Chapter 2

Jets, etc.

1. Semi-continuous functions

Definition 2.1. *Assume that $\mathcal{A} \subset \mathbb{R}^d$ and that $u : \mathcal{A} \to [-\infty, \infty]$. Then*

$$\boxed{\begin{array}{c} u \text{ is upper semicontinuous, or } u \in \mathcal{USC}(\mathcal{A}) \\ \Leftrightarrow \limsup_{\substack{\mathbf{x}\to\mathbf{x}_0 \\ \mathbf{x}\in\mathcal{A}}} u(\mathbf{x}) \le u(\mathbf{x}_0), \quad \mathbf{x}_0 \in \mathcal{A}, \end{array}}$$

and

$$\boxed{\begin{array}{c} u \text{ is lower semicontinuous, or } u \in \mathcal{LSC}(\mathcal{A}) \\ \Leftrightarrow \liminf_{\substack{\mathbf{x}\to\mathbf{x}_0 \\ \mathbf{x}\in\mathcal{A}}} u(\mathbf{x}) \ge u(\mathbf{x}_0), \quad \mathbf{x}_0 \in \mathcal{A} \end{array}}$$

Clearly a function is both upper and lower semicontinuous precisely when it is continuous.

There are several equivalent characterizations of semi-continuity:

Theorem 2.2. *Suppose that $\mathcal{A} \subset \mathbb{R}^d$ and $u : \mathcal{A} \to [-\infty, \infty]$. Then the following conditions are equivalent:*

(a) $u \in \mathcal{USC}(\mathcal{A})$;

(b) *The set* $\{ \mathbf{x} \in \mathcal{A} : u(\mathbf{x}) < \alpha \}$ *is open in* $\mathcal{A}$ *for each* $\alpha \in \mathbb{R}$;

(c) *There is a sequence* $(u_n)_{n=1}^{\infty}$ *of continuous functions in* $\mathcal{A}$ *such* $\lim_{n\to\infty} u_n(\mathbf{x}) = u(\mathbf{x})$ *and* $u_n(\mathbf{x}) \ge u(\mathbf{x})$ *for each* $\mathbf{x} \in \mathcal{A}$ *and* $n \ge 1$.

(d) *There is a set* $\{u_i\}_{i\in\mathcal{I}}$ *of continuous functions in* $\mathcal{A}$ *such that* $u(\mathbf{x}) = \inf_{i\in\mathcal{I}} u_i(\mathbf{x})$ *for every* $x \in \mathcal{A}$;

(e) $-u \in LSC(\mathcal{A})$;

If one has a function that is not necessarily upper or lower semicontinuous one can take the upper and lower semicontinuous enevelopes, see Definition 2.6 below.

2. "Super- and subjets"

Here we introduce some notation which makes it possible to formulate the notion of sub and super-solutions in a slightly different way.

Definition 2.3. *Let $\mathcal{A} \subset \mathbb{R}^d$ and $u : \mathcal{A} \to [-\infty, \infty]$. Then*

$$J^{2,+}_{\mathcal{A},u} = \Big\{ (\mathbf{x}, r, \mathbf{p}, X) \in \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) : \mathbf{x} \in \mathcal{A}, \quad u(\mathbf{x}) = r,$$
$$u(\mathbf{y}) \le u(\mathbf{x}) + \langle p, \mathbf{y} - \mathbf{x} \rangle + \tfrac{1}{2} \langle X(\mathbf{y} - \mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + o(|\mathbf{y} - \mathbf{x}|^2),$$
$$\textit{as} \quad \mathbf{y} \to \mathbf{x}, \quad \mathbf{y} \in \mathcal{A} \Big\},$$

and

$$J^{2,+}_{\mathcal{A},u}(\mathbf{x}) = \{ (\mathbf{p}, X) \in \mathbb{R}^d \times \mathcal{S}(d) : (\mathbf{x}, u(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u} \}.$$

Similarly, one defines

$$J^{2,-}_{\mathcal{A},u} = \Big\{ (\mathbf{x}, r, \mathbf{p}, X) \in \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) : \mathbf{x} \in \mathcal{A}, \quad u(\mathbf{x}) = r,$$
$$u(\mathbf{y}) \ge u(\mathbf{x}) + \langle p, \mathbf{y} - \mathbf{x} \rangle + \tfrac{1}{2} \langle X(\mathbf{y} - \mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + o(|\mathbf{y} - \mathbf{x}|^2),$$
$$\textit{as}\ \mathbf{y} \to \mathbf{x}, \quad \mathbf{y} \in \mathcal{A} \Big\},$$

and

$$J^{2,-}_{\mathcal{A},u}(\mathbf{x}) = \{ (\mathbf{p}, X) \in \mathbb{R}^d \times \mathcal{S}(d) : (\mathbf{x}, u(\mathbf{x}), \mathbf{p}, X) \in J^{2,-}_{\mathcal{A},u} \}.$$

First we give another characterization of these jets in the spirit of Definition 1.3.

Proposition 2.4. *Let $\mathcal{A} \subset \mathbb{R}^d$ and $u : \mathcal{A} \to [-\infty, \infty]$. Then the following conditions are equivalent:*

(i) $(\mathbf{x}, r, \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u}$;

(ii) $(\mathbf{x}, -r, -\mathbf{p}, -X) \in J^{2,-}_{\mathcal{A},-u}$;

(iii) *There is a function $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$ and $s > 0$ such that $D\varphi(\mathbf{x}) = \mathbf{p}$, $D^2\varphi(\mathbf{x}) = X$, and $u(\mathbf{y}) \le \varphi(\mathbf{y}) + u(\mathbf{x}) - \varphi(\mathbf{x})$ for all $\mathbf{y} \in \mathcal{A}$ with $|\mathbf{y} - \mathbf{x}| < s$.*

Proof. It is clear that (i) and (ii) are equivalent and by Taylors's formula that (iii) implies (i). Suppose next that $(\mathbf{x}, r, \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u}$. By definition there is a function h with $\lim_{t \downarrow 0} \frac{h(t}{t^2} = 0$ and $h(0) = 0$ so that

$$u(\mathbf{y}) \le u(\mathbf{x}) + \langle p, \mathbf{y} - \mathbf{x} \rangle + \tfrac{1}{2} \langle X(\mathbf{y} - \mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + h(|\mathbf{y} - \mathbf{x}|),$$

for all $\mathbf{y} \in \mathcal{A}$ with $|\mathbf{y}-\mathbf{x}| \leq s$ for some $s > 0$. We may assume that h is bounded on $[0, 4s]$, and then we can define the function ψ by

$$\psi(t) = 16 \int_0^t (t-\tau)\frac{1}{\tau} \int_{\frac{\tau}{2}}^{\tau} \sup_{0<\sigma\leq 4\eta} \frac{h(\sigma)}{\sigma^2}\, \mathrm{d}\eta\, \mathrm{d}\tau, \quad 0 \leq t \leq s.$$

Then we can easily conlude that $\psi \in \mathcal{C}^2([0,\infty)$ with $\psi(0) = \psi'(0) = \psi''(0) = 0$. Furthermore, since $4\eta \geq t$ when $\eta \geq \frac{\tau}{2}$ and $\tau \geq \frac{t}{2}$ we get

$$\psi(t) \geq 16 \int_{\frac{t}{2}}^t (t-\tau)\frac{1}{\tau} \int_{\frac{\tau}{2}}^{\tau} \frac{h(t)}{t^2}\, \mathrm{d}\eta\, \mathrm{d}\tau = h(t).$$

