Chapter 5

Existence with Perron's method

1. The basic result

Theorem 5.1. *Assume that*

(i) $\mathcal{A} \subset \mathbb{R}^d$ *is locally compact (and nonempty);*

(ii) $F : \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *is degenerate elliptic (nonincreasing in its last argument), i.e, it satisfies* (1.1)*;*

(iii) $\underline{u} : \mathcal{A} \to \mathbb{R}$ *is a subsolution of* $F_* = 0$, $\overline{u} : \mathcal{A} \to \mathbb{R}$ *is a supersolution of* $F^* = 0$ *and* $\underline{u} \leq \overline{u}$, $\underline{u}_* > -\infty$ *and* $\overline{u}^* < \infty$ *in* $\mathcal{A}$*;*

(iv) *Comparison holds in the sense that if in* $\mathcal{A}$, u *is a subsolution of* $F_* = 0$, v *is a supersolution of* $F^* = 0$, $\underline{u}_* \leq u \leq \overline{u}^*$, *and* $\underline{u}_* \leq v \leq \overline{u}^*$, *then* $u \leq v$ *in* $\mathcal{A}$.

(v) *If* $(\mathbf{x}, r, \mathbf{p}, X) \in \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d)$, $F^*(x, r, \mathbf{p}, X) \leq 0$, *and* $(\mathbf{q}, Y) \in J^{2,+}_{\mathcal{A},\chi_\mathcal{A}}(\mathbf{x})$ *then* $F_*(x, r, \mathbf{p} + \mathbf{q}, X + Y) \leq 0$.

Then there is a unique function $u \in \mathcal{C}(\mathcal{A})$ *such that* u *is generalized viscosity solution of* $F = 0$ *in in* $\mathcal{A}$ *(i.e., a subsolution of of* $F_* = 0$ *and a supersolution of* $F^* = 0$*) and* $\underline{u} \leq u \leq \overline{u}$ *in* $\mathcal{A}$.

Recall that $(\mathbf{q}, Y) \in J^{2,+}_{\mathcal{A},\chi_\mathcal{A}}(\mathbf{x})$ for a point $\mathbf{x}$ in the interior of $\mathcal{A}$ if and only if $\mathbf{q} = \mathbf{0}$ and $Y \geq 0$ so $F_*(\mathbf{x}, r, \mathbf{p} + \mathbf{q}, X + Y) \leq F^*(\mathbf{x}, r, \mathbf{p}, X)$ by the basic monotonicity assumption. Thus assumption (v) is a consequence of (ii) if $\mathcal{A}$ is open and we have:

Corollary 5.2. *The conclusion of Theorem* 5.1 *holds if assumptions* (i) *and* (v) *are replaced by the assumption that $\mathcal{A}$ is open.*

This result can be extended to the case where one combines the equation in an open set with the boundary values.

Corollary 5.3. *Assume that*

(i) $\Omega \subset \mathbb{R}^d$ *is open (and nonempty);*

(ii) $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *and* $G : \partial\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *are degenerate elliptic (nonincreasing in their last argument);*

(iii) $\underline{u} : \overline{\Omega} \to \mathbb{R}$ *is a subsolution of* $H_* = 0$, $\overline{u} : \overline{\Omega} \to \mathbb{R}$ *is a supersolution of* $H^* = 0$ *and* $\underline{u} \le \overline{u}$, $\underline{u}_* > -\infty$ *and* $\overline{u}^* < \infty$ *in* $\overline{\Omega}$ *where*

$$H(\mathbf{x}, r, \mathbf{p}, X) = \begin{cases} F(\mathbf{x}, u, \mathbf{p}, X), & \mathbf{x} \in \Omega, \\ G(\mathbf{x}, u, \mathbf{p}, X), & \mathbf{x} \in \partial\Omega. \end{cases}$$

(iv) *Comparison holds in the sense that if in* $\overline{\Omega}$, u *is a subsolution of* $H_* = 0$, v *is a supersolution of* $H^* = 0$, $\underline{u}_* \le u \le \overline{u}^*$, *and* $\underline{u}_* \le v \le \overline{u}^*$, *then* $u \le v$ *in* $\overline{\Omega}$.

