(\mathbf{x}-\mathbf{y}_\diamond),$$

and

$$D^2_{\mathbf{x}}\Psi(\alpha,\mathbf{x}) = \alpha\frac{\psi'(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|}I$$
$$+\left(\alpha^2\frac{\psi''(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|^2} - \alpha\frac{\psi'(\alpha(|\mathbf{x}-\mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}-\mathbf{y}_\diamond|^3}\right)(\mathbf{x}-\mathbf{y}_\diamond)\otimes(\mathbf{x}-\mathbf{y}_\diamond).$$

Now we know that $\mathbf{x}_\alpha\to\mathbf{x}_\diamond$ and $\psi(\alpha(|\mathbf{x}_\alpha-\mathbf{y}_\diamond|-\rho_\diamond))\to 0$ and hence $\alpha(|\mathbf{x}_\alpha-\mathbf{y}_\diamond| - \rho_\diamond)\to 0$ as $\alpha\to\infty$. Thus we see that if we define $\mathbf{n}_\alpha = \frac{1}{|\mathbf{x}_\alpha-\mathbf{y}_\diamond|}(\mathbf{x}_\alpha-\mathbf{y}_\diamond)$

then

$$\begin{aligned}
\mathbf{n}_\alpha &\to \mathbf{n}_\diamond, \\
\psi'(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond)) &\to 1, \\
\frac{\psi'(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond))}{|\mathbf{x}_\alpha - \mathbf{y}_\diamond|} &\to \frac{1}{\rho_\diamond}, \\
\psi''(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond)) &\to -2\eta,
\end{aligned}$$

as $\alpha \to \infty$.

If we let $\lambda = \alpha\psi'(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond))$ then we see that for sufficiently large $\alpha$ (and hence $\lambda$) we have

$$\alpha^2\psi''(\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\diamond| - \rho_\diamond)) \le -\lambda^2\eta.$$

Thus we see (recall that $|\mathbf{x}_\alpha - \mathbf{x}_\diamond| \ge \rho_\diamond$) that

$$D^2_{\mathbf{x}}\Psi(\alpha, \mathbf{x}_\alpha) \le \frac{\lambda}{\rho_\diamond} I - \eta\lambda^2 \mathbf{n}_\alpha \otimes \mathbf{n}_\alpha,$$

Combining this result with the degenerate ellipticity of $F$ and (2) we get a contradiction from inequality (3.1). ☐

The following result is an example of how one for parabolic equations easily sees that if the initial condition is satisfied in the generalized sense, then it is, under certain conditions actually satisfied in a strict sense as well.

**Proposition 3.6.** *Assume that*

(i) $\mathcal{A} \subset \mathbb{R}^d$ *is open and* $0 < T \le \infty$*;*

(ii) $F : (0,T)\times\mathcal{A}\times\mathbb{R}\times\mathbb{R}^d\times\mathcal{S}(d) \to \mathbb{R}$ *is nonincreasing in its fifth variable, and* $F_* > -\infty$

(iii) $g \in \mathcal{C}(\mathcal{A})$*;*

(iv) $u \in \mathcal{USC}([0,T) \times \mathcal{A}; \mathbb{R})$ *and* $u$ *is a viscosity solution of the equation* $H_* = 0$ *in* $[0,T) \times \mathcal{A})$ *where*

$$H((t,\mathbf{x}), r, (a,\mathbf{p}), \tilde{X}) = \begin{cases} a + F(t, \mathbf{x}, r, \mathbf{p}, \tilde{X}(1:d, 1:d)), & t \in (0,T), \quad \mathbf{x} \in \mathcal{A}, \\ r - g(\mathbf{x}), & t = 0, \quad \mathbf{x} \in \mathcal{A}. \end{cases}$$

*Then* $u(0,\mathbf{x}) \le g(\mathbf{x})$ *for all* $\mathbf{x} \in \mathcal{A}$*.*

**Proof of Proposition 3.6.** Assume to the contrary that there is some point $\mathbf{x}_* \in \mathcal{A}$ such that $u(0,\mathbf{x}_*) > g(\mathbf{x}_*)$.

Let $\tau \in (0,T)$. Since $u$ is upper semicontinuous $u$ is bounded from above in $[0,\tau]\times\mathcal{K}$ for every compact subset $\mathcal{K} \subset \mathcal{A}$. Thus we can find a function $\varphi \in \mathcal{C}(\mathbb{R}^d)$ such that

(3.2) $u(0,\mathbf{x}_*) > \varphi(\mathbf{x}_*)$ and if $u(0,\mathbf{x}) > \varphi(\mathbf{x})$, $\mathbf{x} \in \mathcal{A}$, then $\varphi(\mathbf{x}) \ge g(\mathbf{x})$, and such that there is a compact subset $\mathcal{K} \subset \mathcal{A}$ such that

$$\varphi(\mathbf{x}) \geq u(t, \mathbf{x}) \text{ for } t \in [0, \tau] \text{ and } \mathbf{x} \in \mathcal{A} \setminus \mathcal{K}.$$

Since $F_* > -\infty$, $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$, and $[0, \tau] \times \mathcal{K}$ is compact we can choose a number $c > 0$ such that

$$\begin{aligned} c + \inf\{\, F_*(t, \mathbf{x}, r, D_{\mathbf{x}}\varphi(\mathbf{x}), D^2_{\mathbf{x}}\varphi(\mathbf{x})) : t \in [0, \tau], \mathbf{x} \in \mathcal{K}, \\ \min_{\mathbf{x}\in\mathcal{K}} \varphi(\mathbf{x}) \leq r \leq \sup_{s\in[0,\tau],\mathbf{y}\in\mathcal{A}} u(s, \mathbf{y}) \,\} > 0 \end{aligned} \tag{3.3}$$

Let

$$\Phi(t, \mathbf{x}) = u(t, \mathbf{x}) - \varphi(\mathbf{x}) - ct - \frac{\epsilon}{\tau - t}, \quad t \in [0, \tau) \times \mathcal{A},$$

where $0 < \epsilon < \tau(u(0, \mathbf{x}_*) - \varphi(\mathbf{x}_*))$. From our construction of $\varphi$ and the fact that $\Phi(0, \mathbf{x}_*) > 0$ it follows that the function $\Phi$ achieves its largest value in $[0, \tau) \times \mathcal{A}$ in some point $(t^*, \mathbf{x}^*)$ where $\mathbf{x}^* \in \mathcal{K}$. Thus we have $((t^*, \mathbf{x}^*), u(t^*, \mathbf{x}^*), (c + \frac{\epsilon}{(\tau-t^*)^2}, D_{\mathbf{x}}\varphi(\mathbf{x})), \tilde{X}) \in J^{2,+}_{[0,T)\times\mathcal{A},u}$ where $\tilde{X}(1 : d, 1 : d) = D^2_{\mathbf{x}}\varphi(\mathbf{x})$. But since $u$ is a subsolution of $H_* = 0$ and (3.3) holds we see that we must have $t^* = 0$ and $u(0, \mathbf{x}^*) \leq g(x^*)$. But since we must have $u(0, \mathbf{x}^*) > \varphi(\mathbf{x}^*)$ this is a contradiction by (3.2). □

## 3. Comments

Essentially the same question about whether a generalized viscosity solution of a boundary value problem satisfies the boundary values is often formulated in the theory of elliptic and parabolic equations in terms of ”regular” points, see e.g.[**5**]. In the setting of viscosity solutions this problem has been considered in e.g. [**4**].