Now we define the function φ by

$$\varphi(\mathbf{y}) = u(\mathbf{x}) + \langle p, \mathbf{y}-\mathbf{x}\rangle + \tfrac{1}{2}\langle X(\mathbf{y}-\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + \psi(|\mathbf{y}-\mathbf{x}|).$$

This completes the proof since the fact that ψ is twice continuously differentiable with $\psi'(0) = \psi''(0) = 0$ implies that $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$. □

We give here just one further simple result which is relevant for the question of when a smooth classical solution is a viscosity solution:

Proposition 2.5. *Assume that $\mathcal{A} \subset \mathbb{R}^d$ and $\mathbf{x} \in \mathcal{A}$. If $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$, then $(\mathbf{p}, X) \in J^{2,\pm}_{\mathcal{A},\varphi}(\mathbf{x})$ if and only if $(\mathbf{p} - D\varphi(\mathbf{x}), X - D^2\varphi(\mathbf{x})) \in J^{2,\pm}_{\mathcal{A},\chi_{\mathcal{A}}}(\mathbf{x})$ and if $u \in \mathcal{USC}(\mathcal{A})$ and $(\mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u}(\mathbf{x})$ then $(\mathbf{p}+\mathbf{q}, X+Y) \in J^{2,+}_{\mathcal{A},u}(\mathbf{x})$ for every $(\mathbf{q}, Y) \in J^{2,+}_{\mathcal{A},\chi_{\mathcal{A}}}(\mathbf{x})$.*

Proof of Proposition 2.5. We only consider the $J^{2,+}$-jets. Assume that $(\mathbf{p}, X) \in J^{2,+}_{\mathcal{A},\varphi}(\mathbf{x})$. so that

$$\varphi(\mathbf{y}) \leq \varphi(\mathbf{x}) + \langle \mathbf{p}, \mathbf{y}-\mathbf{x}\rangle + \tfrac{1}{2}\langle X(\mathbf{y}-\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + o(|\mathbf{y}-\mathbf{x}|^2),$$

as $\mathbf{y} \to \mathbf{x}$ and $\mathbf{y} \in \mathcal{A}$. By Taylors formula we have

$$\varphi(\mathbf{y}) = \varphi(\mathbf{x}) + \langle D\varphi(\mathbf{x}), \mathbf{y}-\mathbf{x}\rangle + \tfrac{1}{2}\left\langle D^2\varphi(\mathbf{x})(\mathbf{y}-\mathbf{x}), \mathbf{y}-\mathbf{x}\right\rangle + o(|\mathbf{y}-\mathbf{x}|^2),$$

as $\mathbf{y} \to \mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^d$. Subtracting both sides of the equation from those of the inequality and adding $1 = \chi_{\mathcal{A}}(\mathbf{y}) = \chi_{\mathcal{A}}(\mathbf{x})$ we get the desired conclusion. For the converse and the second claim, one argues in a similar manner. □

One can define first order jets as well, and combinations of first and second order jets which could be used for defining so called parabolic jets.

3. Envelopes, limsups, etc.

Definition 2.6. *Let $\mathcal{A} \subset \mathbb{R}^d$ and $u : \mathcal{A} \to [-\infty, \infty]$. Then*

$$u^*(\mathbf{x}) = \lim_{s \downarrow 0} \sup\{\, u(\mathbf{y}) : y \in \mathcal{A}, \quad |\mathbf{y} - \mathbf{x}| \le s \,\},$$
$$u_*(\mathbf{x}) = \lim_{s \downarrow 0} \inf\{\, u(\mathbf{y}) : y \in \mathcal{A}, \quad |\mathbf{y} - \mathbf{x}| \le s \,\}.$$

for $\mathbf{x} \in \mathbb{R}^d$.

In particular, one sees that $u^*(\mathbf{x}) = -\infty$ and $u_* = +\infty$ when $\mathbf{x}$ belongs to the interior of the complement of $\mathcal{A}$.

Definition 2.7. *Let $\mathcal{A} \subset \mathbb{R}^d$ and $(u_n)_{n \in \mathbb{N}}$ be a sequnce of functions: $\mathcal{A} \to [-\infty, \infty]$. Then*

$$(\limsup_{n \to \infty}{}^* u_n)(\mathbf{x}) = \lim_{n \to \infty} \sup\{\, u_m(\mathbf{y}) : m \ge n, \quad y \in \mathcal{A}, \quad |\mathbf{y} - \mathbf{x}| \le \tfrac{1}{n} \,\},$$
$$(\liminf_{n \to \infty}{}_* u_n)(\mathbf{x}) = \lim_{n \to \infty} \inf\{\, u_m(\mathbf{y}) : m \ge n, \quad y \in \mathcal{A}, \quad |\mathbf{y} - \mathbf{x}| \le \tfrac{1}{n} \,\}.$$

for $\mathbf{x} \in \mathbb{R}^d$.

There are several obvious points to be made but we go straight for a harder and more fundamental result.

Theorem 2.8. *Let $\mathcal{A} \subset \mathbb{R}^d$ be locally compact and let $(u_n)_{n \in \mathbb{N}}$ be a sequence of upper semicontinuous functions: $\mathcal{A} \to [-\infty, \infty]$ and let $u_\infty = \limsup^*{}_{n \to \infty} u_n$. Then*

$$J^{2,+}_{\mathcal{A},u_\infty} \subset \cap_{n \in \mathbb{N}} \overline{\cup_{m=n}^{\infty} J^{2,+}_{\mathcal{A},u_n}},$$

that is, if $(\mathbf{x}, u_\infty(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u_\infty}$ then there is a subsequence (n_j) and elements $(\mathbf{x}_j, u_{n_j}(\mathbf{x}_j), \mathbf{p}_j, X_j) \in J^{2,+}_{\mathcal{A},u_{n_j}}$ such that $\mathbf{x}_j \to \mathbf{x}$, $u_{n_j}(\mathbf{x}_j) \to u_\infty(\mathbf{x})$, $\mathbf{p}_j \to \mathbf{p}$, and $X_j \to X$ as $j \to \infty$.

Proof. Assume that $(\mathbf{x}, u_\infty(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathcal{A},u_\infty}$ and note that we may, without loss of generality, assume that $\mathbf{x} = \mathbf{0}$. By the assumption that $\mathcal{A}$ is locally compact and by the definition of $J^{2,+}_{\mathcal{A},u_\infty}$ we know that for every $\delta > 0$ there is a number $s > 0$ such that $N_s \stackrel{\text{def}}{=} \{\, \mathbf{y} \in \mathcal{A} : |\mathbf{y}| \le s \,\}$ is compact and

$$u_\infty(\mathbf{y}) \le u_\infty(\mathbf{0}) + \langle \mathbf{p}, \mathbf{y} \rangle + \tfrac{1}{2} \langle X\mathbf{y}, \mathbf{y} \rangle + \delta |\mathbf{y}|^2, \quad \mathbf{y} \in N_s. \tag{2.1}$$

Let $\varphi(\mathbf{y}) = \langle \mathbf{p}, \mathbf{y} \rangle + \frac{1}{2} \langle X\mathbf{y}, \mathbf{y} \rangle + 2\delta |\mathbf{y}|^2$ for $\mathbf{y} \in \mathbb{R}^d$. By the definition of u_∞ there exists $\mathbf{x}_j \in \mathcal{A}$ such that $(\mathbf{x}_j, u_{n_j}(\mathbf{x}_j)) \to (\mathbf{0}, u_\infty(\mathbf{0}))$ as $j \to \infty$. Let $\mathbf{x}_j^*$ be the point where the function $\mathbf{y} \in N_s \to u_{n_j}(\mathbf{y}) - \varphi(\mathbf{y})$ achieves its maximum so that

$$u_{n_j}(\mathbf{y}) \le u_{n_j}(\mathbf{x}_j^*) + \varphi(\mathbf{y}) - \varphi(\mathbf{x}_j^*), \quad \mathbf{y} \in N_s. \tag{2.2}$$

