Fourier – muunnos

Määritelmä: $f : \mathbb{R} \to \mathbb{R}$ (tai $\mathbb{C}$) integroituva

$$\mathcal{F}f(k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx} f(x)\,dx$$

Amplitudi – vaihe – esitys

$$\hat{f}(k) = |\hat{f}(k)|\, e^{i\Theta(k)}$$

$|\hat{f}(k)|$ = Fourier amplitudi

$\Theta(k)$ = Fourier – vaihe

$|\hat{f}(k)|^2$ = f:n tehospektri (Power spectrum)

Ominaisuuksia: Lineaarisuus

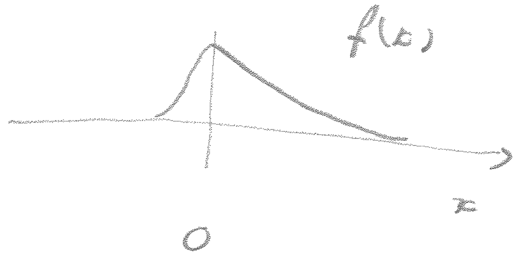
$$\mathcal{F}(f+g)(k) = \mathcal{F}(f)(k) + \mathcal{F}(g)(k)$$

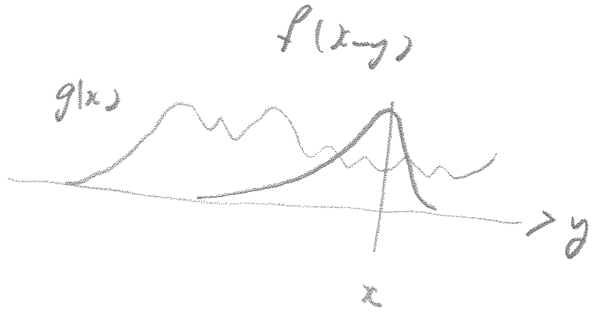
$$\mathcal{F}(\alpha f)(k) = \alpha \mathcal{F} f(k)$$

Konvoluutio: Määritellään konvoluutiotulo

$$f * g(x) = \int f(x-y)\, g(y)\, dy$$

Tulkinta:



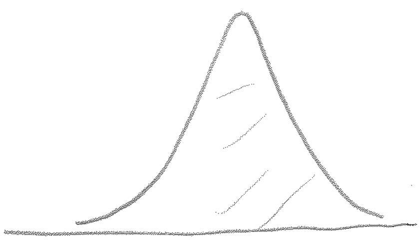


Laitefunktio, impulssiresponssi

Diracin delta l. impulssifunktio

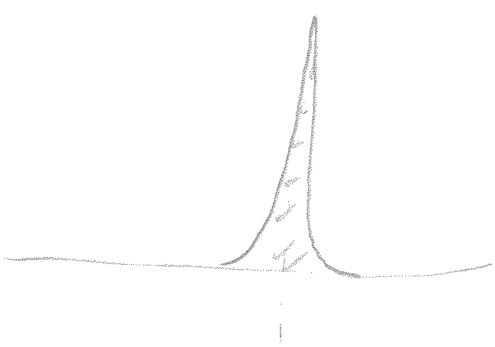
$$\int f(x)\,\delta(x-y)\,dx = f(y)$$

δ rajaprosessina; Olkoon φ jokin kapea funktio



$$\int \varphi(x)\,dx = 1$$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) \quad ; \quad \int \varphi_\varepsilon(x)\,dx = 1$$



$$\int \varphi_\varepsilon(x-y)\,f(y)\,dy \longrightarrow f(x)$$

Voidaan ajatella, että $\delta(x-y) = $ "$\lim_{\varepsilon \to 0+} \varphi_\varepsilon(x-y)$"

$$f * \varphi_\varepsilon(x) \xrightarrow{\varepsilon \to 0} f * \delta(x) = f(x)$$

f = laitteen vaste impulssisyöttöön.

