Chapter 4

Comparison

The basic problem to be considered here is the question when one can say that a supersolution is always greater than or equal to a subsolution of a problem, where one in most cases assume that this inequality holds on the boundary. It will then immediately lead to uniqueness for the Dirichlet problem (where the boundary value are specified). One cannot hope to establish the comparison principle in complete generality because if $\Omega \subset \mathbb{R}^d$ is an open set and f is a continuously differentiable function in $\mathbb{R}^d$ which vanishes on $\partial\Omega$ but not in all of Ω, then the equation $|Du|^2 - |f'(x)|^2 = 0$ with $u = 0$ on $\partial\Omega$, then this equation has two solutions: $\pm f$, and it cannot be the case that a supersolution is always greater than or equal to a subsolution.

1. The simplest cases

First we consider the case where we have a bounded open set and first we state an intermediate result that does not involve the equation at all.

Proposition 4.1. *Assume that* $d \geq 1$ *and that*

(i) $\Omega \subset \mathbb{R}^d$ *is open, bounded, and nonempty;*
(ii) $u \in \mathcal{USC}(\Omega)$ *is such that* $\sup_{\mathbf{x}\in\Omega} u(\mathbf{x}) < \infty$*;*
(iii) $v \in \mathcal{LSC}(\Omega)$ *is such that* $\inf_{\mathbf{x}\in\Omega} v(\mathbf{x}) > -\infty$*;*
(iv) $\sup_{\mathbf{x}\in\Omega}(u(\mathbf{x}) - v(\mathbf{x})) > \sup_{\mathbf{y}\in\partial\Omega}(u^*(\mathbf{y}) - v_*(\mathbf{y}))$*;*

Then there is a point $\mathbf{x}_* \in \Omega$ *such that* $u(\mathbf{x}_*) - v(\mathbf{u}_*) = \sup_{\mathbf{x}\in\Omega}(u(\mathbf{x}) - v(\mathbf{x}))$ *and and for each* $j \geq 1$ *there are points* $\mathbf{x}_j, \mathbf{y}_j \in \Omega$*,* $\mathbf{p}_j \in \mathbb{R}^d$*,* $X_j, Y_j \in \mathcal{S}(d)$ *such that* $\lim_{j\to\infty} \mathbf{x}_j = \lim_{j\to\infty} \mathbf{y}_j = \mathbf{x}_*$*,* $\lim_{j\to\infty} u(\mathbf{x}_j) = u(\mathbf{x}_*)$*,* $\lim_{j\to\infty} v(\mathbf{y}_j) =$

$v(\mathbf{x}_*)$, $(\mathbf{x}_j, u(\mathbf{x}_j), \mathbf{p}_j, X_j) \in \overline{J^{2,+}_{\Omega,u}}$, $(\mathbf{y}_j, v(\mathbf{y}_j), \mathbf{p}_j, Y_j) \in \overline{J^{2,-}_{\Omega,v}}$, $X_j \leq Y_j$, $\lim_{j\to\infty}(|x_j - y_j||\mathbf{p}_j| + |x_j - y_j|^2(|X_j| + |Y_j|)) = 0$.

Proof of Proposition 4.1. Let

$$\Phi(\mathbf{x}, \mathbf{y}) = u^*(\mathbf{x}) - v_*(\mathbf{y}) - \tfrac{1}{2}\alpha|\mathbf{x} - \mathbf{y}|^2, \quad \mathbf{x}, \mathbf{y} \in \overline{\Omega}.$$

Since Φ is upper semicontinuous and $\overline{\Omega} \times \overline{\Omega}$ is compact the maximum of Φ is achieved at some point $(\mathbf{x}_\alpha, \mathbf{y}_\alpha)$. If $\mathbf{z}^*$ is a cluster point of $(\mathbf{x}_\alpha, \mathbf{y}_\alpha)$ then it follows from the assumption $\sup_{\mathbf{x}\in\Omega}(u(\mathbf{x}) - v(\mathbf{x})) > \sup_{\mathbf{y}\in\partial\Omega}(u^*(\mathbf{y}) - v_*(\mathbf{y}))$ and from Lemma 4.9 that $\mathbf{z}^* = (\mathbf{x}_*, \mathbf{x}_*)$ for some $\mathbf{x}_* \in \Omega$. It also follows that $u(\mathbf{x}_*) - v(\mathbf{x}_*) = \sup_{\mathbf{x}\in\Omega}(u(\mathbf{x}) - v(\mathbf{x}))$ and this implies, since u is upper and v is lower semicontinuous that $\lim_{j\to\infty} u(\mathbf{x}_j) = u(\mathbf{x}_*)$ and $\lim_{j\to\infty} v(\mathbf{y}_j) = v(\mathbf{x}_*)$.

By taking subsequences we find $\mathbf{x}_j$ and $\mathbf{y}_j$ such that $\lim_{j\to\infty}\mathbf{x}_j = \lim_{j\to\infty}\mathbf{y}_j = \mathbf{x}_*$. The remaining claims follow from Theorem 2.9 by taking $u_1 = u$, $u_2 = -v$, $\varphi(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\alpha|\mathbf{x} - \mathbf{y}|^2$, $\kappa = \frac{\alpha}{3}$, so that $D_\mathbf{x}\varphi(\mathbf{x}, \mathbf{y}) = -D_\mathbf{y}\varphi(\mathbf{x}, \mathbf{y}) = \alpha(\mathbf{x} - \mathbf{y})$ and $D^2\varphi(\mathbf{x}, \mathbf{y}) = \alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$. □

Now we get a first comparison result (where $t_+ = \max\{0, t\}$):

Theorem 4.2. *Assume that* $d \geq 1$ *and that*

(i) $\Omega \subset \mathbb{R}^d$ *is open, bounded, and nonempty;*

(ii) $F \in \mathcal{C}(\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d); \mathbb{R})$ *is nonincreasing in its fourth argument;*

(iii) *There is a function* $a \in \mathcal{C}(\Omega \times \Omega \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ *such that* $a(\mathbf{x}, \mathbf{x}, r, 0) = 0$ *and*

$$|F(\mathbf{x}, r, \mathbf{p}, X) - F(\mathbf{y}, r, \mathbf{p}, X)| \leq a(\mathbf{x}, \mathbf{y}, r, |\mathbf{x} - \mathbf{y}||\mathbf{p}| + |\mathbf{x} - \mathbf{y}|^2|X|),$$

for all $\mathbf{x}, \mathbf{y} \in \Omega$, $r, s \in \mathbb{R}$, $\mathbf{p} \in \mathbb{R}^d$ *and* $X \in \mathcal{S}(d)$;

(iv) *There is a function* $b \in \mathcal{C}(\Omega \times \mathbb{R} \times [0, \infty); \mathbb{R})$ *such such that*

$$F(\mathbf{x}, r, \mathbf{p}, X) - F(\mathbf{x}, s, \mathbf{p}, X) \geq b(\mathbf{x}, r, s) > 0,$$

for all $\mathbf{x} \in \Omega$, $r > s$, $\mathbf{p} \in \mathbb{R}^d$, *and* $X \in \mathcal{S}(d)$.

(v) $u \in \mathcal{USC}(\Omega)$ *is a subsolution of* $F = 0$ *in* Ω *and* $\sup_{\mathbf{x}\in\Omega} u(\mathbf{x}) < \infty$;

(vi) $v \in \mathcal{LSC}(\Omega)$ *is a subsolution of* $F = 0$ *in* Ω *and* $\inf_{\mathbf{x}\in\Omega} v(\mathbf{x}) > -\infty$;

Then

$$u(\mathbf{x}) - v(\mathbf{x}) \leq \sup_{\mathbf{y}\in\partial\Omega}\big(u^*(\mathbf{y}) - v_*(\mathbf{y})\big)_+, \quad \mathbf{x} \in \Omega.$$

Proof. If the claim does not hold, then the assumptions of Proposition 4.1 hold. Since u is a subsolution, v is a supersolution and F is continuous we see that

$$F(\mathbf{x}_j, u(\mathbf{x}_j), \mathbf{p}_j, X_j) \leq 0 \quad \text{and} \quad F(\mathbf{y}_j, v(\mathbf{x}_j), \mathbf{p}_j, Y_j) \geq 0.$$

Now it follows from the assumptions that

$$\begin{aligned}0 \geq F(\mathbf{x}_j, u(\mathbf{x}_j), \mathbf{p}_j, X_j) &- F(\mathbf{y}_j, v(\mathbf{x}_j), \mathbf{p}_j, Y_j) \\ &\geq F(\mathbf{x}_j, u(\mathbf{x}_j), \mathbf{p}_j, X_j) - F(\mathbf{y}_j, v(\mathbf{x}_j), \mathbf{p}_j, X_j) \\ &= F(\mathbf{x}_j, u(\mathbf{x}_j), \mathbf{p}_j, X_j) - F(\mathbf{x}_j, v(\mathbf{x}_j), \mathbf{p}_j, X_j) \\ &\quad + F(\mathbf{x}_j, v(\mathbf{x}_j), \mathbf{p}_j, X_j) - F(\mathbf{y}_j, v(\mathbf{x}_j), \mathbf{p}_j, X_j) \\ \geq b(\mathbf{x}_j, u(\mathbf{x}_j), v(\mathbf{y}_j)) &- a(\mathbf{x}_j, \mathbf{y}_j, v(\mathbf{x}_j), |\mathbf{x}_j - \mathbf{y}_j||\mathbf{p}_j| + |\mathbf{x}_j - \mathbf{y}_j|^2|X_j|) \\ &\qquad \to b(\mathbf{x}_*, u(\mathbf{x}_*), v(\mathbf{x}_*) - a(\mathbf{x}_*, \mathbf{x}_*, v(\mathbf{x}_*), 0) > 0,\end{aligned}$$

and we have a contradiction.