Suppose that $\hat{\mathbf{x}}_j \to \mathbf{z}$ (taking a subsequence, again denoted by n_j, if necessary). Taking $\mathbf{y} = \mathbf{x}_j$ in (2.2) and $\mathbf{y} = \mathbf{z}$ in (2.1) we conclude since $u_{n_j}(\mathbf{x}_j) \to u_\infty(\mathbf{0})$ and $\liminf_{j\to\infty} u_{n_j}(\mathbf{x}_j^*) \le \limsup_{j\to\infty} u_{n_j}(\mathbf{x}_j^*) \le u_\infty(\mathbf{z})$ that

$$u_\infty(\mathbf{0}) = \lim_{j\to\infty} u_{n_j}(\mathbf{x}_j) \le \liminf_{j\to\infty} u_{n_j}(\mathbf{x}_j^*) + \varphi(\mathbf{0}) - \varphi(\mathbf{z})$$
$$\le u_\infty(\mathbf{z}) - \varphi(\mathbf{z}) \le u_\infty(\mathbf{0}) - \delta|\mathbf{z}|^2.$$

Thus $u_\infty(\mathbf{0}) \le u_\infty(\mathbf{0}) - \delta|\mathbf{z}|^2$ which implies that $\mathbf{z} = 0$ so that $\mathbf{x}_j^* \to 0$ (and there is no need to take a subsequence. But we also have $u_\infty(\mathbf{0}) \le \liminf_{j\to\infty} u_{n_j}(\mathbf{x}_j^*)$ and since $\limsup_{j\to\infty} u_{n_j}(\mathbf{x}_j^*) \le u_\infty(\mathbf{0})$ (by the definition of u_∞, we have $\lim_{j\to\infty} u_{n_j}(\mathbf{x}_j^*) = \mathbf{0}$).

When $|\mathbf{x}_j^*| < s$ (which is the case for sufficiently large j) it follows from (2.2) that $(\mathbf{x}_j^*, u_{n_j}(\mathbf{x}_j^*), D\varphi(\mathbf{x}_j^*), D^2\varphi(\mathbf{x}_j^*)) \in J^{2,+}_{\mathcal{A},u_{n_j}}$ and this in turn implies that $(\mathbf{0}, u_\infty(\mathbf{0}), \mathbf{p}, X + 4\delta I) \in \cap_{n\in\mathbb{N}} \overline{\cup_{m=n}^\infty J^{2,+}_{\mathcal{A},u_n}}$. But δ is arbitrary and the intersection of closed sets is closed, so we have $(\mathbf{0}, u_\infty(\mathbf{0}), \mathbf{p}, X) \in \cap_{n\in\mathbb{N}} \overline{\cup_{m=n}^\infty J^{2,+}_{\mathcal{A},u_n}}$. This completes the proof. □

4. The Theorem on Sums

Theorem 2.9. *Assume that*

(i) $\mathcal{A}_i$ *is a locally compact nonempty subset of* $\mathbb{R}^{d_i}$ *for* $i = 1,\dots,m$, $\mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_m$;

(ii) $u_i \in \mathcal{USC}(\mathcal{A}_i, \mathbb{R})$;

(iii) $\varphi \in C^2(\mathbb{R}^d, \mathbb{R})$ *where* $d = d_1 + \dots + d_m$

(iv) $\mathbf{x}^* = (\mathbf{x}_1^*, \dots, \mathbf{x}_m^*) \in \mathcal{A}$ *is such that for some* $r > 0$

$$\sum_{i=1}^m u_i(\mathbf{x}_i) - \varphi(\mathbf{x}) \le \sum_{i=1}^m u_i(\mathbf{x}_i^*) - \varphi(\mathbf{x}^*),$$
$$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m) \in \mathcal{A}, \quad |\mathbf{x} - \mathbf{x}^*| \le r.$$

Then for each $\kappa > 0$ *with* $\|\kappa D^2\varphi(\mathbf{x}^*)\| < 1$ *there are* $X_i \in \mathcal{S}(d_i)$, $i = 1,\dots,m$ *such that*

$$(\mathbf{x}_i^*, u_i(\mathbf{x}_i^*), D_{\mathbf{x}_i}\varphi(\mathbf{x}^*), X_i) \in \overline{J^{2,+}_{\mathcal{A}_i,u_i}}, \quad i = 1,\dots,m$$

and

$$-\kappa^{-1} \le \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_m \end{pmatrix} \le \left(I - \kappa D^2\varphi(\mathbf{x}^*)\right)^{-1} D^2\varphi(\mathbf{x}^*).$$

Here κ is in general a symmetric matrix with positive eigenvalues, but in most cases on takes it to be κI where κ is then some positive number.

Next we define the so-called "sup convolution".

Definition 2.10. *Let $\kappa \in \mathcal{S}(d)$ with $\kappa > 0$ and assume that $\psi : \mathbb{R}^d \to [-\infty, \infty)$ is such that $\psi \not\equiv -\infty$ and $\limsup_{|\mathbf{x}|\to\infty} \frac{1}{|\mathbf{x}|}\big(\psi(\mathbf{x}) - \frac{1}{2}\left\langle \kappa^{-1}\mathbf{x}, \mathbf{x}\right\rangle\big) = -\infty$. Then*

$$\hat{\psi}_\kappa(\mathbf{x}) = \sup_{\mathbf{y}\in\mathbb{R}^d} \big(\psi(\mathbf{y}) - \tfrac{1}{2}\left\langle \kappa^{-1}(\mathbf{y}-\mathbf{x}), \mathbf{y}-\mathbf{x}\right\rangle\big), \quad \mathbf{x} \in \mathbb{R}^d.$$

Theorem 2.11. *Let $\psi : \mathbb{R}^d \to [-\infty, \infty)$ be upper semicontinuous, $\psi \not\equiv -\infty$, and let $\kappa \in \mathcal{S}(d)$, $\kappa > 0$ be such that $\limsup_{|\mathbf{x}|\to\infty} \frac{1}{|\mathbf{x}|}\big(\varphi(\mathbf{x}) - \frac{1}{2}\left\langle \kappa^{-1}\mathbf{x}, \mathbf{x}\right\rangle\big) = -\infty$. Then*

(i) *If φ satisfies the same conditions as ψ and $\psi \le \varphi$, then $\hat{\psi}_\kappa \le \hat{\varphi}_\kappa$.*

(ii) *$\psi \le \hat{\psi}_\kappa$;*

(iii) *$\hat{\psi}_\kappa(\underline{\mathbf{x}}) + \frac{1}{2}\left\langle \kappa^{-1}\underline{\mathbf{x}}, \underline{\mathbf{x}}\right\rangle$ is convex;*

(iv) *If $(\mathbf{x}, \hat{\psi}_\kappa(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathbb{R}^d,\hat{\psi}_\kappa}$ then $(\mathbf{x}+\kappa\mathbf{p}, \psi(\mathbf{x}+\kappa\mathbf{p}), \mathbf{p}, X) \in J^{2,+}_{\mathbb{R}^d,\psi}$, $\psi(\mathbf{x}+\kappa\mathbf{p}) = \hat{\psi}_\kappa(\mathbf{x}) + \frac{1}{2}\left\langle \kappa\mathbf{p}, \mathbf{p}\right\rangle$, and $X \ge -\kappa^{-1}$;*

One can actually prove a sligthly better result in that X in the statement $(\mathbf{x} + \kappa\mathbf{p}, \psi(\mathbf{x}+\kappa\mathbf{p}), \mathbf{p}, X) \in J^{2,+}_{\mathbb{R}^d,\psi}$ in (iv) can be replaced by a smaller matrix, but we shall not use that here.