Konvoluution Fourier-muunnos:

$$\underline{\mathcal{F}(f*g)(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left(\int f(x-y)\, g(y)\, dy \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} e^{-ikx} f(\underbrace{x-y}_{=t;\ x = t+y})\, dx \; dy$$

$$= \int_{-\infty}^{\infty} g(y)\, e^{-iky} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} f(t)\, dt}_{=\hat{f}(k)} \; dy$$

$$= \hat{f}(k) \int_{-\infty}^{\infty} e^{-iky} g(y)\, dy = \underline{\sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k)}$$

Tulkinta: Suodatus

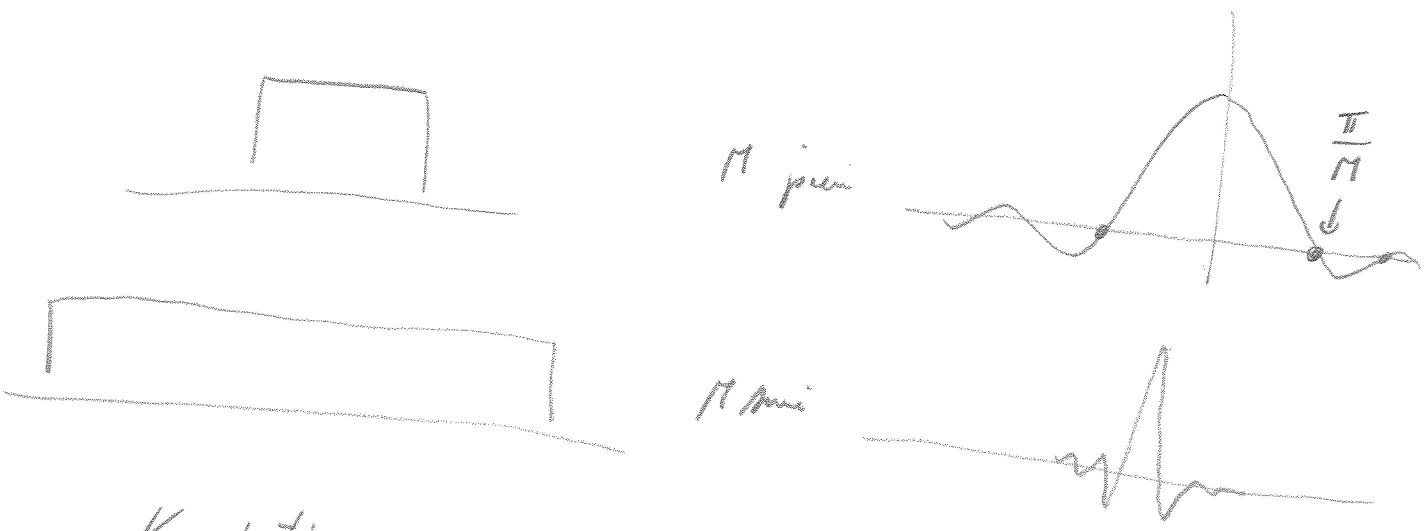
Esimerkki: Laatikkofunktio:

$$g(x) = \begin{cases} 1, & -M \le x \le M \\ 0, & |x| > M \end{cases}$$

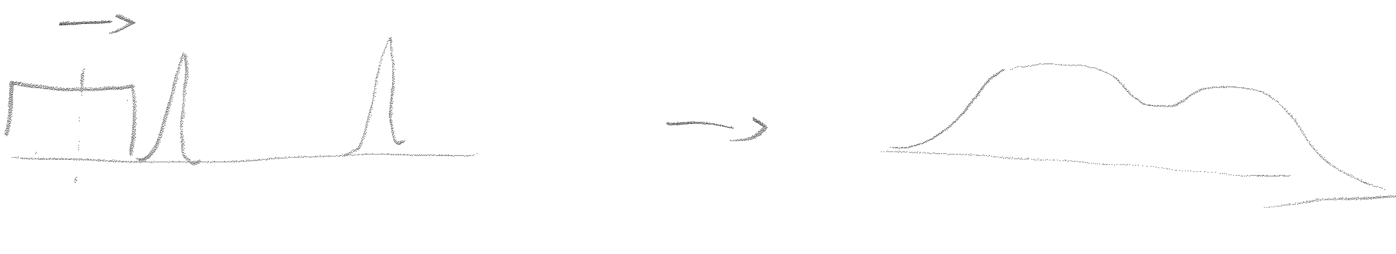
$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-M}^{M} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \Big|_{-M}^{M} \frac{e^{-ikx}}{-ik}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{ik} \left(e^{ikM} - e^{-ikM}\right) = \frac{2}{\sqrt{2\pi}\, k} \sin(kM)$$

