Chapter 1

Introduction

1. The basic definitions

For the definitions of upper and lower semicontinuous functions and their properties, see section 1, but remember that one may to a quite large extent assume that all semicontinuous functions are in fact continuous.

Definition 1.1. *The set of* $d \times d$ *symmetric matrices is denoted by* $\mathcal{S}(d)$ *and* $X \geq Y$ *where* X, $Y \in \mathcal{S}(d)$ *when* $\langle \mathbf{x}, X\mathbf{x} \rangle \geq \langle \mathbf{x}, Y\mathbf{x} \rangle$ *for all* $\mathbf{x} \in \mathbb{R}^d$ *where* $\langle \cdot, \cdot \rangle$ *denotes the standard inner product in* $\mathbb{R}^d$.

Definition 1.2. *Let* $\mathcal{A} \subset \mathbb{R}^d$. *A function* $F : \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ *is said to be degenerate elliptic if*

$$\boxed{\begin{gathered} F(\mathbf{x}, r, \mathbf{p}, X) \leq F(\mathbf{x}, r, \mathbf{p}, Y) \\ \mathbf{x} \in \mathcal{A}, \mathbf{p} \in \mathbb{R}^d, r \in \mathbb{R}, \quad X, Y \in \mathcal{S}(d), X \geq Y. \end{gathered}} \tag{1.1}$$

The function F *said to be proper (i.e., satisfy the basic monotonicity assumptions) if*

$$\boxed{\begin{gathered} F(\mathbf{x}, s, \mathbf{p}, X) \leq F(\mathbf{x}, r, \mathbf{p}, Y) \\ \mathbf{x} \in \mathcal{A}, \mathbf{p} \in \mathbb{R}^d, \quad s, r \in \mathbb{R}, \quad s \leq r, \quad X, Y \in \mathcal{S}(d), X \geq Y. \end{gathered}} \tag{1.2}$$

Now we can define sub- and supersolutions of the equation $F(\mathbf{x}, u, Du, D^2u) = 0$:

Definition 1.3. *Suppose that* $\mathcal{A} \subset \mathbb{R}^d$, $F : \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to \mathbb{R}$ *is degenerate elliptic, i.e, satisfies* (1.1). *A function* $u : \mathcal{A} \to \mathbb{R}$ *is a subsolution of* $F = 0$ *in* $\mathcal{A}$ *if*

$u \in \mathcal{USC}(\mathcal{A})$ and for every $\mathbf{x}_0 \in \mathcal{A}$ one has

$$\boxed{\begin{aligned} &\varphi \in \mathcal{C}^2(\mathbb{R}^d) \text{ and } u(\mathbf{x}) \le \varphi(\mathbf{x}) + u(\mathbf{x}_0) - \varphi(\mathbf{x}_0), \\ &\mathbf{x} \in \mathcal{A},\ |\mathbf{x} - \mathbf{x}_0| < \delta, \text{ for some } \delta > 0 \\ &\Rightarrow \quad F\big(\mathbf{x}_0, u(\mathbf{x}_0), D\varphi(\mathbf{x}_0), D^2\varphi(\mathbf{x}_0)\big) \le 0 \end{aligned}}$$

Similarly, a function $u : \mathcal{A} \to \mathbb{R}$ is a supersolution of $F = 0$ in $\mathcal{A}$ if $u \in \mathcal{LSC}(\mathcal{A})$ and for every $\mathbf{x}_0 \in \mathcal{A}$ one has

$$\boxed{\begin{aligned} &\varphi \in \mathcal{C}^2(\mathbb{R}^d) \text{ and } u(\mathbf{x}) \ge \varphi(\mathbf{x}) + u(\mathbf{x}_0) - \varphi(\mathbf{x}_0), \\ &\mathbf{x} \in \mathcal{A},\ |\mathbf{x} - \mathbf{x}_0| < \delta, \text{ for some } \delta > 0 \\ &\Rightarrow \quad F\big(\mathbf{x}_0, u(\mathbf{x}_0), D\varphi(\mathbf{x}_0), D^2\varphi(\mathbf{x}_0)\big) \ge 0 \end{aligned}}$$

If u is both a sub- and a supersolution, then it is a (viscosity) solution.