The next theorem is a classical result due to Alexandrov.

Theorem 2.12. *If $\phi : \mathbb{R}^d \to \mathbb{R}$ is convex, or ϕ is the difference between two convex functions, then for almost every $\mathbf{x} \in \mathbb{R}^d$ $J^{2,+}_{\mathbb{R}^d,\phi}(\mathbf{x}) \cap J^{2,-}_{\mathbb{R}^d,\phi}(\mathbf{x}) \ne \emptyset$, that is, ϕ is twice differentiable at $\mathbf{x}$ in the sense that there are $D\phi(\mathbf{x}) \in \mathbb{R}^d$ and $D^2\phi(\mathbf{x}) \in \mathcal{S}(d)$ such that*

$$\phi(\mathbf{y}) = \phi(\mathbf{x}) + \left\langle D\phi(\mathbf{x}), \mathbf{y}-\mathbf{x}\right\rangle + \tfrac{1}{2}\left\langle D^2\phi(\mathbf{x})(\mathbf{y}-\mathbf{x}), \mathbf{y}-\mathbf{x}\right\rangle + o(|\mathbf{y}-\mathbf{x}|^2)$$

as $\mathbf{y} \to \mathbf{x}$.

We need the following lemma as well.

Lemma 2.13. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ and $\lambda > 0$ be such that $\phi(\underline{\mathbf{x}}) + \frac{\lambda}{2}|\underline{\mathbf{x}}|^2$ is convex and assume that $r > 0$ and $\mathbf{x}^* \in \mathbb{R}^d$ are such that*

$$\phi(\mathbf{x}) < \phi(\mathbf{x}^*), \quad |\mathbf{x}-\mathbf{x}^*| = r.$$

Then for each $\delta > 0$ the set

$$K = \{\mathbf{x} : |\mathbf{x}-\mathbf{x}^*| < r \text{ and } \phi(\mathbf{y}) - \left\langle \mathbf{p}, \mathbf{y}\right\rangle \le \phi(\mathbf{x}) - \left\langle \mathbf{p}, \mathbf{x}\right\rangle \\ \text{for some } \mathbf{p} \text{ with } |\mathbf{p}| \le \delta \text{ and for all } \mathbf{y} \text{ with } |\mathbf{y}-\mathbf{x}^*| \le r\}$$

has positive measure.

Proof of Theorem 2.9. Let $\epsilon > 0$ be so small that $\|\kappa\|\epsilon + \|\kappa D^2\varphi(\mathbf{x}^*)\| < 1$ and let $A = \epsilon I + D^2\varphi(\mathbf{x}^*)$. Define for each $i = 1, \dots, m$,

$$v_i(\mathbf{x}_i) = \begin{cases} u_i(\mathbf{x}_i^* + \mathbf{x}_i) - u_i(\mathbf{x}_i^*) - \langle D_{\mathbf{x}_i}\varphi(\mathbf{x}^*), \mathbf{x}_i\rangle, & \mathbf{x}_i^* + \mathbf{x}_i \in \mathcal{A}_i, \quad |\mathbf{x}_i| \le r_\epsilon, \\ -\infty, & \text{otherwise,} \end{cases}$$

where $r_\epsilon > 0$ is at least so small that the set $\{\, \mathbf{x} : \mathbf{x} \in \mathcal{A}_i, |\mathbf{x} - \mathbf{x}_i^*| \le r_\epsilon \,\}$ is compact for each i. Furthermore, we may by assumption and the defintion of A choose r_ϵ so small that

$$\sum_{i=1}^m v_i(\mathbf{x}_i) \le \frac{1}{2}\langle A\mathbf{x}, \mathbf{x}\rangle, \quad \mathbf{x} \in \mathbb{R}^d.$$

It is easy to check that if $w(\underline{\mathbf{x}}) = \sum_{i=1}^m v_i(\underline{\mathbf{x}_i})$ then $\hat{w}_\kappa(\underline{\mathbf{x}}) = \sum_{i=1}^m \widehat{(v_i)}_\kappa(\underline{\mathbf{x}_i})$ so that we have by Theorem 2.11

$$\sum_{i=1}^m \widehat{(v_i)}_\kappa(\mathbf{x}_i) \le \tfrac{1}{2}\left\langle (I - \kappa A)^{-1}A\mathbf{x}, \mathbf{x}\right\rangle, \quad \mathbf{x} \in \mathbb{R}^d.$$

Since $0 = v_i(\mathbf{0}) \le \widehat{(v_i)}_\kappa(\mathbf{0})$ we conclude from the inequality above that $\widehat{(v_i)}_\kappa(\mathbf{0}) = 0$. Thus the function

$$\phi(\underline{\mathbf{x}}) = \sum_{i=1}^m \widehat{(v_i)}_\kappa(\underline{\mathbf{x}_i}) - \tfrac{1}{2}\left\langle ((I - \kappa A)^{-1}A + \epsilon I)\underline{\mathbf{x}}, \underline{\mathbf{x}}\right\rangle$$

satisfies the assumptions of lemma 2.13 for arbitrary $r > 0$.

Thus we can for each $\delta > 0$ find a point $\mathbf{x}_\delta = (\mathbf{x}_{1,\delta}, \dots, \mathbf{x}_{k,\delta})$ such that $|\mathbf{x}_\delta| < \delta$, the function $\varphi(\underline{\mathbf{x}}) - \langle \mathbf{p}_\delta, \underline{\mathbf{x}}\rangle$ where $|\mathbf{p}_\delta| \le \delta$, reaches its maximum in $\{\, \mathbf{x} : |\mathbf{x}| \le \delta \,\}$ at $\mathbf{x}_\delta$, and the functions $\widehat{(v_i)}_\kappa$ are twice differentiable at the points $\mathbf{x}_{i,\delta}$ for $i = 1, \dots, m$.

Since the derivative vanishes at a maximum point we have

$$D_{\mathbf{x}_i}\widehat{(v_i)}_\kappa(\mathbf{x}_{i,\delta}) = \mathbf{p}_{i,\delta} + P_i(I - \kappa A)^{-1}A\mathbf{x}_\delta + \epsilon\mathbf{x}_{i,\delta} \stackrel{\text{def}}{=} \mathbf{q}_{i,\delta},$$

where P_i is the projection of $\mathbf{x}$ onto $\mathbf{x}_i$. The maximum property together with the properties of the sup convolution implies that

$$-\kappa^{-1} \le \begin{pmatrix} D^2\widehat{(v_1)}_\kappa(\mathbf{x}_{1,\delta}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D^2\widehat{(v_1)}_\kappa(\mathbf{x}_{1,\delta}) \end{pmatrix} \le (I - \kappa A)^{-1}A + \epsilon I.$$

Thus we see that taking a sequnce $\delta_j \downarrow 0$ we may assume that

$$\lim_{j\to\infty} \widehat{D^2_{\mathbf{x}_i}(v_i)}_\kappa(\mathbf{x}_{i,\delta_j}) = X_j, \quad i = 1, \dots, m,$$

and

$$-\kappa^{-1} \leq \begin{pmatrix} X_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_m \end{pmatrix} \leq (I - \kappa A)^{-1} A + \epsilon I.$$