□

2. Parabolic equations

Just as in the case of elliptic equations there are very many different variants of the comparison results. We start by presenting a basic one. Observe that although this result is given for a general domain we do not have to worry about which points of the boundary belong to the parabolic boundary.

Theorem 4.3. *Assume that* $d \geq 1$ *and that*

(i) $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ *is open and for each* $\tau < T_\Omega \overset{\text{def}}{=} \sup\{\, t \in \mathbb{R} : (t, \mathbf{x}) \in \Omega, \mathbf{x} \in \mathbb{R}^d \,\}$ *the set* $(-\infty, \tau] \times \mathbb{R}^d \cap \Omega$ *is bounded;*

(ii) $F \in \mathcal{C}(\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d); \mathbb{R})$ *is nonincreasing in its last argument;*

(iii) *There is a number* $\lambda \in \mathbb{R}$ *such that* $F(t, \mathbf{x}, r, \mathbf{p}, X) - F(t, \mathbf{x}, s, \mathbf{p}, X) \geq -\lambda(r - s)$, $(t, \mathbf{x}) \in \Omega$, $r > s$, $\mathbf{p} \in \mathbb{R}^d$ *and* $X \in \mathcal{S}(d)$;

(iv) *There is a function* $a \in \mathcal{C}(\Omega_2 \times \mathbb{R} \times \mathbb{R}^+; \mathbb{R})$ *where* $\Omega_2 \overset{\text{def}}{=} \{\, (t, \mathbf{x}, \mathbf{y}) : (t, \mathbf{x}), (t, \mathbf{y}) \in \Omega \,\}$ *such that* $a(t, \mathbf{x}, \mathbf{x}, r, 0) = 0$, $(t, \mathbf{x}) \in \Omega$, $r \in \mathbb{R}$ *and*

$$\begin{aligned}|F(t, \mathbf{x}, r, \mathbf{p}, X) &- F(t, \mathbf{y}, r, \mathbf{p}, X)| \\ &\leq a\big(t, \mathbf{x}, \mathbf{y}, r, |\mathbf{x} - \mathbf{y}||\mathbf{p}| + |\mathbf{x} - \mathbf{y}|^2|X|\big),\end{aligned}$$

for all $(t, \mathbf{x}, \mathbf{y}) \in \Omega_2$, $r \in \mathbb{R}$, $\mathbf{p} \in \mathbb{R}^d$ *and* $X \in \mathcal{S}(d)$;

(v) $u \in \mathcal{USC}(\Omega; [-\infty, \infty))$ *is a subsolution of* $u_t + F(t, \mathbf{x}, u, D_x u, D_x^2 u) = 0$ *in* Ω *and* $\sup_{t \leq \tau, (t, \mathbf{x}) \in \Omega} u(t, \mathbf{x}) < \infty$ *for all* $\tau < T_\Omega$;

(vi) $v \in \mathcal{LSC}(\Omega; (-\infty, \infty])$ *is a supersolution of* $v_t + F(t, \mathbf{x}, v, D_x v, D_x^2 v) = 0$ *in* Ω *and* $\inf_{t \leq \tau, (t, \mathbf{x}) \in \Omega} v(t, \mathbf{x}) > -\infty$ *for all* $\tau < T_\Omega$;

Then

$$u(t, \mathbf{x}) - v(t, \mathbf{x}) \leq \sup_{(s, \mathbf{y}) \in \partial\Omega, s \leq t} \mathrm{e}^{\lambda(t-s)}(u^*(s, y) - v_*(s, y))_+$$

for every $(t, \mathbf{x}) \in \Omega$.

Proof of Theorem 4.3. Let $\tau < T_\Omega$ and $\epsilon > 0$ First we show that $u(t,\mathbf{x}) - v(t,\mathbf{x}) - \frac{\epsilon \mathrm{e}^{\lambda t}}{\tau - t} \le \sup_{s<\tau,(s,\mathbf{y})\in\partial\Omega} \mathrm{e}^{\lambda(t-s)}(u^*(s,y) - v_*(s,y))_+$, for all $(t,\mathbf{x}) \in \Omega$ with $t < \tau$. Suppose that this is not the case. Let

$$\Phi(t,\mathbf{x},\mathbf{y}) = \mathrm{e}^{-\lambda t}\big(u^*(t,\mathbf{x}) - v_*(t,\mathbf{y})\big) - \frac{\epsilon}{\tau - t} - \tfrac{1}{2}\alpha|\mathbf{x}-\mathbf{y}|^2,$$
$$(t,\mathbf{x}),(t,\mathbf{y}) \in \overline{\Omega}, \quad t < \tau.$$

By assumption we know that this function is bounded from above and there is a number $\delta > 0$, independent of α such that a maximum cannot be found at a point where $t > \tau - \delta$. Since the set $\overline{\Omega} \cap (-\infty, \tau - \delta] \times \mathbb{R}^d$ is compact the maximum value is achieved at some point $(t_\alpha, \mathbf{x}_\alpha, \mathbf{y}_\alpha)$.

We may clearly choose a convergent subsequence and since we by Lemma 4.9 know that $\alpha|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2 \to 0$ as $\alpha \to \infty$ we find some point $(t^*, \mathbf{x}^*) \in \overline{\Omega}$ with $t^* \le \tau - \delta$ such that $\lim_{j\to\infty} t_{\alpha_j} = t^*$ and $\lim_{j\to\infty} \mathbf{x}_{\alpha_j} = \lim_{j\to\infty} \mathbf{y}_{\alpha_j} = \mathbf{x}^*$ for some subsequence $(\alpha_j)_{j=1}^\infty$.

Our assumption that $u(t,\mathbf{x}) - v(t,\mathbf{x}) - \frac{\epsilon \mathrm{e}^{\lambda t}}{\tau - t} \le \sup_{s<\tau,(s,\mathbf{y})\in\partial\Omega} \mathrm{e}^{\lambda(t-s)}(u^*(s,y) - v_*(s,y))_+$ does not hold for all $t < \tau$ implies that $(t^*, \mathbf{x}^*) \in \Omega$ and thus we conclude that $(t_\alpha, \mathbf{x}_\alpha)$ and $(t_\alpha, \mathbf{y}_\alpha) \in \Omega$ for α large enough (because otherwise we could choose a suitable subsequence which would give a contradiction).

