A1

$$\Big\{ \overbrace{\begin{pmatrix}0\\1\\2\end{pmatrix}}^{x_1}, \overbrace{\begin{pmatrix}1\\3\\1\end{pmatrix}}^{x_2}, \overbrace{\begin{pmatrix}2\\1\\0\end{pmatrix}}^{x_3} \Big\}$$ on $\mathbb{R}^3$:n kanta,

jos vektorit ovat lineaarisesti riippumattomat. Tällöin seuraavalla yhtälöryhmällä on vain triviaaliratkaisu.

$$\overbrace{\begin{pmatrix}0 & 1 & 2\\ 1 & 3 & 1\\ 2 & 1 & 0\end{pmatrix}}^{=A}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=0$$

$$\iff \det(A)\neq 0$$

$$\det(A) = -\begin{vmatrix}1 & 1\\ 2 & 0\end{vmatrix} + 2\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix} = 2-10=-8 \quad \square$$

Gram-Schmidt:

$$e_1 = \frac{x_1}{\|x_1\|} = \frac{x_1}{\sqrt{1^2+2^2}} = \begin{pmatrix}0\\ \frac{1}{\sqrt5}\\ \frac{2}{\sqrt5}\end{pmatrix}$$

$$\tilde{e}_2 = x_2 - (x_2^T e_1)e_1 = \begin{pmatrix}1\\3\\1\end{pmatrix} - \overbrace{\left(\frac{3}{\sqrt5}+\frac{2}{\sqrt5}\right)}^{=\sqrt5}\begin{pmatrix}0\\ \frac{1}{\sqrt5}\\ \frac{2}{\sqrt5}\end{pmatrix} =$$

$$= \begin{pmatrix}1\\3\\1\end{pmatrix} - \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$e_2 = \frac{\tilde{e}_2}{\|\tilde{e}_2\|} = \frac{\tilde{e}_2}{\sqrt{1^2+2^2+1^2}} = \begin{pmatrix}\frac{1}{\sqrt6}\\ \frac{2}{\sqrt6}\\ -\frac{1}{\sqrt6}\end{pmatrix}$$

$$\tilde{e}_3 = x_3 - (x_3^T e_1)e_1 - (x_3^T e_2)e_2 = \begin{pmatrix}2\\1\\0\end{pmatrix} - \frac{1}{\sqrt5}\begin{pmatrix}0\\ \frac{1}{\sqrt5}\\ \frac{2}{\sqrt5}\end{pmatrix} - \frac{4}{\sqrt6}\begin{pmatrix}\frac{1}{\sqrt6}\\ \frac{2}{\sqrt6}\\ -\frac{1}{\sqrt6}\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}0\\ \frac{1}{5}\\ \frac{2}{5}\end{pmatrix} - \begin{pmatrix}\frac{2}{3}\\ \frac{4}{3}\\ -\frac{2}{3}\end{pmatrix} = \begin{pmatrix}\frac{4}{3}\\ -\frac{8}{15}\\ +\frac{4}{15}\end{pmatrix}$$

$$e_3 = \frac{\tilde{e}_3}{\|\tilde{e}_3\|} = \frac{\tilde{e}_3}{\sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{8}{15}\right)^2+\left(\frac{4}{15}\right)^2}} = \frac{15}{\sqrt{480}}\begin{pmatrix}\frac{4}{3}\\ -\frac{8}{15}\\ -\frac{4}{15}\end{pmatrix} = \frac{1}{\sqrt{480}}\begin{pmatrix}20\\-8\\-4\end{pmatrix}$$

e_1, e_2 ja e_3 muodostavat ortonormaalin kannan

A2

Matriisinormin määritelmä:

$$\|A\|_1 := \max_{\|x\|_1=1} \|Ax\|_1 \quad \text{eli} \quad \|x\|_1 = \sum_{j=1}^{n} |x_j| = 1$$