The assumption that F is degenerate elliptic guarantees that if u is a twice continuously differentiable classical solution in an open set, then it is also a viscosity solution.

Observe in particular that if there is no function φ such that $u(\mathbf{x}) \ge \varphi(\mathbf{x}) + u(\mathbf{x}_0) - \varphi(\mathbf{x}_0)$, then there are no conditions imposed at the point $\mathbf{x}_0$!

There are other equivalent ways of defining sub- and supersolutions and we will use some of them later on.

2. A simple example

Let us see what one can say about the equation $|u'| - 1 = 0$ in $(-1, 1)$ with boundary conditions $u(-1) = u(1) = 0$. We shall later return to boundary conditions in more detail.

It is quite straightforward to check that $u(x) = 1 - |x|$ is a solution of this equation but we shall give a proof that it is the only solution. Later we shall consider comparison principles from which this result follows immediately, and the techinques we use here cannot be directly extended to very many other cases, but the basic ideas are of course the same.

First we consider subsolutions, more precisely let us assume that u is upper semicontinuous on $[-1, 1]$, a subsolution of the equation $|u'| - 1 =$ on $(-1, 1)$ and that $u(-1) \le 0$ and $u(1) \le 0$. Suppose that it is not true that $u(x) \le 1 + x$ but at some point $x_* \in (-1, 1)$ we have $u(x_*) > 1 + x_*$. But then we also have $u(x_*) - c(1 + x_*) > 0$ for $c = 1 + \frac{u(x_*)-1-x_*}{4} > 1$. Let $x_0 \in [-1, 1]$ be the point where $u(x) - c(1 + x)$ achieves its maximum. Cleary, we cannot have $x_0 = \pm 1$ because the maximum is positive. But then we can take $\varphi(x)$ in the definition of a subsolution to be $c(1 + x)$ so that we get the condition $c - 1 = |\varphi'(x_0)| - 1 \le 0$

which is a contradiction. Thus we have shown that $u(x) \leq 1+x$ when $x \in [-1,1]$. A similar argument shows that $u(x) \leq 1-x$ so that we have $u(x) \leq 1-|x|$.

Nest we have to show that if u is a lower semicontinuous function in $[-1,1]$ which is a supersolution of $|u'|-1=0$, and is such that $u(-1) \geq 0$ and $u(1) \geq 0$ then $u(x) \geq 1-|x|$. Suppose that this is not the case. Then there exists some point x_0 such that $u(x_0) < 1-|x_0|$ and we can find a positive number δ such that we have $u(x_0) < \sqrt{1+\delta} - \sqrt{x_0^2+\delta}$ since $\lim_{\delta\downarrow 0}\left(\sqrt{1+\delta}-\sqrt{x^2+\delta}\right) = 1-|x|$ for all x. Clearly $u(x) - \left(\sqrt{1+\delta}-\sqrt{x^2+\delta}\right) \geq 0$ when $x = \pm 1$ so if x_0 is the point where $u(x) - \left(\sqrt{1+\delta}-\sqrt{x^2+\delta}\right)$ achieves it smallest value in $[-1,1]$, we know that $x_0 \in (-1,1)$. Now we can take $\varphi(x) = \sqrt{1+\delta}-\sqrt{x^2+\delta}$ and since $\varphi'(x) = \frac{x}{\sqrt{x^2+\delta}}$ we get from the assumption that u is a upper solution that

$$\left|\frac{x_0}{\sqrt{x_0^2+\delta}}\right| - 1 \geq 0.$$

But this is a contradiction so we conclude that $u(x) \geq 1-|x|$ for a all $x \in [-1,1]$.

Thus we have shown that if u is both a sub- and supersolution, then it must be $1-|x|$.