(v) *If* $(\mathbf{x}, r, \mathbf{p}, X) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d)$, $G^*(x, r, \mathbf{p}, X) \le 0$, *and* $(\mathbf{q}, Y) \in J^{2,+}_{\overline{\Omega}, \chi_{\overline{\Omega}}}(\mathbf{x})$ *then* $G_*(x, r, \mathbf{p} + \mathbf{q}, X + Y) \le 0$ *(thus in particular if* G *does not depend on its last two arguments).*

Then there is a unique function $u \in \mathcal{C}(\mathcal{A})$ *such that* u *is generalized viscosity solution of* $H = 0$ *in in* $\overline{\Omega}$ *(i.e., a subsolution of of* $H_* = 0$ *and a supersolution of* $H^* = 0$*) and* $\underline{u} \le u \le \overline{u}$ *in* $\overline{\Omega}$.

Finally we observe from the proof that the comparison hypothesis can be weakened at the expense of the uniqueness statement. It is, however, not so clear to what extent this weaker comparison property is easier to verify except in some obvious cases as when one can prove that sub- and supersolutions are continuous.

Corollary 5.4. *Let the assumptions of Theorem* 5.1 *hold except that* (iv) *is replaced by*

(iv') *If* u *is a subsolution of* $F_* = 0$, v *is a supersolution of* $F^* = 0$ *in* $\mathcal{A}$, $u \ge \underline{u}$ *and* $v \le \overline{u}$ *in* $\mathcal{A}$ *and either* $u \le v^*$ *in* $\mathcal{A}$ *or* $u_* \le v$ *in* $\mathcal{A}$, *then* $u \le v$ *in* $\mathcal{A}$.

Then the conlusion, except for uniqueness, of Theorem 5.1 *holds.*

Proof of Theorem 5.1. Let

$$\mathcal{F} = \{\, w \mid \underline{u} \le w \le \overline{u} \text{ in } \mathcal{A} \text{ and } w \text{ is a subsolution of } F_* = 0 \text{ in } \mathcal{A} \,\}$$

and

$$W(\mathbf{x}) = \sup_{w \in \mathcal{F}} w(\mathbf{x}), \quad \mathbf{x} \in \mathcal{A}.$$

First we note by Proposition 3.3 that W^* is a subsolution of $F_* = 0$ (and here we use the assumption that $\overline{u}^* < \infty$).

By definition we know that $\underline{u} \le W^* \le \overline{u}^*$ so by the comparison assumption we have $W^* \le \overline{u}$, thus $W^* \in \mathcal{F}$, which by the defintion of W implies that $W^* \le W$ so that $W^* = W$.

Since $\underline{u}_* \le W_* \le \overline{u}$ we know that W_* is finite and it is by defintion lower semicontinuous. Assume next that W_* is not a supersolution of $F^* = 0$. Then there exists a point in $\mathcal{A}$, for simplicity assumed to be $\mathbf{0}$, such that

$$F^*(\mathbf{0}, W_*(\mathbf{0}), \mathbf{p}, X) < 0 \quad \text{where} \quad (\mathbf{p}, X) \in J^{2,-}_{\mathcal{A},W_*}(\mathbf{0}) \tag{5.1}$$

Now $W_*(\mathbf{x}) \le \overline{u}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{A}$ and if $W_*(\mathbf{0}) = \overline{u}(\mathbf{0})$ then $(\mathbf{p}, X) \in J^{2,-}_{\mathcal{A},\overline{u}}(\mathbf{0})$. But then $F^*(\mathbf{0}, W_*(\mathbf{0}), \mathbf{p}, X) = F^*(\mathbf{0}, \overline{u}(\mathbf{0}), \mathbf{p}, X) \ge 0$ and we have a contradiction. Thus $W_*(\mathbf{0}) < \overline{u}(\mathbf{0})$.