By the differentiablity we have

$$(\mathbf{x}_{i,\delta}, \widehat{(v_i)}_\kappa(\mathbf{x}_{i,\delta}), \mathbf{q}_{i,\delta}, D^2_{\mathbf{x}_i} \widehat{(v_i)}_\kappa(\mathbf{x}_{i,\delta_j})) \in J^{2,+}_{\mathbb{R}^{d_i}, \widehat{(v_i)}_\kappa},$$

and therefore by Lemma 2.11 we know that

$$(\mathbf{x}_{i,\delta} + \kappa \mathbf{q}_{i,\delta}, v_i(\mathbf{x}_{i,\delta} + \kappa \mathbf{q}_{i,\delta}), \mathbf{q}_{i,\delta}, D^2_{\mathbf{x}_i} \widehat{(v_i)}_\kappa(\mathbf{x}_{i,\delta})) \in J^{2,+}_{\mathbb{R}^{d_i}, v_i},$$

or when $\delta < r_\epsilon$ that

$$(\mathbf{x}_{i,\delta} + \kappa \mathbf{q}_{i,\delta} + \mathbf{x}^*_i, u_i(\mathbf{x}_{i,\delta} + \kappa \mathbf{q}_{i,\delta} + \mathbf{x}^*_i), \mathbf{q}_{i,\delta} + D_{\mathbf{x}_i}\varphi(\mathbf{x}^*), D^2_{\mathbf{x}_i} \widehat{(v_i)}_\kappa(\mathbf{x}_{i,\delta})) \in J^{2,+}_{\mathcal{A}_i, u_i}.$$

Since $\widehat{(v_i)}_\kappa(\mathbf{x}_{i,\delta}) = v_i(\mathbf{x}_{i,\delta} + \kappa \mathbf{q}_{i,\delta}) - \frac{1}{2} \langle \kappa \mathbf{q}_{i,\delta}, \mathbf{q}_{i,\delta} \rangle$ and $\widehat{(v_i)}_\kappa(\mathbf{0}) = 0$ we conclude when we take $\delta = \delta_j$ and let $j \to \infty$ that

$$(\mathbf{x}^*_i, u_i(\mathbf{x}^*_i), D_{\mathbf{x}_i}\varphi(\mathbf{x}^*), X_i) \in \overline{J^{2,+}_{\mathcal{A}_i, u_i}}, \quad i = 1, \ldots, m.$$

Finally we see that the only term depending on ϵ is X_i and when we let $\epsilon \downarrow 0$ we may choose a convergent subsequence, and since we are dealing with a closed set we get the desired conclusion. □

Proof of Theorem 2.11. Claims (i) and (ii) are left to the reader. To see that (iii) holds we observe that

$$\hat{\psi}_\kappa(\underline{\mathbf{x}}) + \tfrac{1}{2} \left\langle \kappa^{-1} \underline{\mathbf{x}}, \underline{\mathbf{x}} \right\rangle = \sup_{\mathbf{y} \in \mathbb{R}^d} \left(\psi(\mathbf{y}) - \tfrac{1}{2} \left\langle \kappa^{-1} \mathbf{y}, \mathbf{y} \right\rangle + \left\langle \kappa^{-1} \mathbf{y}, \underline{\mathbf{x}} \right\rangle \right),$$

and the supremum of convex or affine functions is convex.

For the proof of (iv) assume that $(\mathbf{x}, \hat{\psi}_\kappa(\mathbf{x}), \mathbf{p}, X) \in J^{2,+}_{\mathbb{R}^d, \hat{\psi}_\kappa}$. We may assume that there is a function $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$ such that

$$\hat{\psi}_\kappa(\mathbf{y}) - \varphi(\mathbf{y}) \leq \hat{\psi}_\kappa(\mathbf{x}) - \varphi(\mathbf{x}), \quad \mathbf{y} \in \mathbb{R}^d, \tag{2.3}$$

such that $\mathbf{p} = D\varphi(\mathbf{x})$ and $X = D^2\varphi(\mathbf{x})$. Since ψ is upper semicontinuous there is a point $\mathbf{x}^*$ such that $\hat{\psi}_\kappa(\mathbf{x}) = \psi(\mathbf{x}^*) - \frac{1}{2} \left\langle \kappa^{-1}(\mathbf{x} - \mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \right\rangle$. Thus we get from (2.3) and the definition of $\hat{\psi}_\kappa$ that

$$\begin{aligned} \psi(z) - \tfrac{1}{2} \left\langle \kappa^{-1}(\mathbf{z} - \mathbf{y}), \mathbf{z} - \mathbf{y} \right\rangle - \varphi(\mathbf{y}) \leq \hat{\psi}_\kappa(\mathbf{y}) - \varphi(\mathbf{y}) &\leq \hat{\psi}_\kappa(\mathbf{x}) - \varphi(\mathbf{x}) \\ = \psi(\mathbf{x}^*) - \tfrac{1}{2} \left\langle \kappa^{-1}(\mathbf{x}^* - \mathbf{x}), \mathbf{x}^* - \mathbf{x} \right\rangle &- \varphi(\mathbf{x}). \end{aligned} \tag{2.4}$$

First we choose $\mathbf{z} = \mathbf{x}^*$ so that we get

$$\varphi(\mathbf{y}) + \tfrac{1}{2} \left\langle \kappa^{-1}(\mathbf{x}^* - \mathbf{y}), \mathbf{x}^* - \mathbf{y} \right\rangle \geq \varphi(\mathbf{x}) + \tfrac{1}{2} \left\langle \kappa^{-1}(\mathbf{x}^* - \mathbf{x}), \mathbf{x}^* - \mathbf{x} \right\rangle, \quad \mathbf{y} \in \mathbb{R}^d,$$

that is, the function $\varphi(\underline{\mathbf{y}}) + \frac{1}{2}\left\langle \kappa^{-1}(\mathbf{x}^* - \underline{\mathbf{y}}), \mathbf{x}^* - \underline{\mathbf{y}}\right\rangle$ has a minimum at $\mathbf{x}$. Therefore the first derivative is zero and the second nonnegative at $\mathbf{x}$ so that

$$\text{(2.5)} \qquad \mathbf{x}^* = \mathbf{x} + \kappa D\varphi(\mathbf{x}) = \mathbf{x} + \kappa\mathbf{p} \quad \text{and} \quad D^2\varphi(\mathbf{x}) \geq -\kappa^{-1}.$$

Next we choose $\mathbf{y} = \mathbf{z} - \mathbf{x}^* + \mathbf{x}$ in (2.4). This yields

$$\psi(\mathbf{z}) - \varphi(\mathbf{z} - \mathbf{x}^* + \mathbf{x}) \leq \psi(\mathbf{x}^*) - \varphi(\mathbf{x}), \quad \mathbf{z} \in \mathbb{R}^d,$$

so that

$$(\mathbf{x}^*, \psi(\mathbf{x}^*), D\varphi(\mathbf{x}), D^2\varphi(\mathbf{x})) = (\mathbf{x}^*, \psi(\mathbf{x}^*), \mathbf{p}, X) \in J^{2,+}_{\mathbb{R}^d,\psi}.$$

□

Proof of Lemma 2.13. Let $\phi_{\mathbf{p}}(\underline{\mathbf{x}}) = \phi(\underline{\mathbf{x}}) - \langle \mathbf{p}, \underline{\mathbf{x}}\rangle$ and $\overline{B(\mathbf{z}, s)} = \{\, \mathbf{y} : |\mathbf{y} - \mathbf{z}| \leq s \,\}$. We may decrease δ so that $\delta < \frac{1}{r}\inf\{\, \phi(\mathbf{x}^*) - \phi(\mathbf{x}) : |\mathbf{x} - \mathbf{x}^*| = r \,\}$. It follows then that the largest value of $\phi_{\mathbf{p}}$ in the set $\{\, \mathbf{x} : |\mathbf{x} - \mathbf{x}^*| \,\}$ is achieved at an interior point for each $\mathbf{p} \in \overline{B(0, \delta)}$.