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \overbrace{|a_{ij} x_j|}^{=|a_{ij}||x_j|}$$

$$= \sum_{j=1}^{n} \left(|x_j| \underbrace{\sum_{i=1}^{n} |a_{ij}|}_{\leq \max_k \sum_{i=1}^{n} |a_{ik}|} \right) \leq \max_k \sum_{i=1}^{n} |a_{ik}| \underbrace{\sum_{j=1}^{n} |x_j|}_{=1} = \max_k \sum_{i=1}^{n} |a_{i}$$

j:nnen sarakkeen alkioiden itseisarvojen summa

suurin itseisarvojen sarakesumma

Osoitetaan vielä, että $\|Ax\|_1$ voi saada arvon $\max_k \sum_{i=1}^{n} |a_{ik}|$

Jos l on indeksi, joka maksimoi itseisarvoj sarakesumman. Eli

$$\sum_{i=1}^{n} |a_{il}| = \max_k \sum_{i=1}^{n} |a_{ik}|$$

Silloin

$$\|Ae^l\|_1 = \sum_{i=1}^{n} |a_{il}| \qquad e^l = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow l\text{:s alkio}$$

$$\Longrightarrow \|A\|_1 = \max_k \sum_{i=1}^{n} |a_{ik}|$$

A3) a)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

A:n ominaisarvot:

$$\det(A-\lambda I) = (1-\lambda)(2-\lambda)^2 \implies \lambda_1 = 1 \text{ ja } \lambda_2 = \lambda_3 = 2$$

A:n ominaisvektorit:

$$\lambda = 1 \quad \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \longleftrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right) \implies v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda = 2 \quad \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \longleftrightarrow \left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \implies v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Vain 2 ominaisvektoria $\implies$ A ei diagonalisoidu.

Ratkaistaan Jordan-muoto:

tuntematon

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} (v_1 \;\; v_2 \;\; w) = (v_1 \;\; v_2 \;\; w) \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{=J}$$

$$\implies Aw = v_2 + 2w$$

$$(A-2I)w = v_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \implies w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

A 3) (a) jatkuu

$$e^A = X e^J X^{-1}$$

Lasketaan Jordanlohkon eksponentti

$$e^{\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}} = e^{2I + \overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{=N}} = e^{2I} \cdot e^{N} = e^{2I}(I+N)$$

$$= \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & e^2 \\ 0 & e^2 \end{pmatrix}$$

$$e^A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e & 0 & 0 \\ e & 0 & e^2 \\ -2e & e^2 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} e & 0 & 0 \\ e-e^2 & e^2 & 0 \\ -2e+e^2 & e^2 & e^2 \end{pmatrix}}}$$

HUOM! Jordan-muodon laskemista ei edellytetä tällä kurssilla.

A3) b) $A = \begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix}$

A:n ominaisarvot:

$$\det(A - \lambda I) = (4-\lambda)(-3-\lambda) + 12$$
$$= \lambda^2 - \lambda = \lambda(\lambda - 1) \implies \lambda_1 = 0 \quad \lambda_2 = 1$$

A:n ominaisvektorit:

- $\lambda_1 = 0$ $\quad \left(\begin{array}{cc|c} 4 & -6 & 0 \\ 2 & -3 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda_2 = 1$ $\quad \left(\begin{array}{cc|c} 3 & -6 & 0 \\ 2 & -4 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$X = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \qquad X^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$

- $e^A = X e^{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} X^{-1} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 2e \\ 2 & e \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -3+4e & 6-6e \\ -2+2e & 4-3e \end{pmatrix}}}$$

$\boxed{L1}$ a) Olkoon A ja B ortogonaalisia ja $C = AB$

$\Longrightarrow AA^T = I$ ja $BB^T = I$

$$CC^T = AB(AB)^T = A\underbrace{BB^T}_{I}A^T = AA^T = I$$

C on siis myös ortogonaalinen. $\square$

b)

A ja B ovat alakolmiomatriiseja ja $C = AB$

$\Longrightarrow a_{ij} = 0$ jos $i < j$

$b_{ij} = 0$ jos $i < j$

Tutkitaan C:n alkioita, jotka ovat diagonaalin yläpuolella. Siis $i < j$

$$c_{ij} = [AB]_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j-1} a_{ik}\underbrace{b_{kj}}_{=0,\text{ koska } k<j} + \sum_{k=j}^{n} \underbrace{a_{ik}}_{=0,\text{ koska } i<j} b_{kj} = 0$$

C on siis alakolmiomatriisi.

c)

Tarkastellaan käänteismatriisin laskemista Gaussin algoritmilla.