$$= \sqrt{\frac{2}{\pi}}\, M \underbrace{\frac{\sin(kM)}{kM}}_{\equiv\, \mathrm{sinc}(kM)} = \sqrt{\frac{2}{\pi}}\, M\, \mathrm{sinc}(kM)$$



Konvoluutio laatikkofunktiolla: Jos M on suuri, pienet piirteet signaalista katoavat



Fourier-muunnos ja derivaatta

Olkoon f ja f' integroituvia $(f, f' \to 0$, kun $|x| \to$

$$\mathcal{F}(f')(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f'(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \Big|_{-\infty}^{\infty} e^{-ikx} f(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{d}{dx}\left(e^{-ikx}\right) dx$$

$$= ik \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = ik \hat{f}(k)$$

Siis

$$\boxed{\mathcal{F}(f')(k) = ik \hat{f}(k)}$$

Induktiivisesti

$$\mathcal{F}(f'')(k) = ik\, \mathcal{F}(f') = (ik)^2 \hat{f}(k) = -k^2 \hat{f}(k)$$

ja yleisemmin

$$\mathcal{F}(f^{(n)})(k) = (ik)^n \hat{f}(k)$$

Diff. yhtälöiden ratkaisu Fourier-muunnoksen avulla:

$$a f''(x) + b f'(x) + c f(x) = g(x)$$

$$\Rightarrow \quad (-ak^2 + ibk + c) \hat{f}(k) = \hat{g}(k)$$

Tästä ratkaistaan $\hat{f}$. Miten saadaan f?

Fourier-käänteiskaava: Tavoitteena on löytää kaava, jolla lasketaan $f(x)$, kun $\hat{f}(k)$ on annettu.

Aputuloksena tarkastellaan Gaussin funktiota

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Pätee:

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1,$$

mikä nähdään napakoordinaattitempulla:

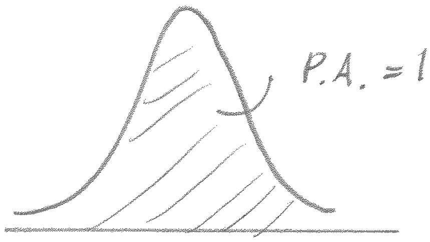
$$\left(\int_{-\infty}^{\infty} \varphi(x)\,dx\right)^2 = \int_{-\infty}^{\infty} \varphi(x)\,dx \int_{-\infty}^{\infty} \varphi(y)\,dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\overbrace{(x^2+y^2)}^{=r^2}/2}\,dx\,dy$$

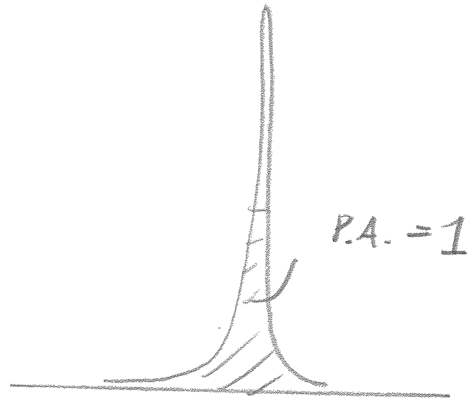
$$= \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{-r^2/2}\,d\varphi\; r\,dr$$

$$= \int_0^{\infty} e^{-r^2/2}\, r\,dr = -\Big|_0^{\infty} e^{-r^2/2} = 1.$$

Funktiota φ voidaan käyttää Diracin deltan eli impulssifunktion konstruointiin:



$\varphi(x)$



$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(x)\,dx = 1.$$

$$\lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} \varphi_\varepsilon(x) f(x)\,dx = f(0)$$

ja yleisemmin,

$$\lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} \varphi_\varepsilon(x-y) f(y)\,dy = f(x) = \int_{-\infty}^{\infty} \delta(x-y) f(y)\,dy$$

δ = Diracin delta.