Suppose that $\phi \in \mathcal{C}^2(\mathbb{R}^d)$. If $\mathbf{x} \in K$ then $D\phi(\mathbf{x}) - \mathbf{p} = D\phi_{\mathbf{p}}(\mathbf{x}) = \mathbf{0}$ and $D^2\phi(\mathbf{x}) = D^2\phi_{\mathbf{p}}(\mathbf{x}) \leq 0$. It follows from the first equality that $D\phi$ is a continous map from K onto $\overline{B(\mathbf{0}, \delta)}$. It is also clear that K is closed hence measurable. Moreover, since $\phi(\underline{\mathbf{x}}) + \frac{\lambda}{2}|\underline{\mathbf{x}}|^2$ is convex, we have $D^2\phi(\underline{x}) \geq -\lambda I$ for all $\mathbf{x}$. Combining these two inequalites we see that $|\det(D^2\phi(\mathbf{x}))| \leq \lambda^d$. Thus

$$\operatorname{meas}(\{\, \mathbf{p} : |\mathbf{p}| \leq \delta \,\}) = \operatorname{meas}(D\phi(K)) \leq \int_K |\det(D^2\phi(\mathbf{x}))| \,\mathrm{d}\mathbf{x} \leq \operatorname{meas}(K)\lambda^d,$$

so that

$$\operatorname{meas}(K) \geq \lambda^{-d}\operatorname{meas}(\{\, \mathbf{p} : |\mathbf{p}| \leq \delta \,\}).$$

Finally if ϕ is not in $\mathcal{C}^2(\mathbb{R}^d)$ we take ψ to be a nonnegative, infinitely many times differentiable function with support in the unit ball such that $\int_{\mathbb{R}^d} \psi(\mathbf{y})\,\mathrm{d}\mathbf{y} = 1$ and let $\phi_n(\underline{\mathbf{x}}) = \int_{\mathbb{R}^d} \phi(\underline{\mathbf{x}} - \mathbf{y}) n^d \psi(n\mathbf{y})\,\mathrm{d}\mathbf{y}$. Since $\int_{\mathbb{R}^d}(\phi(\underline{\mathbf{x}} - \mathbf{y}) + \frac{\lambda}{2}|\underline{\mathbf{x}} - \mathbf{y}|^2) n^d \psi(n\mathbf{y})\,\mathrm{d}\mathbf{y}$ and $\frac{\lambda}{2}|\underline{\mathbf{x}}|^2 - \int_{\mathbb{R}^d} \frac{\lambda}{2}|\underline{\mathbf{x}} - \mathbf{y}|^2) n^d \psi(n\mathbf{y})\,\mathrm{d}\mathbf{y}$ are convex, we conclude that $\phi_n(\underline{\mathbf{x}}) + \frac{\lambda}{2}|\underline{\mathbf{x}}|^2$ is convex for every $n \geq 1$.

Since $\phi_n \to \phi$ uniformly on compact sets we see that we can require $\delta > 0$ to be such that $\delta < \frac{1}{r}\inf\{\, \phi_n(\mathbf{x}^*) - \phi_n(\mathbf{x}) : |\mathbf{x} - \mathbf{x}^*| = r \,\}$ for all sufficiently large n. If we define K_n as K with ϕ replaced by ϕ_n then we can conclude the measure of K_n is bounded from below by a positive constant independent of n.

If now $\mathbf{x} \in \cap_{m=1}^{\infty}\overline{\cup_{n=m}^{\infty} K_n}$ then there exists points $\mathbf{x}_{n_j} \in K_{n_j}$ such that $\lim_{j\to\infty} \mathbf{x}_{n_j} \to \mathbf{x}$ (and $\lim_{j\to\infty} n_j = \infty$). We may assume that the vectors $\mathbf{p}_{n_j} = D\phi_{n_j}(\mathbf{x}_{n_j})$ converge to some $\mathbf{p}$ with $|\mathbf{p}| \leq \delta$. Then we see that $\mathbf{x} \in K$ (using the fact that the maximum is achieved in the interior of the ball $\overline{B(\mathbf{x}^*, r)}$), so that $K \supset \cap_{m=1}^{\infty}\overline{\cup_{n=m}^{\infty} K_n}$ and K has positive measure. □

5. Parabolic jets

For parabolic equations there are at least two approaches one can take. One is to define the second order jets as above and simply disregard all second order partial derivatives involving the "time"-variable. The other approach is to reformulate the defintion of the jets and there eliminate all unneeded partal derivatives and so that the error term is of the form $o(|s-t|+|\mathbf{y}-\mathbf{x}|^2)$. This approach is in some ways more natural but has the weakness that many theorems should be redone with only minor modifications. Here we take the first approach.

Definition 2.14. *Let $\mathcal{B} \subset \mathbb{R}^{d+1}$ and $u : \mathcal{B} \to [-\infty, \infty]$. Then*

$$P^{2,\pm}_{\mathcal{B},u} = \Big\{ \big(t, \mathbf{x}, r, a, \mathbf{p}, X\big) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) : \\ \big((t,\mathbf{x}), r, (a,\mathbf{p}), \tilde{X}\big) \in J^{2,\pm}_{\mathcal{B},u}, \quad \tilde{X}(2:d+1, 2:d+1) = X \Big\}$$

and

$$P^{2,\pm}_{\mathcal{B},u}(t,\mathbf{x}) = \{ (a,\mathbf{p},X) \in \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) : (t,\mathbf{x},u(t,\mathbf{x}),a,\mathbf{p},X) \in P^{2,\pm}_{\mathcal{B},u} \}.$$

Usually we take $\mathcal{B} = \mathcal{T} \times \mathcal{A}$ where $\mathcal{T}$ is an interval.

With this definiton we immediately have the following version of Theorem 2.8.

Corollary 2.15. *Let $\mathcal{B} \subset \mathbb{R}^{d\times 1}$ be locally compact and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of upper semicontinuous functions: $\mathcal{B} \to [-\infty,\infty]$ and let $U = \limsup^*{}_{n\to\infty} u_n$. Then*

$$P^{2,+}_{\mathcal{B},U} \subset \cap_{n\in\mathbb{N}} \overline{\cup_{m=n}^{\infty} P^{2,+}_{\mathcal{B},u_n}},$$

that is, if $(t,\mathbf{x},U(t,\mathbf{x}),a,\mathbf{p},X) \in P^{2,+}_{\mathcal{B},U}$ then there is a subsequence (n_j) and elements $(t_{n_j},\mathbf{x}_{n_j},u_{n_j}(t_{n_j},\mathbf{x}_{n_j}),a_{n_j},\mathbf{p}_{n_j},X_{n_j}) \in P^{2,+}_{\mathcal{B},u_{n_j}}$ such that $t_{n_j} \to t$, $\mathbf{x}_{n_j} \to \mathbf{x}$, $u_{n_j}(t_{n_j},\mathbf{x}_{n_j}) \to U(\mathbf{x})$, $a_{n_j} \to a$, $\mathbf{p}_{n_j} \to \mathbf{p}$, and $X_{n_j} \to X$ as $j \to \infty$.

The theorem on sums requires a new proof since we want to have it in a form where all the functions have the same t-variable. If this is not the case, then no new theorem is needed.

Next we formulate and prove the parabolic version of the theorem on sums.