By definition we have

$$u(t,\mathbf{x}) - v(t,\mathbf{y}) \le \mathrm{e}^{\lambda(t-t_\alpha)}\big(u(t_\alpha,\mathbf{x}_\alpha) - v(t_\alpha,\mathbf{y}_\alpha)\big) + \mathrm{e}^{\lambda t}\left(\frac{\epsilon}{\tau - t} - \frac{\epsilon}{\tau - t_\alpha}\right)$$
$$+ \mathrm{e}^{\lambda t}\left(\tfrac{1}{2}\alpha|\mathbf{x}-\mathbf{y}|^2 - \tfrac{1}{2}\alpha|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2\right),$$

for all $(t,\mathbf{x})$ and $(t,\mathbf{y}) \in \Omega$ with $t < \tau$. Therefore we know, by Theorem 2.16 (and the fact that $P^{2,+}_{\mathcal{B},-v}(t,\mathbf{x}) = -P^{2,-}_{\mathcal{B},v}(t,\mathbf{x})$) that there are

$$(t_\alpha, \mathbf{x}_\alpha, u(t_\alpha,\mathbf{x}_\alpha), a_\alpha, \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X_\alpha) \in \overline{P^{2,+}_{[0,\tau)\times\overline{\Omega},u}},$$
$$(t_\alpha, \mathbf{y}_\alpha, u(t_\alpha,\mathbf{y}_\alpha), b_\alpha, \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y_\alpha) \in \overline{P^{2,-}_{[0,\tau)\times\overline{\Omega},v}},$$

such that

$$a_\alpha - b_\alpha = \frac{\epsilon \mathrm{e}^{\lambda t_\alpha}}{(\tau - t_\alpha)^2} + \lambda(u(t_\alpha,\mathbf{x}_\alpha) - v(t_\alpha,\mathbf{y}_\alpha)), \tag{4.1}$$

and

$$-3\alpha\mathrm{e}^{\lambda t_\alpha} I \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3\alpha\mathrm{e}^{\lambda t_\alpha}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Thus we have $X_\alpha \le Y_\alpha$ and $\|X_\alpha\| \le 3\alpha\mathrm{e}^{\lambda t_\alpha}$. Provided α is so large that $(t_\alpha, \mathbf{x}_\alpha)$ and $(t_\alpha, \mathbf{y}_\alpha) \in \Omega$ we have by the assumption that u is a sub- and v a supersolution that

$$a_\alpha + F(t_\alpha, \mathbf{x}_\alpha, u(t_\alpha,\mathbf{x}_\alpha), \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X_\alpha) \le 0,$$

and

$$b_\alpha + F(t_\alpha, \mathbf{y}_\alpha, v(t_\alpha, \mathbf{y}_\alpha), \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y_\alpha) \ge 0.$$

By subtracting these equations from each other, using (4.1), the monotonicity assumptions on F together with the facts that $u(t_\alpha, \mathbf{x}_\alpha) > v(t_\alpha, \mathbf{y}_\alpha)$ and $X_\alpha \le Y_\alpha$ we conclude that

$$\begin{aligned}
0 \ge \frac{\epsilon \mathrm{e}^{\lambda t_\alpha}}{(\tau - t_\alpha)^2} &+ \lambda\big(u(t_\alpha, \mathbf{x}_\alpha) - v(t_\alpha, \mathbf{y}_\alpha)\big) \\
&+ F(t_\alpha, \mathbf{x}_\alpha, u(t_\alpha, \mathbf{x}_\alpha), \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X_\alpha) \\
&\quad - F(t_\alpha, \mathbf{y}_\alpha, v(t_\alpha, \mathbf{y}_\alpha), \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), Y_\alpha) \\
\ge \frac{\epsilon \mathrm{e}^{\lambda t_\alpha}}{(\tau - t_\alpha)^2} &+ F(t_\alpha, \mathbf{x}_\alpha, u(t_\alpha, \mathbf{x}_\alpha), \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X_\alpha) \\
&- F(t_\alpha, \mathbf{y}_\alpha, u(t_\alpha, \mathbf{x}_\alpha), \mathrm{e}^{\alpha t_\alpha}\alpha(\mathbf{x}_\alpha - \mathbf{y}_\alpha), X_\alpha) \\
\ge \frac{\epsilon \mathrm{e}^{\lambda t_\alpha}}{(\tau - t_\alpha)^2} &- a(t_\alpha, \mathbf{x}_\alpha, \mathbf{y}_\alpha, u(t_\alpha, \mathbf{x}_\alpha), \mathrm{e}^{\alpha t_\alpha}\alpha|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2 + |\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2|X_\alpha|)).
\end{aligned}$$

If we now let $\alpha \to \infty$ (and choose a subsequnce if necessary) we get a contradiction because $\lim_{\alpha\to\infty} \mathrm{e}^{\alpha t_\alpha}\alpha|\mathbf{x}_\alpha - \mathbf{y}_\alpha|^2 = 0$, $|X_\alpha| \le 3\alpha \mathrm{e}^{\lambda t_\alpha}$, and t_α stays in a bounded interval.

Thus we have shown that for each $\epsilon > 0$ and $\tau < T_\Omega$ we have

$$u(t, \mathbf{x}) - v(t, \mathbf{x}) - \frac{\epsilon \mathrm{e}^{\lambda t}}{\tau - t} \le \sup_{s<\tau,(s,\mathbf{y})\in\partial\Omega} \mathrm{e}^{\lambda(t-s)}(u^*(s, y) - v_*(s, y))_+,$$

for all $(t, \mathbf{x}) \in \Omega$ with $t < \tau$. Since ϵ is arbitrary we get

$$u(t, \mathbf{x}) - v(t, \mathbf{x}) \le \sup_{s<\tau,(s,\mathbf{y})\in\partial\Omega} \mathrm{e}^{\lambda(t-s)}(u^*(s, y) - v_*(s, y))_+.$$

The desired inequality now follows by letting $\tau \downarrow t$ and using the fact that $u^* - v_*$ is upper semicontinuous. ☐

It would not be difficult to let the constant λ in (iii) depend on r and s but we only formulate this result for a non-quantitative comparison result.

Corollary 4.4. *Let the assumptions of Theorem* 4.3 *hold except that* (iii) *is replaced by the assumption*

(iii)’ *For each $C < \infty$ there is a number $\lambda \in \mathbb{R}$ such that $F(t, \mathbf{x}, r, \mathbf{p}, X) - F(t, \mathbf{x}, s, \mathbf{p}, X) \ge -\lambda(r - s)$, $(t, \mathbf{x}) \in \Omega$, $C > r > s > -C$, $\mathbf{p} \in \mathbb{R}^d$ and $X \in \mathcal{S}(d)$;*

Then $u(t, x) \le v(t, x)$ where $(t, \mathbf{x}) \in \Omega$ provided $u^*(s, \mathbf{y}) \le v_*(s, \mathbf{y})$ for all $(s, \mathbf{y}) \in \partial\Omega$ with $s \le t$.

A similar trivial (and thus rather silly) extension is that we may let λ depend on $|\mathbf{p}|$ and $\|X\|$ provided we know that these are bounded. Thus we get a rather strong comparison result for smooth solutions.

Corollary 4.5. *Let the assumptions of Theorem* 4.3 *hold except that* (iii) *is replaced by the assumption*

(iii)" *For each $C < \infty$ there is a number $\lambda \in \mathbb{R}$ such that $F(t, \mathbf{x}, r, \mathbf{p}, X) - F(t, \mathbf{x}, s, \mathbf{p}, X) \geq -\lambda(r - s)$, $(t, \mathbf{x}) \in \Omega$, $C > r > s > -C$, $\mathbf{p} \in \mathbb{R}^d$ with $|\mathbf{p}| < C$ and $X \in \mathcal{S}(d)$ with $\|X\| < C$.;*

and we, in addition, assume that

(vii) *For each $\tau < T_\Omega$ we have* $\sup\{\, |\mathbf{p}| + (\mu)_- : (t, \mathbf{x}, u(t, \mathbf{x}), a, \mathbf{p}, X) \in P^{2,+}_{\Omega,u}, t \leq \tau, \mu \in \operatorname{eig}(X)\,\} < \infty$ *and* $\sup\{\, |\mathbf{p}| + (\mu)_+ : (t, \mathbf{x}, v(t, \mathbf{x}), a, \mathbf{p}, X) \in P^{2,-}_{\Omega,v}, t \leq \tau, \mu \in \operatorname{eig}(X)\,\} < \infty$ *where* $\operatorname{eig}(X)$ *denotes the eigenvalues of X.*

Then $u(t, x) \leq v(t, x)$ where $(t, \mathbf{x}) \in \Omega$ provided $u^*(s, \mathbf{y}) \leq v_*(s, \mathbf{y})$ for all $(s, \mathbf{y}) \in \partial\Omega$ with $s \leq t$.

To see that some extra smoothness assumption as above is needed one has only to consider the following example.

Example 4.6. *There are at least two solutions to the equation*

$$u_t - (1 - u)_+ u_{xx} = 0, \quad t > 0, \quad -1 < x < 1, \tag{4.2}$$

with boundary conditions $u(t, -1) = u(1, 1) = 0$ and initial condition $u(0, x) = 1 - |x|$.

Proof. One solution is clearly $u(t, x) = 1 - |x|$, $|x| \leq 1$ and $t \geq 0$. (Note that the fact that the coefficient of u_{xx} vanishes at $x = 0$ is important only when showing that u is a subsolution.) The other solution will be constructed as the limit of smooth solutions.

