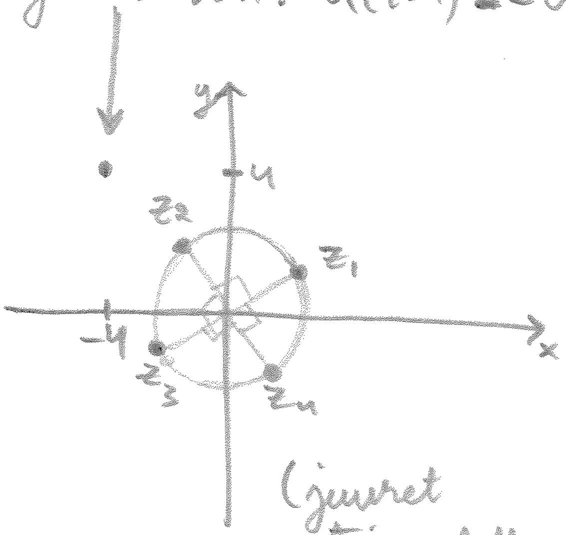
1. juurrettava: $4(i-1) = -4+4i$



(juuret etäisyydellä $32^{1/8} \approx 1{,}54$ origosta)

$$|-4+4i| = \sqrt{(-4)^2+4^2} = 4\sqrt{2}$$

$$\mathrm{Arg}(-4+4i) = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

Kaikki juuret $\sqrt[4]{4(i-1)}$:

$$z_1 = \sqrt[4]{4\sqrt{2}}\, e^{i\frac{3\pi}{16}} = 32^{1/8} e^{i\frac{3\pi}{16}}$$

$$z_2 = 32^{1/8} e^{i\left(\frac{3\pi}{16}+\frac{2\pi}{4}\right)} = 32^{1/8} e^{i\frac{11\pi}{16}}$$

$$z_3 = 32^{1/8} e^{i\left(\frac{11\pi}{16}+\frac{\pi}{2}\right)} = 32^{1/8} e^{i\frac{19\pi}{16}}$$

$$z_4 = 32^{1/8} e^{i\left(\frac{19\pi}{16}+\frac{\pi}{2}\right)} = 32^{1/8} e^{i\frac{27\pi}{16}}$$

2.

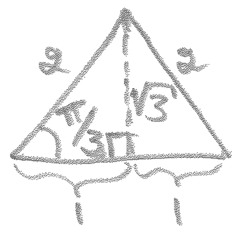
a) $\ln z = \ln|z| + i(\mathrm{Arg}(z) + 2\pi k)$, $k \in \mathbb{Z}$

$z = -1 + i\sqrt{3}$

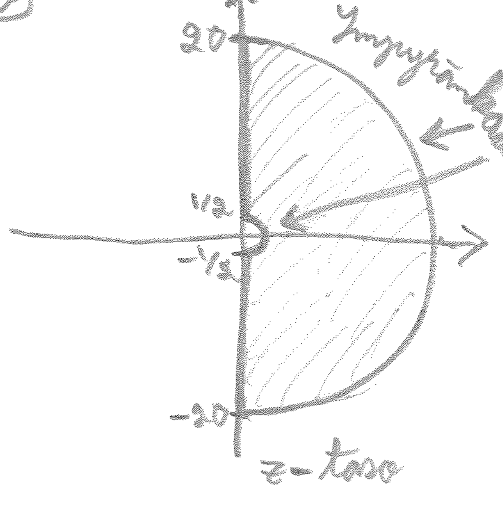
$|z| = \sqrt{(-1)^2 + \sqrt{3}^2} = 2$

$\mathrm{Arg}(z) = \arctan\left(\frac{\sqrt{3}}{-1}\right) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

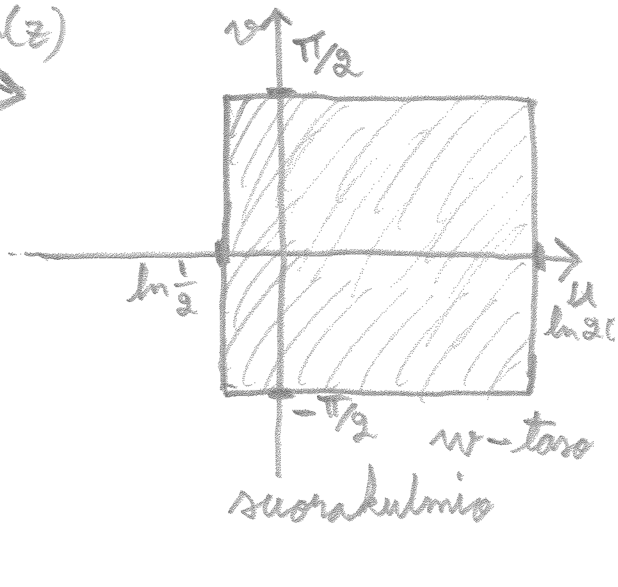
$$\Rightarrow \underline{\underline{\ln(-1+i\sqrt{3}) = \ln 2 + i\left(\frac{2\pi}{3} + 2\pi k\right),\ k \in \mathbb{Z}}}$$



b)



