Mat-1.413, Matematiikan peruskurssi C3

Harjoitus 4

Harjoitukset viikolla 40, 2005. **kt**=kotitehtävä, **ht**=harjoitustehtävä.

Kotitehtävät palautetaan oman ryhmän assistentille loppuviikon laskuharjoituksissa.

Käytä paperikokoa A4 ja nido arkit nippuun, jos palautat useamman kuin yhden.

Muista laittaa kotitehtäväpaperiin **nimi, opiskelijanumero ja harjoitusryhmä!**

1) **(kt)** Olkoon

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

a) Laske e^A diagonalisoimalla A ensin.

b) Laske e^{At}, kun $t \in \mathbb{R}$.

2) **(kt)** Ortonormeeraa Gram-Schmidtin menetelmällä vektorit

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \text{ja} \quad \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

3) **(kt)** Laske e^A kun A on

$$\text{a)} \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix} \qquad \text{b)} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

4) **(ht)** a) Miksi A ei voi koskaan olla similaarinen matriisin $A + I$ kanssa?

b) Etsi diagonaalinen M, jonka avulla voit osoittaa, että

$$\text{a)} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad \text{b)} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

ovat similaariset.

C3 Harjoitus 4

1. $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$$\det(A - I\lambda) = (-1-\lambda)^2 - 1\cdot 1 = 0$$

$$1+\lambda = \pm 1$$

$$\lambda = -1 \pm 1$$

$\lambda_1 = -2 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$\lambda_2 = 0 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$A = XDX^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

a) $e^A = Xe^DX^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+e^{-2} & 1-e^{-2} \\ 1-e^{-2} & 1+e^{-2} \end{pmatrix}$

b) $e^{At} = Xe^{Dt}X^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+e^{-2t} & 1-e^{-2t} \\ 1-e^{-2t} & 1+e^{-2t} \end{pmatrix}$

~~2.~~ 3.

a) $A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$ merk $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$BC = CB$$

$$\Rightarrow e^A = e^{(B+C)} = e^B e^C$$

$C^0 = I$, $\quad C^{2k+1} \underset{k\geq 0}{=} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $\quad C^{2k} \underset{k\geq 1}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$e^C = \sum_{k=0}^{\infty} \frac{C^k}{k!} = \sum_{k=0}^{\infty}\left(\frac{C^{2k}}{(2k)!} + \frac{C^{2k+1}}{(2k+1)!}\right) = \begin{pmatrix} \sum \frac{1}{(2k)!} & 0 & 0 & \sum\frac{1}{(2k+1)!} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sum\frac{1}{(2k+1)!} & 0 & 0 & \sum\frac{1}{(2k)!} \end{pmatrix}$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{k!} + \frac{(-1)^k}{k!}\right) = \frac{1}{2}\left(e^1 + e^{-1}\right) = \cosh$$

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)!} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{k!} - \frac{(-1)^k}{k!}\right) = \frac{1}{2}\left(e^1 - e^{-1}\right) = \sinh 1$$

$$\Rightarrow \quad e^A = e^B e^C$$

$$= \begin{pmatrix} e^2 & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} \cosh 1 & 0 & 0 & \sinh 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh 1 & 0 & 0 & \cosh 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^2\cosh 1 & 0 & 0 & e^2 \sinh 1 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ e^2 \sinh 1 & 0 & 0 & e^2\cosh 1 \end{pmatrix}$$

b)

$$A = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$BC = CB \qquad C^{2k} = I, \quad C^{2k+1} = C, \quad k \geq 0$$

$$e^C = \sum_{k=0}^{\infty} \frac{C^k}{k!} = \sum_{k=0}^{\infty}\left(\frac{C^{2k}}{(2k)!} + \frac{C^{2k+1}}{(2k+1)!}\right)$$

$$c_1 := \sum \frac{1}{(2k)!} = \cosh 1$$

$$s_1 := \sum \frac{1}{(2k+1)!} = \sinh 1$$

$$= \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ 0 & c_1 & 0 & s_1 \\ s_1 & 0 & c_1 & 0 \\ 0 & s_1 & 0 & c_1 \end{pmatrix}$$

$$e^A = e^{B+C} = e^B e^C = \begin{pmatrix} e^2 & & & \\ & e^2 & & \\ & & e^2 & \\ & & & e^2 \end{pmatrix} \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ 0 & c_1 & 0 & s_1 \\ s_1 & 0 & c_1 & 0 \\ 0 & s_1 & 0 & c_1 \end{pmatrix}$$

$$= \begin{pmatrix} e^2\cosh 1 & 0 & e^2\sinh 1 & 0 \\ 0 & e^2\cosh 1 & 0 & e^2 \sinh 1 \\ e^2 \sinh 1 & 0 & e^2\cosh 1 & 0 \\ 0 & e^2\sinh 1 & 0 & e^2\cosh 1 \end{pmatrix}$$

(2.)

$$x_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad x_2 = \begin{pmatrix}1\\2\\4\end{pmatrix}, \quad x_3 = \begin{pmatrix}1\\3\\5\end{pmatrix}$$

$$e_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\tilde{e}_2 = x_2 - (x_2^T e_1)e_1 = \begin{pmatrix}1\\2\\4\end{pmatrix} - \frac{17}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} = \frac{1}{14}\begin{pmatrix}-3\\-6\\5\end{pmatrix}$$

$$e_2 = \frac{\tilde{e}_2}{\|e_2\|} = \left(\frac{1}{14}\sqrt{(-3)^2+(-6)^2+5^2}\right)^{-1} \cdot \frac{1}{14}\begin{pmatrix}-3\\-6\\5\end{pmatrix}$$

$$= \frac{14}{\sqrt{70}} \cdot \frac{1}{14}\begin{pmatrix}-3\\-6\\5\end{pmatrix} = \frac{1}{\sqrt{70}}\begin{pmatrix}-3\\-6\\5\end{pmatrix}$$

$$\tilde{e}_3 = x_3 - (x_3^T e_1)e_1 - (x_3^T e_2)e_2$$

$$= \begin{pmatrix}1\\3\\5\end{pmatrix} - \frac{22}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{4}{\sqrt{70}} \cdot \frac{1}{\sqrt{70}}\begin{pmatrix}-3\\-6\\5\end{pmatrix}$$

$$= \begin{pmatrix}1\\3\\5\end{pmatrix} - \frac{22}{14}\begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{4}{70}\begin{pmatrix}-3\\-6\\5\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

$$e_3 = \frac{\tilde{e}_3}{\|e_3\|} = \left(\frac{1}{5}\sqrt{2^2+1^2+0^2}\right)^{-1} \cdot \frac{1}{5}\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

$$= \frac{5}{\sqrt{5}} \cdot \frac{1}{5}\begin{pmatrix}-2\\1\\0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix}$$

$$\Bigg\{ \overset{e_1}{\frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}}, \overset{e_2}{\frac{1}{\sqrt{70}}\begin{pmatrix}-3\\-6\\5\end{pmatrix}}, \overset{e_3}{\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix}} \Bigg\}$$