Let $n \geq 2$ and define $u_{n,0}(x)$ to be such that it is inifinitely many times differentiable, even and concave, and such that $u_{n,0}(0) = \frac{n-1}{n}$ and $u_{n,0}(x) = 1 - |x|$ when $1 \geq |x| \geq \frac{1}{n-1}$. For each n we may define a inifintely many times differentiable strictly decreasing function g_n such that $g_n(r) \geq \frac{1}{n+1}$ and $g_n(r) = 1 - r$ when $r \leq \frac{n-1}{n}$. Now it follows from standard theory, see e.g. [**6**, Thm. 4.2 Chap. VI] that there exista a solution u_n to the equation

$$u_t(t, x) = g_n(u(t, x))u_{xx}(t, x), \quad -1 < x < 1, \quad t > 0,$$

$u(0, x) = u_{n,0}(x)$, such that $(u_n)_{xx}(t, x)$ is continuous in $\mathbb{R}^+ \times [-1, 1]$. Thus we may apply Corollary 4.5 and first we conclude that $u_n(t, x) \leq \frac{n-1}{n}$ so we do not need to introduce the function g_n at all in the equation. Thus we can again use Corollary 4.5 and conclude that $u_{n+1}(t, x) \geq u_n(t, x)$, that $\max_{x \in [-1,1]} u_n(t, x)$

is nonincreasing, and that $u_n(t,x) \le 1-x$ and $u_n(t,x) \le 1+x$ for all $t \ge 0$ and $x \in [-1,1]$.

The second solution we are looking for is of course $\lim_{n\to\infty} u_n(t)$ but in order to show that it is a solution and that it is different from the first one we need some estimates. For this purpose we investigate the properties of classical solutions to (4.2) with $u(t,x) < 1$. We muliply both sides of the equation by $u_{xx}(t,x)$ and integrate over $[0,T]$ and $[-1,1]$. By an integration by parts (with respect to the x-variable) where we use the boundary conditions we get

$$\begin{aligned}(4.3)\quad \frac{1}{2}\int_{-1}^{1} u_x(0,x)^2\,\mathrm{d}x - \frac{1}{2}\int_{-1}^{1} u_x(t,x)^2\,\mathrm{d}x \\ = \int_0^T \int_{-1}^{1} (1-u(t,x))u_{xx}(t,x)^2\,\mathrm{d}x \\ \ge \int_0^T \bigl(1 - \max_{x\in[-1,1]} u(t,x)\bigr) \int_{-1}^{1} u_{xx}(t,x)^2\,\mathrm{d}x.\end{aligned}$$

Now fix $t \in [0,T]$, let $\max_{x\in[-1,1]} u(t,x) = 1-c$ and suppose that $c \le \frac{1}{3}$. Assume that the maximum is achieved at a point x_0. (One can show that x_0 must be 0 be we do not go into these details here.) Since $u(t,-1)=0$ and $c \le \frac{1}{3}$ there is a point $x_1 < x_0$ so that $u(t,x_1) = 1-3c$. Since $u(t,x) \le 1-|x|$ we have $x_1 \ge -3c$. Now we conclude, since $u_x(t,x_0)=0$, that

$$\begin{aligned}2c &= \int_{x_1}^{x_0} u_x(t,x)\,\mathrm{d}x = \int_{x_1}^{x_0} (u_x(t,x) - u_x(t,x_0))\,\mathrm{d}x \\ &= \int_{x_1}^{x_0}\int_{x}^{x_0} (-u_{xx}(t,y)\,\mathrm{d}y\,\mathrm{d}x = \int_{x_1}^{x_0} (-u_{xx}(t,y)) \int_{x_1}^{y} \mathrm{d}x\,\mathrm{d}y \\ &= \int_{x_1}^{x_0} (-u_{xx}(t,y))(y-x_1)\,\mathrm{d}y \le \sqrt{\int_{x_1}^{x_0} u_{xx}(t,y)^2\,\mathrm{d}y \int_{x_1}^{x_0} (y-x_1)^2\,\mathrm{d}y}.\end{aligned}$$

Because

$$\int_{x_1}^{x_0} (y-x_1)^2\,\mathrm{d}y = \frac{1}{3}(x_0-x_1)^3 \le \frac{1}{3}(x_0+3c)^3,$$

we conclude that

$$\int_{x_1}^{x_0} u_{xx}(t,x)^2\,\mathrm{d}x \ge \frac{12c^2}{(x_0+3c)^3}.$$

By the same argument we get

$$\int_{x_0}^{x_2} u_{xx}(t,x)^2\,\mathrm{d}x \ge \frac{12c^2}{(x_0-3c)^3},$$

where $x_2 > x_0$ is such that $u(t,x_2) = 1-3c$. We add these inequalities and observe that the smallest value is achieved when $x_0 = 0$ so that

$$\int_{-1}^{1} u_{xx}(t,x)^2 \,\mathrm{d}x \geq \frac{8}{9c}.$$

Thus we conlude provided $\max_{x\in[-1,1]} u_n(t,x) \geq \frac{2}{3}$ we have

$$\frac{1}{2}\int_{-1}^{1} (u_n)_x(0,x)^2 \,\mathrm{d}x - \frac{1}{2}\int_{-1}^{1} (u_n)_x(t,x)^2 \,\mathrm{d}x \geq \frac{8t}{9}.$$

On the other hand, we easily get (by an argument similar to the one used above) that

$$\max_{x\in[-1,1]} u_n(t,x) \leq \sqrt{\frac{1}{2}\int_{-1}^{1} (u_n)_x(t,x)^2 \,\mathrm{d}x},$$

so that as long as $\max_{x\in[-1,1]} u_n(t,x) \geq \frac{2}{3}$ we have

$$\max_{x\in[-1,1]} u_n(t,x) \leq \sqrt{1 - \frac{8t}{9}}. \tag{4.4}$$

From (4.3) we conclude that

$$\int_0^T \int_{-1}^{1} (1 - u_n(t,x)(u_n)_{xx}(t,x)^2 \,\mathrm{d}x \,\mathrm{d}t \leq 1$$

and using this inequality we get from equation (4.2) and Hölder's inequality that

$$\begin{aligned}\int_{-1}^{1} |u_n(t,x) - u_n(\tau,x)|^2 \,\mathrm{d}x &\leq \int_{-1}^{1} \left| \int_{\tau}^{t} (u_n)_t(s,x)\,\mathrm{d}s \right|^2 \mathrm{d}x \\ &\leq \int_{-1}^{1} |t-\tau| \int_{\tau}^{t} (u_n)_t(s,x)^2 \,\mathrm{d}s \,\mathrm{d}x \\ &= |t-\tau| \int_{-1}^{1}\int_{\tau}^{t} (1-u_n(s,x))^2 (u_n)_{xx}(s,x)^2 \,\mathrm{d}s\,\mathrm{d}x \leq |t-\tau|.\end{aligned}$$

When we combine this result with the fact that $\int_{-1}^{1} (u_n)_x(t,x)^2 \,\mathrm{d}x \leq 1$ we conclude that the functions $u_n(t,x)$ are equicontinuous, which implies that we get $\limsup^*{}_{n\to\infty} u_n = \liminf_{*n\to\infty} u_n$ and this implies by Theorem 3.2 that this limit is a sollution to equation (4.2). From inequality (4.4) we see that this solution cannot be the same as the first one. □

3. Further comparison results

First we prove a version of the strong maximum principle.

Theorem 4.7. *Assume that* $d \geq 1$ *and that*

(i) $\Omega \subset \mathbb{R}^d$ *is open and nonempty;*

(ii) *$F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}(d) \to [-\infty, \infty]$ is nonincreasing in its fourth variable;*

(iii) *For every $\mathbf{x} \in \Omega$ there is $\rho > 0$ such that for every $\lambda > 0$ there are $\mu > 0$ and $\delta > 0$ such that*

$$F\left(\mathbf{y}, s, \mathbf{p}, \lambda|\mathbf{p}|I - \frac{\mu}{|\mathbf{p}|}\mathbf{p} \otimes \mathbf{p}\right) > 0$$

$$\left(or \quad F\left(\mathbf{y}, s, \mathbf{p}, -\lambda|\mathbf{p}|I + \frac{\mu}{|\mathbf{p}|}\mathbf{p} \otimes \mathbf{p}\right) < 0\right)$$

for all $\mathbf{y} \in \Omega$ with $|\mathbf{y} - \mathbf{x}| < \rho$, $s > s_0 \geq -\infty$, (or $s < s_0 \leq \infty$), and $0 < |\mathbf{p}| < \delta$.