$\# =$ jotain

$$\left(\begin{array}{cccc|cccc} a_{11} & & & 0 & 1 & & & 0 \\ a_{12} & a_{22} & & & & 1 & & \\ \vdots & & \ddots & & & & \ddots & \\ a_{n1} & \cdots & & a_{nn} & 0 & & & 1 \end{array}\right) \hookrightarrow \left(\begin{array}{cccc|cccc} a_{11} & & & 0 & 1 & & & 0 \\ 0 & a_{22} & & & \# & 1 & & \\ \vdots & \# & \ddots & & \vdots & 0 & \ddots & \\ 0 & & & a_{nn} & \# & & & 1 \end{array}\right)$$

$$\hookrightarrow \left(\begin{array}{cccc|cccc} a_{11} & & & 0 & 1 & & & 0 \\ & a_{22} & & & & 1 & & \\ 0 & & \ddots & & & \# & \ddots & \\ & & & a_{nn} & & & & 1 \end{array}\right) \hookrightarrow \left(\begin{array}{cccc|c} 1 & & & 0 & \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & 1 & \underbrace{\begin{matrix} \# & & & 0 \\ \# & \# & & \\ \vdots & & \ddots & \\ \# & \cdots & \cdots & \# \end{matrix}}_{A^{-1}} \end{array}\right)$$

A^{-1} on alakolmiomatriisi.

L2) Oletetaan, että N diagonalisoituu

$$N = XD_N X^{-1}$$

$$\Rightarrow N^2 = \underbrace{X D_N^2 X^{-1}}_{=A} \qquad D_N = \begin{pmatrix} \lambda_{N1} & 0 \\ 0 & \lambda_{N2} \end{pmatrix}$$

Diagonalisoidaan $A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

A:n ominaisarvot:

$$\det(A - \lambda I) = (5-\lambda)^2 - 4^4 = \lambda^2 - 10\lambda + 9 = (\lambda - 1)(\lambda - 9)$$

$$\Rightarrow \lambda_1 = 1 \text{ ja } \lambda_2 = 9$$

A:n ominaisvektorit:

$$\lambda = 1 \quad \left(\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda = 9 \quad \left(\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right) \hookrightarrow \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad X^{-1} = \frac{1}{1-(-1)} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \lambda_{N1}^2 = 1 \\ \lambda_{N2}^2 = 9 \end{cases} \Rightarrow \begin{cases} \lambda_{N1} = \pm 1 \\ \lambda_{N2} = \pm 3 \end{cases}$$

N ei ole yksikäsitteinen

Lasketaan N arvoilla $\lambda_{N1} = 1$ ja $\lambda_{N2} = 3$

$$N = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

L3 Tarkastellaan tulon $A(t)B(t)$ yksittäistä alkiota.

$$\left[\frac{d}{dt}(A(t)B(t))\right]_{ij} = \frac{d}{dt}\sum_{k=1}^{n} a_{ik}(t)b_{kj}(t)$$

$$= \sum_{k=1}^{n} \frac{d}{dt}\left(a_{ik}(t)b_{kj}(t)\right) = \sum_{k=1}^{n}\left(a'_{ik}(t)b_{kj}(t) + a_{ik}(t)b'_{kj}(t)\right)$$

$$= \underbrace{\sum_{k=1}^{n} a'_{ik}(t)b_{kj}(t)}_{=[A'(t)B(t)]_{ij}} + \underbrace{\sum_{k=1}^{n} a_{ik}(t)b'_{kj}(t)}_{=[A(t)B'(t)]_{ij}} = \left[A'(t)B(t) + A(t)B'(t)\right]_{ij}$$

$$\Longrightarrow \frac{d}{dt}(A(t)B(t)) = A'(t)B(t) + A(t)B'(t)$$

L4

Lause: Similaarisilla matriiseilla on samat ominaisarvot ~~ja -vektorit.~~ (Kreyszig Ed.8 s.392)

Olkoon λ A:n ominaisarvo ja x A:n ominaisvektori, jolloin $Ax = \lambda x$.

$$(A+I)x = Ax + x = \lambda x + x = \underbrace{(\lambda+1)}_{A+I\text{:n ominaisarvo}}x$$

A+I:n ominaisarvot ovat $\lambda_i + 1$, joten ne eivät ole samoja kuin A:n ominaisarvot λ_i.