Mikä on φ:n Fourier-muunnos?

$$\hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \varphi(x)\,dx$$

$$\frac{d}{dk}\hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ix) e^{-ikx} \varphi(x)\,dx$$

Toisaalta

$$\frac{d}{dx}\varphi(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx}\left(e^{-x^2/2}\right) = -x\,\varphi(x)$$

joten

$$\frac{d}{dk}\hat{\varphi}(k) = i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\,\varphi'(x)\,dx$$

$$= \frac{i}{\sqrt{2\pi}}\Big|_{-\infty}^{\infty} e^{-ikx}\varphi(x) - \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d}{dx}\left(e^{-ikx}\right)\varphi(x)\,dx$$

$$= -k\,\hat{\varphi}(k)$$

Ratkaistaan saatu diff. yhtälö:

$$\int\frac{d\hat{\varphi}}{\hat{\varphi}} = -\int k\,dk$$

eli

$$\log\hat{\varphi}(k) - \log\hat{\varphi}(0) = -\frac{1}{2}k^2$$

mistä

$$\hat{\varphi}(k) = \hat{\varphi}(0)\,e^{-\frac{1}{2}k^2}$$

Toisaalta

$$\hat{\varphi}(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)\,dx = \frac{1}{\sqrt{2\pi}}$$

Siis:

$$\boxed{\hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}k^2}}$$

Gaussisen funktion Fourier-muunnos on se itse!

Funktion φ_ε Fourier-muunnos:

$$\hat{\varphi}_\varepsilon(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) dx \qquad ; \; y = \frac{x}{\varepsilon}, \quad dy = \frac{dx}{\varepsilon}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\varepsilon y} \varphi(y)\, dy$$

$$= \hat{\varphi}(\varepsilon k)$$

$\hat{\varphi}(k)$ $\qquad\qquad$ $\hat{\varphi}(\varepsilon k)$

Tarvitsemme nyt tutkia integraalia

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, dk = ?$$

Approksimoidaan $\hat{f}(k)$:ta $\sqrt{2\pi}\,\hat{\varphi}(\varepsilon k)\hat{f}(k)$:lla, ja merkitään

$$I_\varepsilon(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{\varphi}(\varepsilon k) \hat{f}(k)\, dk$$

Koska $\sqrt{2\pi}\,\hat{\varphi}(\varepsilon k) = e^{-\varepsilon^2 k^2/2} \xrightarrow{\varepsilon \to 0+} 1$, niin

$$I_\varepsilon(x) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, dk.$$

$$I_\varepsilon(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \, e^{-\varepsilon^2 k^2/2} \, \frac{1}{\sqrt{2\pi}} \int e^{-iky} f(y)\,dy\,dk$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} \, e^{-\varepsilon^2 k^2/2} \, dk \right]}_{= k_\varepsilon(x-y)} f(y)\,dy$$

$$k_\varepsilon(x-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} \, e^{-\varepsilon^2 k^2/2} \, dk \qquad \varkappa = \varepsilon k; \quad d\varkappa = \varepsilon\, dk$$

$$= \frac{1}{2\pi} \, \frac{1}{\varepsilon} \int_{-\infty}^{\infty} e^{i\varkappa((x-y)/\varepsilon)} \, e^{-\varkappa^2/2} \, d\varkappa$$

$$= \frac{1}{\varepsilon} \hat{\varphi}\left(\frac{y-x}{\varepsilon}\right) = \frac{1}{\varepsilon} \varphi\left(\frac{x-y}{\varepsilon}\right) = \rho_\varepsilon(x-y)$$

Siis

$$I_\varepsilon(x) \longrightarrow f(x)$$

Fourier käänteiskaava:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\,dk = \mathcal{F}^{-1}(\hat{f})(x)}$$