(iv) *$u \in \mathcal{USC}(\Omega)$ (or $\mathcal{LSC}(\Omega)$) is a subsolution (or supersolution) of $F(\mathbf{x}, u, Du, D^2u) = 0$ in Ω and u achieves a local maximum (or minimum) at the point $\mathbf{x}_0 \in \Omega$ with $u(\mathbf{x}_0) > s_0$ (or $u(\mathbf{x}_0) < s_0$).*

Then u is constant in a neighbourhood of $\mathbf{x}_0$.

Proof. We choose $r > 0$ so small that $\rho(\mathbf{x}_0) < r$ and $B(\mathbf{x}_0, r) \subset \Omega$ and $u(\mathbf{x}) \leq u(\mathbf{x}_0)$ for all $\mathbf{x} \in B(\mathbf{x}_0, r)$. If u is not a constant in $B(\mathbf{x}_0, r)$, then there is, of course, a point $\mathbf{z}_0 \in B(\mathbf{x}_0, r)$ such that $u(\mathbf{z}_0) < u(\mathbf{x}_0)$ but we claim that $\mathbf{z}_0$ can be chosen so that $u(\mathbf{x}) < u(\mathbf{x}_0)$ when $|\mathbf{x} - \mathbf{z}_0| < \frac{3}{4}R$ and $u(\mathbf{y}_0) = u(\mathbf{x}_0)$ for some $\mathbf{y}_0 \in B(\mathbf{z}_0, R)$ where $R < r - |\mathbf{z}_0 - \mathbf{x}_0|$. To see that this is the case we observe that since u is upper semi-continuous it follows that the set $\{\, \mathbf{x} \in \Omega : u(\mathbf{x}) < u(\mathbf{x}_0) \,\}$ is open so we may certainly choose $R < r - |\mathbf{z}_0 - \mathbf{x}_0|$ such that $u(\mathbf{x}) < u(\mathbf{x}_0)$ when $|\mathbf{x} - \mathbf{z}_0| < \frac{3}{4}R$. If there is no $\mathbf{y}_0 \in B(\mathbf{z}_0, R)$ such that $u(\mathbf{y}_0) = u(\mathbf{x}_0)$ we can replace $\mathbf{z}_0$ by $\mathbf{z}_0 + \frac{R}{4|\mathbf{x}_0 - \mathbf{z}_0|}(\mathbf{x}_0 - \mathbf{z}_0)$ and repeat this procedure until we find such a point $\mathbf{y}_0$. Using once more the fact that u is upper semicontinuous we see that there is a number $\epsilon > 0$ such that $u(\mathbf{x}) \leq u(\mathbf{x}_0) - \epsilon$ when $|\mathbf{x} - \mathbf{z}_0| = \frac{1}{2}R$. Furthermore we may choose $\epsilon < u(\mathbf{x}_0) - s_0$.

Next define the function ψ by

$$\psi(\underline{\mathbf{x}}) = u(\mathbf{x}_0) + \mathrm{e}^{-\alpha R^2} - \mathrm{e}^{-\alpha|\underline{\mathbf{x}} - \mathbf{z}_0|^2},$$

where $\alpha > 0$. We have

$$D\psi(\underline{\mathbf{x}}) = 2\alpha \mathrm{e}^{-\alpha|\underline{\mathbf{x}} - \mathbf{z}_0|^2}(\underline{\mathbf{x}} - \mathbf{z}_0),$$

and

$$D^2\psi(\underline{\mathbf{x}}) = \alpha \mathrm{e}^{-\alpha|\underline{\mathbf{x}} - \mathbf{z}_0|^2}\big(2I - 4\alpha(\underline{\mathbf{x}} - \mathbf{z}_0) \otimes (\underline{\mathbf{x}} - \mathbf{z}_0)\big).$$

If now $|\mathbf{x} - \mathbf{z}_0| \geq \frac{1}{2}R$ and $\mathbf{p} = D\psi(\mathbf{x})$ then we have

$$D^2\psi(\mathbf{x}) \leq \frac{2|\mathbf{p}|}{R}I - \frac{\alpha R}{|\mathbf{p}|}\mathbf{p} \otimes \mathbf{p}.$$

Thus we take $\lambda = \frac{2}{R}$ in (iii) and if we choose $\alpha > \frac{\mu}{R}$ so large that $|\mathbf{p}| < \delta$, then we conclude from (ii) and (iii) that, provided $u(\mathbf{x}) > s_0$,

$$F(\mathbf{x}, u(\mathbf{x}), D\psi(\mathbf{x}), D^2\psi(\mathbf{x})) > 0, \quad \mathbf{x} \in B(\mathbf{z}_0, R) \setminus \overline{B(\mathbf{z}_0, \tfrac{R}{2})}.$$

Now $\psi(\mathbf{x}) = u(\mathbf{x}_0) \geq u(\mathbf{x})$ for all $\mathbf{x}$ with $|\mathbf{x} - \mathbf{z}_0| = R$ and by choosing α large enough we have $\psi(\mathbf{x}) = u(\mathbf{x}_0) + \mathrm{e}^{-\alpha R^2} - \mathrm{e}^{-\frac{1}{4}\alpha R^2} \geq u(\mathbf{x}_0) - \epsilon \geq u(\mathbf{x})$ for all $\mathbf{x}$ with $|\mathbf{x} - \mathbf{z}_0| = \frac{1}{2}R$. But we assumed that there is a point $\mathbf{y}_0$ with $\frac{R}{2} < |\mathbf{y}_0 - \mathbf{z}_0| < R$ such that $u(\mathbf{y}_0) = u(\mathbf{x}_0)$ so that $\psi(\mathbf{y}_0) < u(\mathbf{x}_0) = u(\mathbf{y}_0)$. Thus we see that the maximum of the function $u - \psi$ in $\{\, \mathbf{x} : \frac{R}{2} \leq |\mathbf{y}_0 - \mathbf{z}_0| \leq R \,\}$ is achieved at some interior point $\mathbf{x}_*$ and since the maximum is positive we have $u(\mathbf{x}_*) > \psi(\mathbf{x}_*) > s_0$. But since u is a subsolution and $F(\mathbf{x}_*, u(\mathbf{x}_*), D\psi(\mathbf{x}_*), D^2\psi(\mathbf{x}_*)) > 0$ we get a contradiction.

This implies that u is a constant in $B(\mathbf{x}_0, r)$.

□

Theorem 4.8. *Assume that* $d \geq 1$ *and that*

(i) $\Omega \subset \mathbb{R}^d$ *is open, bounded, and nonempty;*

(ii) $F : \mathbb{R}^d \setminus \{\mathbf{0}\} \times \mathcal{S}(d) \to \mathbb{R}$ *is nonincreasing in its second variable and Lipschitz-continuous on compact subsets of* $\mathbb{R}^d \setminus \{\mathbf{0}\} \times \mathcal{S}(d)$*;*

(iii) *For every* $\lambda > 0$ *there are* $\mu > 0$ *and* $\delta > 0$ *such that*

$$F\left(\mathbf{p}, \lambda|\mathbf{p}|I - \frac{\mu}{|\mathbf{p}|}\mathbf{p} \otimes \mathbf{p}\right) > 0 \quad \textit{and} \quad F\left(\mathbf{p}, -\lambda|\mathbf{p}|I + \frac{\mu}{|\mathbf{p}|}\mathbf{p} \otimes \mathbf{p}\right) < 0,$$

for all $\mathbf{p} \in \mathbb{R}^d$ *with* $0 < |\mathbf{p}| < \delta$*;*

(iv) *For each compact set* $\mathcal{K} \subset \mathbb{R}^d \setminus \{\mathbf{0}\} \times \mathcal{S}(d)$ *there is a positive constant* $\gamma_{\mathcal{K}}$ *such that*

$$F(\mathbf{p}, M + \alpha\mathbf{p} \otimes \mathbf{p}) - F(\mathbf{p}, M + \beta\mathbf{p} \otimes \mathbf{p}) \geq \gamma_{\mathcal{K}}(\beta - \alpha),$$

for all $(\mathbf{p}, M) \in \mathcal{K}$ *and* $\quad -1 < \alpha < \beta < 1$*;*

(v) $u \in \mathcal{USC}(\Omega)$ *is a subsolution of* $F_*(Du, D^2u) = 0$*,* $v \in \mathcal{LSC}(\Omega)$ *is a supersolution of the equation* $F^*(Dv, D^2v) = 0$ *in* Ω*,* $\sup_{\mathbf{x}\in\Omega} u(\mathbf{x}) < \infty$*, and* $\inf_{\mathbf{x}\in\Omega} v(\mathbf{x}) > -\infty$*;*

Then

$$u(\mathbf{x}) - v(\mathbf{x}) \leq \sup_{\mathbf{y}\in\partial\Omega} \big(u^*(\mathbf{y}) - v_*(\mathbf{y})\big), \quad \mathbf{x} \in \Omega.$$

Proof. The fact that the equation does not involve u implies that if u is a subsolution (or a supersolution) then $u + c$ is a subsolution (supersolution) as well for any constant c. If the claim of the theorem does not hold we may therefore assume that

$$\sup_{\mathbf{x}\in\Omega}(u(\mathbf{x}) - v(\mathbf{x})) > 0 \quad \text{and} \quad \sup_{\mathbf{y}\in\partial\Omega}(u^*(\mathbf{y}) - v_*(\mathbf{y})) < 0. \tag{4.5}$$

Since $\overline{\Omega}$ is compact it follows from the second inequality that there is a number $\delta > 0$ such that the set

$$\big\{\, (\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in \Omega,\ u(\mathbf{x}) - v(\mathbf{y}) \geq 0,\ |\mathbf{x} - \mathbf{y}| \leq 2\delta \,\big\},$$

is a compact subset of $\Omega \times \Omega$.