A ja A+I eivät ole similaarisia.

L5 Yhtälön $\dot{y} = Ay$ yleinen ratkaisu voidaan esittää matriisieksponentin avulla muodossa:

$$y = e^{tA} x_0 \quad , \text{ missä } x_0 = y(0)$$

A:n ominaisarvot:

$$\det(A - \lambda I) = (-2-\lambda)(-2-\lambda) + 4$$

$$= \lambda^2 + 4\lambda + 8 \Longrightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4 \cdot 8}}{2} = \begin{cases} -2+2i \\ -2-2i \end{cases}$$

A:n ominaisvektorit:

$$\lambda = -2+2i \quad \left(\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right) \xrightarrow{\cdot i} \left(\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Longrightarrow v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\lambda = -2-2i \quad \left(\begin{array}{cc|c} 2i & 2 & 0 \\ -2 & 2i & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Longrightarrow v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \qquad X^{-1} = \frac{1}{-2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{pmatrix}$$

$$e^{tA} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{(-2+2i)t} & 0 \\ 0 & e^{(-2-2i)t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{pmatrix}$$

$$= e^{-2t} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^{2it} & -\frac{1}{2}ie^{2it} \\ \frac{1}{2}e^{-2it} & \frac{1}{2}ie^{-2it} \end{pmatrix}$$

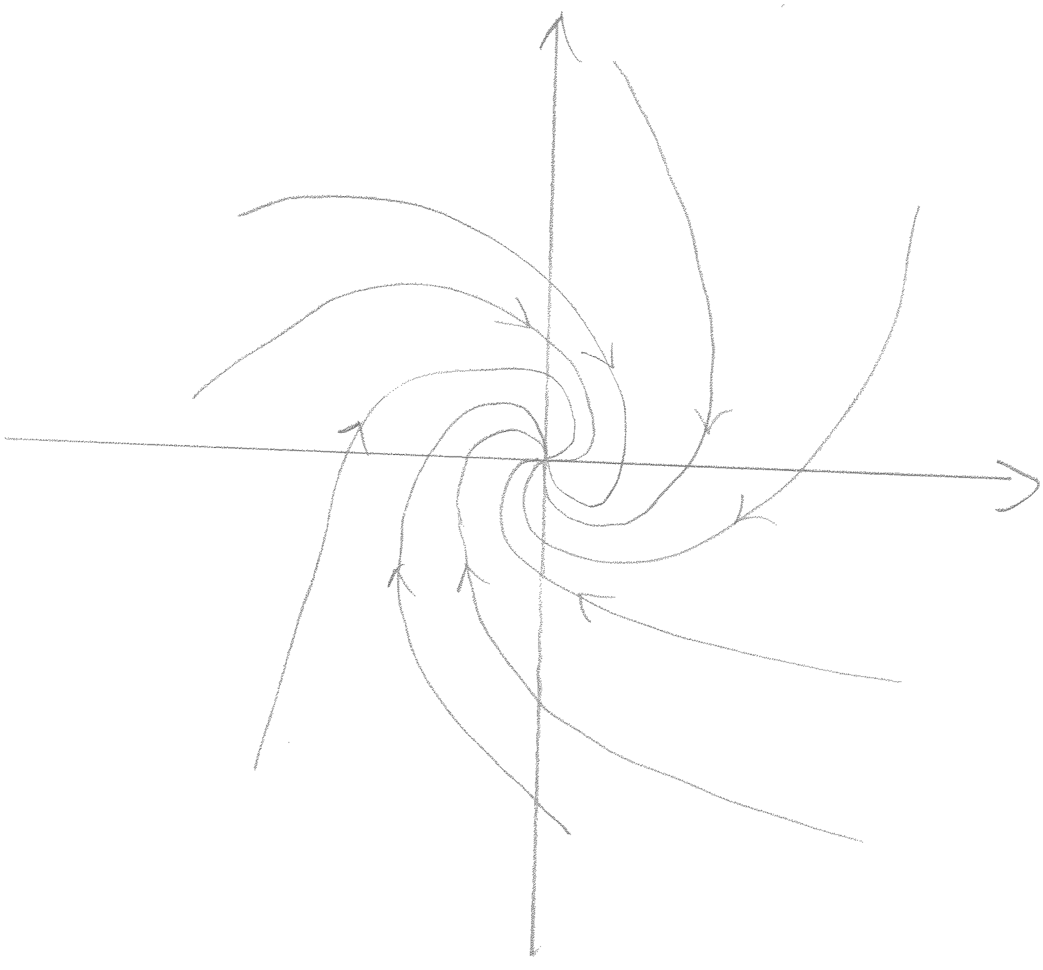
$$= e^{-2t} \begin{pmatrix} \frac{1}{2}(e^{2it} + e^{-2it}) & -\frac{i}{2}(e^{2it} - e^{-2it}) \\ \frac{i}{2}(e^{2it} - e^{-2it}) & \frac{1}{2}(e^{2it} + e^{-2it}) \end{pmatrix}$$