Let

$$u_{\delta,\gamma}(\mathbf{x}) = W_*(\mathbf{0}) + \delta + \langle \mathbf{p}, \mathbf{x} \rangle + \tfrac{1}{2} \langle X\mathbf{x}, \mathbf{x} \rangle - \gamma |\mathbf{x}|^2, \quad \mathbf{x} \in \mathbb{R}^d.$$

By (5.1) and the facts that $W_*(\mathbf{0}) < \overline{u}(\mathbf{0})$, F^* is upper and $\overline{u}$ is lower semicontinuous we conclude that there are positive numbers δ_0, r, and γ such that

(5.2)
$$F^*(\mathbf{x}, u_{\delta,\gamma}(\mathbf{x}), Du_{\delta,\gamma}(\mathbf{x}), D^2 u_{\delta,\gamma}(\mathbf{x})) < 0, \quad 0 < \delta \le \delta_0, \quad \mathbf{x} \in \mathcal{A}, \quad |\mathbf{x}| \le r,$$

and

$$u_{\delta,\gamma}(\mathbf{x}) \le \overline{u}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{A}, \quad |\mathbf{x}| \le r. \tag{5.3}$$

Furthermore, since we have

$$W(\mathbf{x}) \ge W_*(\mathbf{x}) \ge W_*(\mathbf{0}) + \langle \mathbf{p}, \mathbf{x} \rangle + \tfrac{1}{2} \langle X\mathbf{x}, \mathbf{x} \rangle + o(|\mathbf{x}|^2),$$

as $\mathbf{x} \to \mathbf{0}$ and $\mathbf{x} \in \mathcal{A}$ it follows that we may in addition choose r so small that when we take $\delta = \min\{\delta_0, \frac{r^2\gamma}{8}\}$ we get

$$W(\mathbf{x}) > u_{\delta,\gamma}(\mathbf{x}), \quad \tfrac{1}{2} r \le |\mathbf{x}| \le r, \quad \mathbf{x} \in \mathcal{A},$$

Now we define

$$U(\mathbf{x}) = \begin{cases} \max\{W(\mathbf{x}), u_{\delta,\gamma}(\mathbf{x})\}, & \text{if } \mathbf{x} \in \mathcal{A} \text{ and } |\mathbf{x}| < r, \\ W(\mathbf{x}), & \text{if } \mathbf{x} \in \mathcal{A} \text{ and } |\mathbf{x}| \ge r. \end{cases}$$

Clearly, U is upper semicontinuous and by (5.3) and the fact that $W = W^* \in \mathcal{F}$ we have $\underline{u} \le U \le \overline{u}$. In order to show that U is a subsolution of $F_* = 0$ we assume that $(\mathbf{x}, U(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},U}$. If $U(\mathbf{x}) = W(x)$ we have $(\mathbf{p}, X) \in J^{2,+}_{\mathcal{A},W}(\mathbf{x})$ so that $F_*(\mathbf{x}, U(\mathbf{x}), \mathbf{p}, X) \le 0$. Thus it remains to consider the case where $U(\mathbf{x}) = u_{\delta,\gamma}(\mathbf{x})$. Then we have $|\mathbf{x}| < \frac{r}{2}$ and $(\mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u_{\delta,\gamma}}(\mathbf{x})$. By Proposition 2.5

we know that $(\mathbf{p} - D\varphi(\mathbf{x}), X - D^2\varphi(\mathbf{x})) \in J^{2,+}_{\mathcal{A},\chi_\mathcal{A}}(\mathbf{x})$. By (5.2) and assumption (v) we conclude that $F_*(\mathbf{x}, U(\mathbf{x}), \mathbf{p}, X) \leq 0$. Thus U is a subsolution and by the defintion of W we have $U(\mathbf{x}) \leq W(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{A}$. Thus $u_{\delta,\gamma}(\mathbf{x}) \leq W(\mathbf{x})$ when $\mathbf{x} \in \mathcal{A}$ and $|\mathbf{x}| < r$ so that $u_{\delta,\gamma}(\mathbf{0}) = (u_{\delta,\gamma})_*(\mathbf{0}) \leq W_*(\mathbf{0})$ but this is a contradiction because $u_{\delta,\gamma}(\mathbf{0}) = W_*(\mathbf{0}) + \delta$.

Thus we have shown that W_* is a supersolution of $F^* = 0$ and by comparison we have $W \leq W_*$ so that $W = W_*$ and we have found a solution. The uniqueness of this solution is a consequence of the comparison assumption.

$\square$