We define

$$a = \inf_{\mathbf{x} \in \Omega} v(\mathbf{x}) - 1 \quad \text{and} \quad b = \sup_{\mathbf{x} \in \Omega} u(\mathbf{x}).$$

Let

$$h(t) = \sup_{\mathbf{x}, \mathbf{y} \in \Omega, |\mathbf{x} - \mathbf{y}| \leq t} (u(\mathbf{x}) - v(\mathbf{y})), \quad t \geq 0.$$

We claim that we have $h(t) > h(0)$ for $t > 0$. Suppose that this is not the case. Since Ω is bounded, (4.5) holds and $u - v$ is upper semi-continuous there is a point $\mathbf{x}_0 \in \Omega$ is such that $u(\mathbf{x}_0) - v(\mathbf{x}_0) = \sup_{\mathbf{x} \in \Omega}(u(\mathbf{x}) - v(\mathbf{x})) = h(0)$. Since $h(t_0) = 0$ for some $t_0 > 0$ it follows that $\mathbf{x}_0$ is a local maximum for u and a local minimum for v so that by Theorem 4.7 we know that u and v are constants in a neighbourhood of $\mathbf{x}_0$. But this implies that the set where $u(\mathbf{x}) - v(\mathbf{x})$ achieves its maximum is open and since it is, by the semi-continuity assumptions, closed, we conclude that this set is the whole of Ω which is a contradiction in view of (4.5) and we conclude in particular that $h(\delta) - h(0) > 0$.

Since h is upper semi-continuous and nondecreasing we have $\lim_{t \downarrow 0} h(t) = h(0)$ so there exists a number $\mu > 0$ so that

$$h(\mu) - h(0) < \frac{1}{8}(h(\delta) - h(0)).$$

The compact set $\mathcal{K} \subset \mathbb{R}^d \setminus \{\mathbf{0}\} \times \mathcal{S}(d)$ is defined by

$$\mathcal{K} = \left\{ (\mathbf{p}, M) : \frac{h(\delta) - h(0)}{4\delta} \leq |\mathbf{p}| \leq Q, \quad \|M\| \leq \frac{6(b-a)}{\delta^2} + \frac{6Q}{\mu} + \frac{Q^2}{8} \right\},$$

where

$$Q = \frac{h(\delta) - h(0)}{2\delta} + \frac{3(b-a)}{\delta}.$$

We choose

$$\varphi(\underline{t}) = t + \mathrm{e}^{-2n^2 - n\underline{t}},$$

where n is a positive integer chosen so that

$$\begin{aligned}
& -n \leq a < b \leq n, \\
& \frac{16n^2 \mathrm{e}^{-n^2}}{1 - n\mathrm{e}^{-n^2}} \leq \min\big\{1, h(\delta) - h(0)\big\}, \\
& n > \frac{L_{\mathcal{K}}}{\gamma_{\mathcal{K}}} \sup_{(\mathbf{p}, M) \in \mathcal{K}} \big(|\mathbf{p}| + \|M\|\big),
\end{aligned}$$

where $L_{\mathcal{K}}$ is the Lipschitz-constant of F in $\mathcal{K}$. Furthermore, let

$$\eta = \varphi^{-1}, \quad \tilde{u} = \eta(u), \tilde{v} = \eta(v),$$

and

$$\tilde{F}(r, \mathbf{q}, N) = F\big(\varphi'(r)\mathbf{q}, \varphi'(r)N + \varphi''(r)\mathbf{q} \otimes \mathbf{q}\big). \tag{4.6}$$

Then we see that $\tilde{u}$ is a subsolution of $\tilde{F}_* = 0$ and $\tilde{v}$ a supersolution of $\tilde{F}^* = 0$ and also that

$$\inf_{\mathbf{x}\in\Omega} \tilde{v}(\mathbf{x}) \geq a \quad \text{and} \quad \sup_{\mathbf{x}\in\Omega} \tilde{u}(\mathbf{x}) \leq b.$$

If we define $\tilde{h}(t) = \sup_{\mathbf{x},\mathbf{y}\in\Omega,|\mathbf{x}-\mathbf{y}|\leq t}(\tilde{u}(\mathbf{x}) - \tilde{v}(\mathbf{y}))$ for $t \geq 0$ then we conclude that

$$\begin{aligned}\tilde{h}(\delta) - \tilde{h}(0) &\geq h(\delta) - \frac{1}{1 - n\mathrm{e}^{-n^2}} h(0) \\ &\geq h(\delta) - h(0) - \frac{(b-a)n\mathrm{e}^{-n^2}}{1 - n\mathrm{e}^{-n^2}} \geq \frac{7}{8}(h(\delta) - h(0)),\end{aligned}$$

and

$$\begin{aligned}\tilde{h}(\mu) - \tilde{h}(0) &\leq \frac{1}{1 - n\mathrm{e}^{-n^2}} h(\mu) - h(0) \\ &\leq \frac{1}{8}(h(\delta) - h(0)) + \frac{(b-a)n\mathrm{e}^{-n^2}}{1 - n\mathrm{e}^{-n^2}} \leq \frac{1}{4}(h(\delta) - h(0))\end{aligned}$$

because $1 \leq \eta'(t) \leq \frac{1}{1 - n\mathrm{e}^{-2n^2 - na}}$ when $\eta(t) \geq a$ and $a \geq -n$.

Next we construct a function Ψ as follows:

$$\Psi(t) = \begin{cases} 0, & t \leq 0, \\ \frac{(h(\delta)-h(0))}{4\delta\mu^3}(2\mu t^3 - t^4), & 0 < t \leq \mu, \\ \left(t - \frac{\mu}{2}\right) \frac{h(\delta)-h(0)}{2\delta}, & \mu < t \leq \delta, \\ \left(t - \frac{\mu}{2}\right) \frac{h(\delta)-h(0)}{2\delta} + \frac{(b-a)(t-\delta)^3}{\delta^3}, & t > \delta. \end{cases}$$

We see that we have a function $\Psi \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ such that $\tilde{h}(t) \leq b - a - (b-a) = 0$ when $t \geq 2\delta$, $\tilde{h}(\delta) - \Psi(\delta) \geq \tilde{h}(0) + \frac{3}{8}(h(\delta) - h(0))$, and $\tilde{h}(t) - \Psi(t) \leq \tilde{h}(0) + \frac{1}{4}(h(\delta) - h(0))$ when $0 \leq t \leq \mu$. Thus the maximum of the the function $\tilde{h} - \Psi$ is achieved at a point $t_* \in (\mu, 2\delta)$ and

$$\begin{aligned}&\frac{h(\delta) - h(0)}{2\delta} \leq \Psi'(t_*) \leq \frac{h(\delta) - h(0)}{2\delta} + \frac{3(b-a)}{\delta}, \\ &0 \leq \Psi''(t_*) \leq \frac{6(b-a)}{\delta^2}.\end{aligned}$$

Now we proceed to a standard argument for viscosity solutions and define

$$\Phi(\mathbf{x}, \mathbf{y}) = \tilde{u}(\mathbf{x}) - \tilde{v}(\mathbf{y}) - \Psi(|\mathbf{x} - \mathbf{y}|), \quad \mathbf{x}, \mathbf{y} \in \Omega.$$