$\xrightarrow{w = \ln(z)}$



3.

a) $\sin(z) = \sin(x+iy) = \underbrace{\sin(x)\cosh(y)}_{=u(x,y)} + i\underbrace{\cos(x)\sinh(y)}_{=v(x,y)}$

Cauchy-Riemann: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

Tarkistetaan:

$\left.\begin{array}{l} u_x = \cos(x)\cosh(y) \\ v_y = \cos(x)\cosh(y) \end{array}\right\} \Rightarrow u_x = v_y \quad (1)$

$\left.\begin{array}{l} u_y = \sin(x)\sinh(y) \\ v_x = -\sin(x)\sinh(y) \end{array}\right\} \Rightarrow u_y = -v_x \quad (2)$

(1) & (2) $\Rightarrow$ CR pätee (kaikkialla). Lisäksi osittaisderivaatat u_x, u_y, v_x, v_y ovat jatkuvat (kaikkialla), joten $\sin(z)$ on analyyttinen ~~koko~~ tasossa.

$f'(z) = u_x + iv_x = \cos x \cosh y - i \sin x \sinh y = \cos z$

b) Ei ole konforminen pisteissä, joissa derivaatta on nolla.

$\frac{d}{dz}\sin(z) = \cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y) = 0$

$\Leftrightarrow \begin{cases} \cos(x)\cosh(y) = 0 \Leftrightarrow \cos(x) = 0 \quad (\cosh(y) \geq 1) \\ \sin(x)\sinh(y) = 0 \end{cases}$

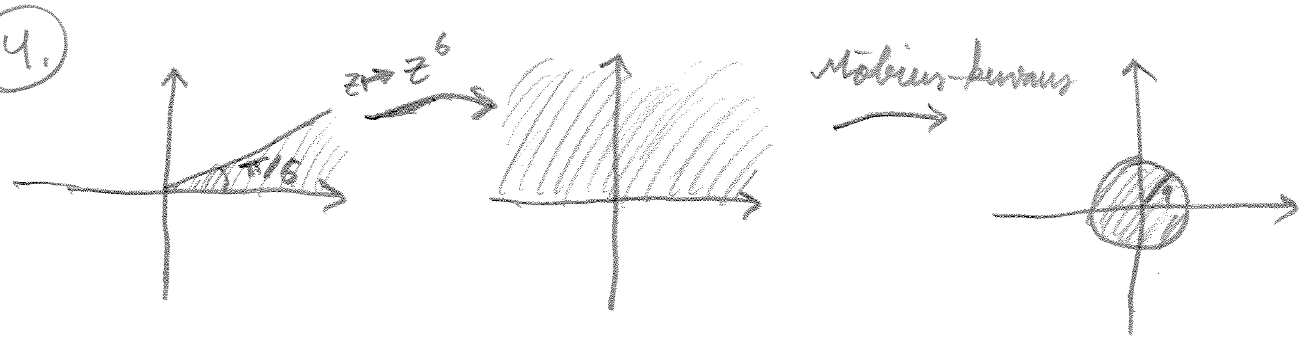
$\cos(x) = 0 \Leftrightarrow x = \frac{\pi}{2} + \pi k, \; k \in \mathbb{Z}$

sijoitetaan $\sin(x)\sinh(y) = 0$:aan $\Rightarrow (-1)^k \sinh(y) = 0 \Leftrightarrow \sinh(y) = 0$

$\Leftrightarrow y = 0$

Siis: $\sin(z)$ ei ole konforminen pisteissä $z = \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z}$

4.



$z_1 = 0 \longrightarrow -i = w_1$

$z_2 = 1 \longrightarrow 1 = w_2$

$z_3 = \infty \longrightarrow i = w_3$

$$\frac{w+i}{w-i}\cdot\underbrace{\frac{1-i}{1+i}}_{=-i} = \frac{z-0}{\cancel{(z-\infty)}}\cdot\frac{\cancel{(1-\infty)}}{1-0} \iff \frac{w+i}{w-i} = iz$$

$$\iff w+i = izw + z \iff w(1-iz) = z-i$$

$$\iff w = \frac{z-i}{-iz+1}$$

$z = i \longrightarrow w = \frac{i-i}{-i\cdot i+1} = 0$, joten ylempi puolitaso kuvautuu ympyrän <u>sisälle</u>. <u>Vast:</u> $w = \underline{\underline{\frac{z^6-i}{-iz^6+1}}}$