Theorem 2.16. *Assume that $d_i \geq 1$, $i = 1,\dots,m$ and that*

(i) *$\mathcal{B}_i \subset \mathbb{R}^{d_i+1}$ is locally compact and nonempty for $i = 1,\dots,m$ and $\mathcal{B} = \{ (t,\mathbf{x}) : \mathbf{x} = (\mathbf{x}_1,\dots,\mathbf{x}_m), \quad (t,\mathbf{x}_i) \in \mathcal{B}_i, i = 1,\dots,m \}$ is nonempty;*

(ii) *$u_i \in \mathcal{USC}(\mathcal{B}_i)$;*

(iii) *$\varphi \in \mathcal{C}^2(\mathbb{R}^{d+1})$ where $d = d_1 + \ldots + d_m$*

(iv) $(t^*,\mathbf{x}^*)\in\mathcal{B}$ *is such that for some* $r>0$

$$\sum_{i=1}^m u_i(t,\mathbf{x}_i)-\varphi(t,\mathbf{x})\le\sum_{i=1}^m u_i(t^*,\mathbf{x}_i^*)-\varphi(t^*,\mathbf{x}^*),$$
$$(t,\mathbf{x})\in\mathcal{B},\quad \mathbf{x}=(\mathbf{x}_1,\ldots,\mathbf{x}_m),\quad |t-t^*|\le r,\quad |\mathbf{x}-\mathbf{x}^*|\le r.$$

(v) *for each* $M<\infty$ *there is a number* $\delta>0$ *and a number* $C<\infty$ *such that if* $1\le i\le m$, $|t-t^*|\le\delta$, $|\mathbf{x}_i-\mathbf{x}_i^*|\le\delta$, $(t,\mathbf{x}_i,u_i(t,\mathbf{x}_i),a_i,\mathbf{p}_i,X_i)\in P^{2,+}_{\mathcal{B}_i,u_i}$, $|u_i(t,\mathbf{x}_i)-u_i(t^*,\mathbf{x}_i^*)|\le\delta$, $|\mathbf{p}_i-D_{\mathbf{x}_i}\varphi(t^*,\mathbf{x}^*)|\le\delta$, *and* $\|X_i\|\le M$ *then* $a_i\le C$.

Then for each $\kappa>0$ *with* $\|\kappa D^2\varphi(\mathbf{x}^*)\|<1$ *there are* $a_i\in\mathbb{R}$ *and* $X_i\in\mathcal{S}(d_i)$, $i=1,\ldots,m$ *such that*

$$(t^*,\mathbf{x}_i^*,u_i(t^*,\mathbf{x}_i^*),a_i,D_{\mathbf{x}_i}\varphi(\mathbf{x}^*),X_i)\in\overline{P^{2,+}_{\mathcal{B}_i,u_i}},\quad i=1,\ldots,m,$$

$$\sum_{i=1}^m a_i=\varphi_t(t^*,\mathbf{x}^*),$$

and

$$-\kappa^{-1}\le\begin{pmatrix}X_1&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&X_m\end{pmatrix}\le\big(I-\kappa D^2_{\mathbf{x}}\varphi(t^*,\mathbf{x}^*)\big)^{-1}D^2_{\mathbf{x}}\varphi(t^*,\mathbf{x}^*).$$

In the general case where very little is known about the functions u_i it could be exceedingly difficult to verify assumption (v). But if u_i is a subsoltuion of the equation $u_t+F_i(t,\mathbf{x},u,Du,D^2u)=0$, then it is quite easy to check that it holds provided F_i is lower semicontinuous.

Proof of Theorem 2.16. We are going to use the number r in (iv) as a parameter that at several stages in the proof must be chosen to be sufficiently small. To begin with we require that r is so small that the sets $\{\,(t,\mathbf{x}_i)\in\mathcal{B}_i : |t-t^*|\le r, |\mathbf{x}_i-\mathbf{x}_i^*|\le r\,\}$ is compact for each i. (If r is decreased this proerty will clearly not be lost.) Define, for each $i=1,\ldots,m$

$$v_i(t,\mathbf{x}_i)=\begin{cases}u_i(t,x_i)-(t-t^*)D_{t\mathbf{x}_i}\varphi(t^*,\mathbf{x}^*)(\mathbf{x}_i-\mathbf{x}_i^*), & (t,\mathbf{x}_i)\in\mathcal{B}_i,\\ & |t-t^*|\le r,\\ & |\mathbf{x}_i-\mathbf{x}_i^*|\le\frac{r}{\sqrt{m}},\\ -\infty, & \text{otherwise.}\end{cases}$$

Let $\eta \in (0,1)^2$ with both components sufficiently small and define

$$\begin{aligned}\tilde{\varphi}_\eta(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2,\dots,t_m,\mathbf{x}_m) = \varphi(t_1,\mathbf{x}) \\ - (t_1 - t^*)\sum_{i=1}^m D_{t\mathbf{x}_i}\varphi(\mathbf{t}^*,\mathbf{x}^*)(\mathbf{x}_i - \mathbf{x}_i^*) \\ + \frac{1}{\eta_1}\sum_{1\le i<j\le m}|t_i - t_j|^2 + \eta_2|\mathbf{x}-\mathbf{x}^*|^2 + \eta_2|t_1 - t^*|^2,\end{aligned}$$

where $\mathbf{x} = (\mathbf{x}_1,\dots,\mathbf{x}_m)$ and finally let $\tilde{\kappa}_\eta \in \mathcal{S}(m+d)$ be defined by

$$\tilde{\kappa}_\eta(i,j) = \begin{cases} \eta_1^2, & \text{if } i = j = \sum_{n=1}^{p-1} d_n + p,\ p = 1,\dots,m, \\ \kappa(i-p,j-q), & \text{if } \sum_{n=1}^{p-1} d_n + p < i \le \sum_{n=1}^{p} d_n \text{ and }, \\ & \quad \sum_{n=1}^{q-1} d_n + q < j \le \sum_{n=1}^{q} d_n, \\ 0, & \text{otherwise.}\end{cases}$$

If κ is a diagonal matrix (e.g. a scalar) then this just means that the diagonal is broken up into blocks and new elements η_1^2 inserted in front of each block.

Now the function $\sum_{i=1}^m v_i(t_i,\mathbf{x}_i) - \tilde{\varphi}_\eta(t_1,\mathbf{x}_1,t_2,\mathbf{x}_2,\dots,t_m,\mathbf{x}_m)$ achieves its maximum at some point $(t_{1,\eta},\mathbf{x}_{1,\eta},\dots,t_{m,\eta},\mathbf{x}_{m,\eta}) = \tilde{\mathbf{x}}_\eta$ in the set where $(t_i,\mathbf{x}_i) \in \mathcal{B}_i$, $|t_i - t^*| \le r$ and $|\mathbf{x}-\mathbf{x}^*| \le r$ and we may therefore apply Theorem 2.9 (provided η_1, η_2 and r are sufficiently small so that $\|\tilde{\kappa}_\eta D^2\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)\| < 1$). It follows that there are $\tilde{X}_i^\eta \in \mathcal{S}(d_i+1)$, for each $i = 1,\dots,m$, such that

$$((t_{i,\eta},\mathbf{x}_{i,\eta}), v_i(t_{i,\eta},\mathbf{x}_{i,\eta}), (D_{t_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta), D_{\mathbf{x}_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)), \tilde{X}_{i,\eta}) \in \overline{J^{2,+}_{\mathcal{B}_i,v_i}}, \tag{2.6}$$

and

$$-\tilde{\kappa}_\eta^{-1} \le \begin{pmatrix} \tilde{X}_{1,\eta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{X}_{m,\eta}\end{pmatrix} \le \left(I - \tilde{\kappa}_\eta D^2\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)\right)^{-1} D^2\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta).$$

Next we define the matrices $X_{i,\eta}$ by $X_{i,\eta} = \tilde{X}_{i,\eta}(\sum_{n=1}^{i-1} d_n + i + 1 : \sum_{n=1}^{i} d_n + i, \sum_{n=1}^{i-1} d_n + i + 1 : \sum_{n=1}^{i} d_n + i)$. For these matrices we get the inequality

$$-\kappa^{-1} \le \begin{pmatrix} X_{1,\eta} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{m,\eta}\end{pmatrix} \le \left(\left(I - \tilde{\kappa}_\eta D^2\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)\right)^{-1} D^2\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)\right)(J,J), \tag{2.7}$$

where J is the index set $\cup_{i=1}^m(\sum_{n=1}^{i-1} d_n + i + 1 : \sum_{n=1}^{i} d_n + i)$.