It follows from our choice of δ, the definition of $\tilde{h}$, the fact that $t_* < 2\delta$, and from the semi-continuity assumptions that the maximum of Φ is achieved at some point $(\mathbf{x}_*, \mathbf{y}_*) \in \Omega \times \Omega$ with $|\mathbf{x}_* - \mathbf{y}_*| = t_*$. Define

$$\begin{aligned}
\mathbf{q} &= \frac{\Psi'(|\mathbf{x}_* - \mathbf{y}_*|)}{|\mathbf{x}_* - \mathbf{y}_*|}(\mathbf{x}_* - \mathbf{y}_*),\\
N &= \left(\frac{\Psi''(|\mathbf{x}_* - \mathbf{y}_*|)}{|\mathbf{x}_* - \mathbf{y}_*|^2} - \frac{\Psi'(|\mathbf{x}_* - \mathbf{y}_*|)}{|\mathbf{x}_* - \mathbf{y}_*|^3}\right)(\mathbf{x}_* - \mathbf{y}_*) \otimes (\mathbf{x}_* - \mathbf{y}_*)\\
&\quad + \frac{\Psi'(|\mathbf{x}_* - \mathbf{y}_*|)}{|\mathbf{x}_* - \mathbf{y}_*|} I.
\end{aligned}$$

By Theorem 2.9 we have $(\mathbf{x}_*, \tilde{u}(\mathbf{x}_*), \mathbf{q}, X) \in \overline{J^{2,+}_{\Omega,\tilde{u}}}$ and $(\mathbf{y}_*, \tilde{v}(\mathbf{y}_*), \mathbf{q}, Y) \in \overline{J^{2,-}_{\Omega,\tilde{u}}}$ such that

$$-\kappa^{-1} I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \left(I - \kappa \begin{pmatrix} N & -N \\ -N & N \end{pmatrix}\right)^{-1} \begin{pmatrix} N & -N \\ -N & N \end{pmatrix},$$

where κ is e.g. $\frac{1}{3\|N\|}$. From this we conclude that $X \leq Y$ and that $\|X\| \leq 3\|N\|$. Using the facts that $\tilde{u}$ and $\tilde{v}$ are, respectively, sub and supersolutions we have

$$\tilde{F}(\tilde{u}(\mathbf{x}_*), \mathbf{q}, X) \leq 0 \quad \text{and} \quad \tilde{F}(\tilde{v}(\mathbf{y}_*), \mathbf{q}, Y) \geq 0.$$

Since $\tilde{F}$ is nonincreasing in its third argument this implies in particular that

$$\tilde{F}(\tilde{u}(\mathbf{x}_*), \mathbf{q}, X) \leq \tilde{F}(\tilde{v}(\mathbf{y}_*), \mathbf{q}, X).$$

Now it remains to show that this is a contradiction, using the fact that

$$\tilde{u}(\mathbf{x}_*) > \tilde{u}(\mathbf{y}_*).$$

We have in other words to show that the function $r \mapsto \tilde{F}(r, \mathbf{q}, X)$ is strictly increasing on the interval (a, b). Taking into account definition (4.6), the facts that $\frac{1}{2} \leq \varphi'(t) \leq 1$ and $|\varphi''(t)| \leq \frac{1}{8}$ when $t \in (a, b)$, $|\mathbf{q}| \leq Q$, $\|X\| \leq 3\|N\|$ and $\|N\| \leq \frac{6(b-a)}{\delta^2} + 2\frac{Q}{\mu}$ we have

$$\begin{aligned}
&\tilde{F}(r, \mathbf{q}, X) - \tilde{F}(s, \mathbf{q}, X)\\
&\quad = F\big(\varphi'(r)\mathbf{q}, \varphi'(r)X + \varphi''(r)\mathbf{q} \otimes \mathbf{q}\big) - F\big(\varphi'(s)\mathbf{q}, \varphi'(s)X + \varphi''(r)\mathbf{q} \otimes \mathbf{q}\big)\\
&\quad + F\big(\varphi'(s)\mathbf{q}, \varphi'(s)X + \varphi''(r)\mathbf{q} \otimes \mathbf{q}\big) - F\big(\varphi'(s)\mathbf{q}, \varphi'(s)X + \varphi''(s)\mathbf{q} \otimes \mathbf{q}\big)\\
&\qquad \geq -L_{\mathcal{K}}|\varphi'(r) - \varphi'(s)|\big(|\mathbf{q}| + \|X\|\big) + (\varphi''(s) - \varphi''(r))\gamma_{\mathcal{K}}\\
&\qquad\quad = n\mathrm{e}^{-2n^2}(\mathrm{e}^{-ns} - \mathrm{e}^{-nr})\Big(-L_{\mathcal{K}}(|\mathbf{q}| + \|X\|) + \gamma_{\mathcal{K}} n\Big), \quad r > s.
\end{aligned}$$

Thus it follows from our choice of n that the function $r \mapsto \tilde{F}(r, \mathbf{q}, X)$ is strictly increasing and the proof is thus complete. □

4. Some auxiliary results

Lemma 4.9. *Assume that* $\mathcal{A} \subset \mathbb{R}^d$, $w : \mathcal{A} \to \mathbb{R}$ *is upper and* $\Psi : \mathcal{A} \to [0, \infty)$ *lower semicontinuous. Suppose furthermore that*

$$\mathcal{N} = \{\, \mathbf{z} \in \mathcal{A} : \Psi(\mathbf{z}) = 0 \,\} \neq \emptyset,$$

and that

$$\sup_{\mathbf{z} \in \mathcal{A}} \bigl(w(\mathbf{z}) - \Psi(\mathbf{z})\bigr) < \infty.$$

Let $M_\alpha = \sup_{\mathbf{z} \in \mathcal{A}}(w(\mathbf{z}) - \alpha\Psi(\mathbf{z}))$ *for* $\alpha \geq 1$ *and assume that* $\mathbf{z}_\alpha \in \mathcal{A}$ *is such that*

$$\lim_{\alpha \to \infty} (M_\alpha - (w(\mathbf{z}_\alpha) - \alpha\Psi(\mathbf{z}_\alpha))) = 0.$$

Then

$$\lim_{\alpha \to \infty} \alpha\Psi(\mathbf{z}_\alpha) = 0. \tag{4.7}$$

Moreover, if $\mathbf{z}_*$ *is a cluster point of* $\mathbf{z}_\alpha$ *as* $\alpha \to \infty$, *then* $\mathbf{z}_* \in \mathcal{N}$ *and* $w(\mathbf{z}) \leq w(\mathbf{z}_*)$ *for all* $\mathbf{z} \in \mathcal{N}$.

We also need the following variant.

Lemma 4.10. *Assume that* $\mathcal{A} \subset \mathbb{R}^d$, $w : \mathcal{A} \to \mathbb{R}$ *is upper and* $\Psi : [1, \infty)^m \times \mathcal{A} \to [0, \infty)$ *nondecreasing in its first* m *arguments and lower semicontinuous with respect to the last one. Suppose furthermore that*

$$\mathcal{N} = \cap_{\boldsymbol{\alpha} \geq \mathbf{1}} \{\, \mathbf{z} \in \mathcal{A} : \Psi(\boldsymbol{\alpha}, \mathbf{z}) = 0 \,\} \neq \emptyset,$$
$$\lim_{\boldsymbol{\alpha} \to \boldsymbol{\infty}} \Psi(\boldsymbol{\alpha}, \mathbf{z}) = \infty, \quad \mathbf{z} \in \mathcal{A} \setminus \mathcal{N},$$

and that

$$\sup_{\mathbf{z} \in \mathcal{A}} \bigl(w(\mathbf{z}) - \Psi(\mathbf{1}, \mathbf{z})\bigr) < \infty.$$

Let $M_{\boldsymbol{\alpha}} = \sup_{\mathbf{z} \in \mathcal{A}}(w(\mathbf{z}) - \Psi(\boldsymbol{\alpha}, \mathbf{z}))$ *for* $\boldsymbol{\alpha} \geq 1$ *and assume that* $\mathbf{z}_{\boldsymbol{\alpha}} \in \mathcal{A}$ *is such that*

$$\lim_{\boldsymbol{\alpha} \to \boldsymbol{\infty}} (M_{\boldsymbol{\alpha}} - (w(\mathbf{z}_{\boldsymbol{\alpha}}) - \Psi(\boldsymbol{\alpha}, \mathbf{z}_{\boldsymbol{\alpha}}))) = 0.$$