$\boxed{5}$

$$= e^{-2t}\begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}$$

$$Y = \begin{pmatrix} e^{-2t}\cos 2t & e^{-2t}\sin 2t \\ -e^{-2t}\sin 2t & e^{-2t}\cos 2t \end{pmatrix} X_0$$

Koska ominaisarvojen reaaliosa on negatiivinen, origo on stabiili fokus (spiral poin



L6 a)

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -4 \\ 1 & -5 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

A:n ominaisarvot:

$$\det(A - \lambda I) = -\lambda(-5-\lambda) + 4 = \lambda^2 + 5\lambda + 4$$

$$= (\lambda+1)(\lambda+4) \implies \lambda_1 = -1 \text{ ja } \lambda_2 = -4$$

A:n ominaisvektorit:

$$\lambda = -1 \quad \left(\begin{array}{cc|c} 1 & -4 & 0 \\ 1 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right) \implies v_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\lambda = -4 \quad \left(\begin{array}{cc|c} 4 & -4 & 0 \\ 1 & -1 & 0 \end{array}\right) \implies v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

yleinen ratkaisu: (Kreyszig Ed.8 s.163)

$$y = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-4t}$$

$$\begin{cases} y_1 = c_1 4 e^{-t} + c_2 e^{-4t} \\ y_2 = c_1 e^{-t} + c_2 e^{-4t} \end{cases}$$

L6 b) $\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

A:n ominaisarvot:

$$\det(A-\lambda I) = (1-\lambda)(1-\lambda)+1 = \lambda^2 - 2\lambda + 2$$

$$\lambda = \frac{2 \pm \sqrt{4-4\cdot 2}}{2} = 1 \pm i$$

A:n ominaisvektorit:

$\lambda = 1+i$ $\quad \left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \Longrightarrow v_1 = \begin{pmatrix} -1 \\ i \end{pmatrix}$

$\lambda = 1-i$ $\quad \left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \Longrightarrow v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

yleinen ratkaisu:

$$y = c_1 \begin{pmatrix} -1 \\ i \end{pmatrix} e^{(1+i)t} + c_2 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1-i)t}$$

$$= \begin{pmatrix} -c_1 e^t(\cos t + i\sin t) + c_2 e^t(\cos t - i \sin t) \\ c_1 i e^t(\cos t + i \sin t) + c_2 i e^t(\cos t - i\sin t) \end{pmatrix}$$

$$= \begin{pmatrix} (c_2 - c_1)e^t \cos t - i(c_1 + c_2)e^t \sin t \\ i(c_1+c_2)e^t \cos t + (c_2 - c_1)e^t \sin t \end{pmatrix}$$

L6) b)

Valitaan c_1 ja c_2 s.e.

$$\begin{aligned} c_2 - c_1 &= d_1 \\ i(c_1 + c_2) &= d_2 \end{aligned} \quad \text{ja } d_1, d_2 \in \mathbb{R}$$

$$Y = \begin{pmatrix} d_1 e^t \cos t - d_2 e^t \sin t \\ d_2 e^t \cos t + d_1 e^t \sin t \end{pmatrix}$$