We observe that the matrix $\left(I - \tilde{\kappa}_\eta D^2\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)\right)^{-1}$ is uniformly bounded in η and this implies that the whole right-hand side of (2.7) is bounded uniformly in η. It follows that we can find a constant M independent of r and η such that

$$\|X_{i,\eta}\| \le M, \quad i = 1,\dots,m. \tag{2.8}$$

Now we have the number M that determines the number δ in (v).

It follows from the definitions above that a maximum of the function

$$\sum_{i=1}^{m} u_i(t_i,\mathbf{x}_i) - \varphi(t_1,\mathbf{x}) - \sum_{i=2}^{m}(t_i - t_1)D_{t,\mathbf{x}_i}\varphi(t^*,\mathbf{x}^*)(\mathbf{x}_i - \mathbf{x}_i^*) - \frac{1}{\eta_1}\sum_{1\le i<j\le m}|t_i - t_j|^2 - \eta_2|\mathbf{x}-\mathbf{x}^*|^2 - \eta_2|t_1 - t^*|^2, \tag{2.9}$$

is achieved at $\tilde{\mathbf{x}}_\eta = (t_{1,\eta},\mathbf{x}_{1,\eta},\ldots,t_{m,\eta},\mathbf{x}_{m,\eta})$. It follows from this fact, assumption (iv), and from the assumption that the functions u_i are upper semicontinuous, that if r is small enough we have

$$|u_i(t_{i,\eta},\mathbf{x}_{i,\eta}) - u_i(t^*,\mathbf{x}_i^*)| < \delta, \quad i = 1,\ldots,m. \tag{2.10}$$

Furthermore, it follows from the fact that $\tilde{\mathbf{x}}_\eta$ is a maximum point of the function in (2.9) that by choosing η sufficiently small we conlcude that $|\mathbf{x}_{i,\eta} - \mathbf{x}_i^*| < r$ and $|t - t^*| < r$ (and not equality, but for more details see tha argument below). Thus we get from (2.6) and the definitions of v_i and $X_{i,\eta}$ that for $i = 1,\ldots,m$ we have

$$\Big(t_{i,\eta},\mathbf{x}_{i,\eta},u_i(t_{i,\eta},\mathbf{x}_{i,\eta}), D_{t_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta) + D_{t\mathbf{x}_i}\varphi((t^*,\mathbf{x}^*)(\mathbf{x}_{i,\eta}-\mathbf{x}_i^*), D_{\mathbf{x}_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta) + (t_{i,\eta}-t^*)D_{t\mathbf{x}_i}\varphi(t^*,\mathbf{x}^*),, X_{i,\eta}\Big) \in \overline{P^{2,+}_{\mathcal{B}_i,u_i}}, \tag{2.11}$$

Clearly, by choosing r and η sufficiently small we have

$$|D_{\mathbf{x}_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta) + (t_{i,\eta}-t^*)D_{t\mathbf{x}_i}\varphi((t^*,\mathbf{x}^*) - D_{\mathbf{x}_i}\varphi(t^*,\mathbf{x}^*)| < \delta, \quad i = 1,\ldots,m,$$

and thus we know by assumption (v), (2.8), (2.10), and (2.11) that the numbers $D_{t_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)$ are bounded from above uniformly with respect to η. Now

$$\sum_{i=1}^{m} D_{t_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta) = D_t\varphi(t_{1,\eta},\mathbf{x}_{1,\eta},\ldots,\mathbf{x}_{m,\eta}) - \sum_{i=1}^{m} D_{t\mathbf{x}_i}\varphi(t^*,\mathbf{x}^*)(\mathbf{x}_{i,\eta}-\mathbf{x}_i^*) + 2\eta_2(t_{1,\eta}-t^*),$$

and it follows that the numbers $D_{t_i}\tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)$ are bounded from below as well, uniformly with respect to η.

First we let $\eta_1 \downarrow 0$ and using compactness we can pick convergent subsequences. We denote the limit points with the subindex η_2. By Lemma 4.9 we conclude that $\frac{1}{\eta_1}\sum_{1\le i<j\le m}|t_{i,\eta} - t_{j,\eta}|^2 \to 0$ so that we have $t_{i,\eta_2} = t_{j,\eta_2} \stackrel{\text{def}}{=} t_{\eta_2}$

for all i and $j = 1, \ldots, m$. By the second conclusion of Lemma 4.9 we also get

$$\sum_{i=1}^{m} u_i(t, \mathbf{x}_i) - \varphi(t, \mathbf{x}) - \eta_2 |\mathbf{x} - \mathbf{x}^*|^2 - \eta_2 |t - t^*|$$
$$\leq \sum_{i=1}^{m} u_i(t_{\eta_2}, \mathbf{x}_i) - \varphi(t_{\eta_2}, \mathbf{x}_{\eta_2}) - \eta_2 |\mathbf{x}_{\eta_2} - \mathbf{x}^*|^2 - \eta_2 |t_{\tau_2} - t^*|^2$$
$$\leq \sum_{i=1}^{m} u_i(t^*, \mathbf{x}_i^*) - \varphi(t^*, \mathbf{x}^*),$$

when $|t_i - t^*| \leq r$ and $|\mathbf{x}_i - \mathbf{x}_i^*| \leq \frac{r}{\sqrt{m}}$. Since the second inequality is strict when $t_{\eta_2} \neq t^*$ or $\mathbf{x}_{\eta_2} \neq \mathbf{x}^*$ we see, that we must have $t_{\eta_2} = t^*$ and $\mathbf{x}_{\eta_2} = \mathbf{x}^*$.

Thus we see that (a subsequence of) the numbers $D_{t_i} \tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)$ converge to some numbers a_{i,η_2} with the property that $\sum_{i=1}^{m} a_{i,\eta_2} = D_t \varphi(t^*, \mathbf{x}^*)$, but the critical point to check is that the matrix $\Big(\big(I - \tilde{\kappa}_\eta D^2 \tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta)\big)^{-1} D^2 \tilde{\varphi}_\eta(\tilde{\mathbf{x}}_\eta) \Big)(J, J)$ converges to $\big(I - \kappa(D^2_{\mathbf{x}} \varphi(\mathbf{x}^*) + 2\eta_2 I)\big)^{-1} (D^2_{\mathbf{x}} \varphi(\mathbf{x}^*) + 2\eta_2 I)$. To see that this is the case the crucial point to observe is that our construction guarantees that

$$\lim_{\eta_1 \downarrow 0} D_{t_j \mathbf{x}_i} \tilde{\varphi}_\eta(t^*, \mathbf{x}_1^*, \ldots, t^*, \mathbf{x}_m^*) = 0$$

for all i and j.

Finally we can use compactness arguments again to pass to the limit $\eta_2 \downarrow 0$, but this is a much simpler procedure. Thus we get the desired conclusion.

□

6. Comments

The definition of the jets in Defintion 2.3 is an extrapolation of the definitions in [**3**] and [**2**] and differs from those only in notation. However, Definition 2.14 differs from those in [**3**] and [**2**] although it is stated in [**3**] that ”it is not hard to see that the two notions are equivalent”.