If $\mathbf{z}_*$ *is a cluster point of* $\mathbf{z}_{\boldsymbol{\alpha}}$ *as* $\boldsymbol{\alpha} \to \boldsymbol{\infty}$, *then* $\mathbf{z}_* \in \mathcal{N}$ *and* $w(\mathbf{z}) \leq w(\mathbf{z}_*)$ *for all* $\mathbf{z} \in \mathcal{N}$ *and if* $\lim_{j \to \infty} \mathbf{z}_{\boldsymbol{\alpha}_j} = \mathbf{z}_*$ *then* $\lim_{j \to \infty} \Psi(\boldsymbol{\alpha}_j, \mathbf{z}_{\boldsymbol{\alpha}_j}) = 0$. *Moreover, if* $\mathcal{A}$ *is compact, then*

$$\lim_{\boldsymbol{\alpha} \to \boldsymbol{\infty}} \Psi(\boldsymbol{\alpha}, \mathbf{z}_{\boldsymbol{\alpha}}) = 0.$$

Proof of Lemma 4.9. Since Ψ is nonnegative, M_α is a nonincreasing function on $[1, \infty)$. On the other hand $\sup_{\mathbf{z} \in \mathcal{N}} w(\mathbf{z}) \leq M_\alpha$ and $\mathcal{N}$ is nonempty, so M_α is bounded from below and $M_\infty \overset{\text{def}}{=} \lim_{\alpha \to \infty} M_\alpha$ exists (and is finite). Let $g(\alpha) = M_\alpha - (w(\mathbf{z}_\alpha) - \alpha\Psi(\mathbf{z}_\alpha))$. If now $\mu, \alpha \geq 1$, then

$$\begin{aligned} M_\mu - (\alpha - \mu)\,\Psi(\mathbf{z}_\alpha) &\geq (w(\mathbf{z}_\alpha) - \mu\Psi(\mathbf{z}_\alpha)) - (\alpha - \mu)\,\Psi(\mathbf{z}_\alpha) \\ &= w(\mathbf{z}_\alpha) - \alpha\Psi(\mathbf{z}_\alpha) = M_\alpha - g(\alpha). \end{aligned}$$

Now we take $\mu = \frac{\alpha}{2}$ so that we have

$$M_{\frac{\alpha}{2}} - \frac{\alpha}{2}\Psi(\mathbf{z}_\alpha) \geq M_\alpha - g(\alpha) \quad \Rightarrow \quad \alpha\Psi(\mathbf{z}_\alpha) \leq 2\big(M_{\frac{\alpha}{2}} - M_\alpha + g(\alpha)\big).$$

Since we know that $\lim_{\alpha\to\infty} M_\alpha$ exists and by assumption $\lim_{\alpha\to\infty} g(\alpha) = 0$ the right hand side of this inequality tends to zero and since Ψ is nonnegative we get the claim (4.7).

Finally, assume that $\mathbf{z}_{\alpha_j} \to \mathbf{z}_* \in \mathcal{A}$ where $\alpha_j \to \infty$. By (4.7) and the lower semicontiuity of Ψ we have $0 = \lim_{j\to\infty} \Psi(\mathbf{z}_{\alpha_j}) \geq \Psi(\mathbf{z}_*)$ and since Ψ is nonnegative we have $\mathbf{z}_* \in \mathcal{N}$. Finally, since w is upper semicontinuous and $\sup_{\mathbf{z}\in\mathcal{N}} w(\mathbf{z}) \leq M_\alpha$ we get

$$\begin{aligned} w(\mathbf{z}_*) &\geq \lim_{j\to\infty} \big(w(\mathbf{z}_{\alpha_j}) - \alpha_j\Psi(\mathbf{z}_{\alpha_j})\big) \\ &= \lim_{j\to\infty} \big(M_{\alpha_j} - g(\alpha_j)\big) = M_\infty \geq \sup_{\mathbf{z}\in\mathcal{N}} w(\mathbf{z}). \end{aligned}$$

☐

Proof of Lemma 4.10. If $\boldsymbol{\alpha} \geq \boldsymbol{\beta} \geq \mathbf{1}$ then we have $\Psi(\boldsymbol{\alpha}, \mathbf{z}) \geq \Psi(\boldsymbol{\beta}, \mathbf{z})$ for all $\mathbf{z} \in \mathcal{A}$ and hence $M_{\boldsymbol{\alpha}} \leq M_{\boldsymbol{\beta}}$ as well. On the other hand, $\mathcal{N}$ is nonempty and $\sup_{\mathbf{z}\in\mathcal{N}} w(\mathbf{z}) \leq M_{\boldsymbol{\alpha}}$, so $M_{\boldsymbol{\alpha}}$ is bounded from below and $M_{\boldsymbol{\infty}} \overset{\text{def}}{=} \lim_{\boldsymbol{\alpha}\to\boldsymbol{\infty}} M_{\boldsymbol{\alpha}}$ exists (and is finite).

Assume next that $\mathbf{z}_{\boldsymbol{\alpha}_j} \to \mathbf{z}_\diamond \in \mathcal{A}$ where $\boldsymbol{\alpha}_j \to \boldsymbol{\infty}$. If $\mathbf{z}_\diamond \notin \mathcal{N}$ then there exists a vector $\boldsymbol{\alpha}_\diamond$ such that $\Psi(\boldsymbol{\alpha}_\diamond, \mathbf{z}_\diamond) \geq w(\mathbf{z}_\diamond) - M_{\boldsymbol{\infty}} + 2$. Since $\mathbf{z} \to \Psi(\boldsymbol{\alpha}_\diamond, \mathbf{z})$ is lower semicontinuous and w is upper semicontinuous we see that for sufficiently large j we have $\Psi(\boldsymbol{\alpha}_\diamond, \mathbf{z}_{\boldsymbol{\alpha}_j}) > w(\mathbf{z}_{\boldsymbol{\alpha}_j}) - M_{\boldsymbol{\infty}} + 1$. Since $\boldsymbol{\alpha}_j \geq \boldsymbol{\alpha}_\diamond$ for sufficiently large values of j and Ψ is nondecreasing in the first variables we see that $\Psi(\boldsymbol{\alpha}_j, \mathbf{z}_{\boldsymbol{\alpha}_j}) > w(\mathbf{z}_{\boldsymbol{\alpha}_j}) - M_{\boldsymbol{\infty}} + 1$ for all sufficiently large j. But this contradicts the assumption that $0 = \lim_{\boldsymbol{\alpha}\to\boldsymbol{\infty}}(M_{\boldsymbol{\alpha}} - w(\mathbf{z}_{\boldsymbol{\alpha}}) + \Psi(\boldsymbol{\alpha}, z_{\boldsymbol{\alpha}})) = M_{\boldsymbol{\infty}} - \lim_{\boldsymbol{\alpha}\to\boldsymbol{\infty}}(w(\mathbf{z}_{\boldsymbol{\alpha}}) - \Psi(\boldsymbol{\alpha}, z_{\boldsymbol{\alpha}}))$.

Since Ψ is nonnegative and w is upper semicontinuous we have

$$\begin{aligned} &w(\mathbf{z}_\diamond) \geq \limsup_{j\to\infty} \big(w(\mathbf{z}_{\boldsymbol{\alpha}_j}) - \Psi(\boldsymbol{\alpha}_j, \mathbf{z}_{\boldsymbol{\alpha}_j})\big) \\ &= \limsup_{j\to\infty} \big(w(\mathbf{z}_{\boldsymbol{\alpha}_j}) - \Psi(\boldsymbol{\alpha}_j, \mathbf{z}_{\boldsymbol{\alpha}_j}) - M_{\boldsymbol{\alpha}_j}\big) + \lim_{j\to\infty} M_{\boldsymbol{\alpha}_j} = M_{\boldsymbol{\infty}} \geq \sup_{\mathbf{z}\in\mathcal{N}} w(\mathbf{z}). \end{aligned}$$

Since $\mathbf{z}_\diamond \in \mathcal{N}$ and $w(\mathbf{z}_\diamond) \geq \limsup_{j\to\infty} w(\mathbf{z}_{\boldsymbol{\alpha}_j})$ this inequality implies in addition that $\lim_{j\to\infty} \Psi(\boldsymbol{\alpha}_j, \mathbf{z}_{\boldsymbol{\alpha}_j}) = 0$.

Finally, if we assume that $\mathcal{A}$ is compact then every subsequence $(\boldsymbol{\alpha}_j)_{j=1}^\infty$ has a subsequence for which $\lim_{k\to\infty} \mathbf{z}_{\boldsymbol{\alpha}_{j_k}} \to \mathbf{z}_\diamond \in \mathcal{A}$ and the claim follows from the results already proven. ☐

5. Comments

The proofs of Theorems 4.7 and 4.8 are modifications of the proofs of these results in [**1**]