Palataan diff. yhtälö esimerkkiin

$$a f''(x) + b f'(x) + c f(x) = g(x)$$

$$\Longrightarrow \qquad \hat{f}(k) = \frac{\hat{g}(k)}{-ak^2 + ibk + c}$$

ja siten

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{\hat{g}(k)}{-ak^2 + ikb + c} \, dk}$$

Ratkaisu käytännössä joskus residylauseella

<u>Esim:</u> $\qquad f''(x) - f(x) = \delta(x)$

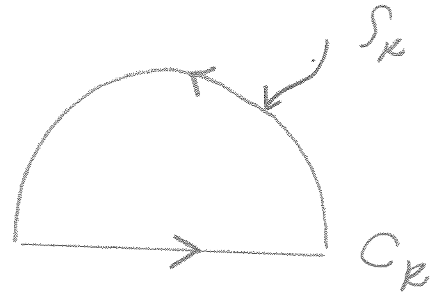
Huom: $\qquad \hat{\delta}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x) \, dx = \frac{1}{\sqrt{2\pi}}$

$$(-k^2 - 1) \hat{f}(k) = \frac{1}{\sqrt{2\pi}}$$

$$f(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+1} \, dk = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k+i)(k-i)} \, dk$$

Integroidaan pitkin suljettua polkua joko ylemmässä puolitasossa $(x > 0)$ tai alemmassa $(x < 0)$.

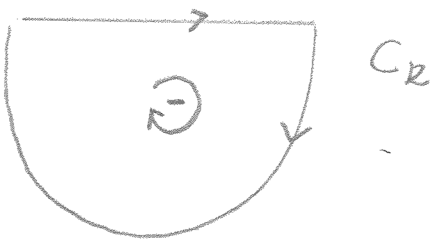
<u>$x > 0$</u> :



$$\oint_{C_R} \frac{e^{ikx}}{(k+i)(k-i)}\,dk = \underbrace{\int_{-R}^{R} \dots\, dk}_{\to \int_{-\infty}^{\infty} \dots\, dk} + \underbrace{\int_{S_R} \dots\, dk}_{\to 0}$$

$$= 2\pi i \operatorname*{Res}_{k=i} \frac{e^{ikx}}{(k+i)(k-i)} = 2\pi i \,\frac{e^{-x}}{2i} = \pi e^{-x}$$

<u>$x < 0$</u> :



$$\oint_{C_R} \frac{e^{ikx}}{(k+i)(k-i)}\,dk = -2\pi i \operatorname*{Res}_{k=-i} \frac{e^{ikx}}{(k+i)(k-i)}$$

$$= -2\pi i \,\frac{e^{x}}{-2i} = \pi e^{x} = \pi e^{-|x|}$$

Sei

$$f(x) = -\frac{1}{2} e^{-|x|}$$

E.o. esimerkin ratkaisuna saadaan ns. perusratkaisuksi eli Greenin funktioksi. Jos yleisemmin pitäisi ratkaista

$$\tilde f''(x) - \tilde f(x) = g(x),$$

saadaan

$$\tilde f(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \frac{1}{k^2+1} \hat g(k)\, dk$$

$$= \mathcal{F}^{-1}\left(\left(-\frac{1}{\sqrt{2\pi}} \frac{1}{k^2+1} \right) \hat g(k) \right)(x)$$

Muistetaan konvoluutiolause:

$$\mathcal{F}(u * v)(k) = \sqrt{2\pi}\, \hat u(k)\, \hat v(k),$$

eli kääntäen,

$$\mathcal{F}^{-1}(\hat u\, \hat v)(x) = \frac{1}{\sqrt{2\pi}}\, u * v(x)$$

Niinpä

$$\mathcal{F}^{-1}\left(\left(-\frac{1}{\sqrt{2\pi}} \frac{1}{k^2+1} \right) \hat g(k) \right)(x)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \left[\mathcal{F}^{-1}\left(-\frac{1}{\sqrt{2\pi}} \frac{1}{k^2+1} \right) \right]}_{= f(x)} * g(x)$$

$$= f * g(x) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} g(y)\, dy$$