C3-II, HARJOITUS 2

A1)

a) $f(t) = \sin^2 4t = \frac{1}{2}(1 - \cos 8t)$

$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$, $\mathcal{L}(1) = 1/s$

$$\Rightarrow \mathcal{L}(f) = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+64}\right)$$

$$= \frac{1}{2}\left(\frac{s^2+64-s^2}{s^3+64s}\right)$$

$$= \frac{32}{s^3+64s}$$

b) $f(t) = e^{-t}\sinh 5t$

$$= e^{-t}\frac{1}{2}(e^{5t} - e^{-5t}) = \frac{1}{2}(e^{4t} - e^{-6t})$$

Taulukosta: $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, joten

$$\mathcal{L}(f) = \frac{1}{2}\left(\frac{1}{s-4} - \frac{1}{s+6}\right) = \frac{1}{2}\left(\frac{s+6-s+4}{s^2+2s-24}\right)$$

$$= \frac{5}{s^2+2s-24} \qquad \left(= \frac{5}{(s+1)^2-25}\right)$$

Vaihtoehtoisesti sintolaunetta käyttäen:

$\mathcal{L}(e^{at}f(t)) = F(s-a)$

$\mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$

$$\Rightarrow \mathcal{L}(e^{-t}\sinh 5t) = \frac{5}{(s+1)^2-25}$$

A2 $f(t) = \begin{cases} t, & 0 < t \leq 1 \\ -t, & 1 < t \leq 2 \\ 0, & t > 2 \end{cases}$

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 t e^{-st}\,dt + \int_1^2 -t e^{-st}\,dt$$

Omittelsintegraal: $\int_a^b t e^{-st}\,dt = \Big|_a^b \frac{t e^{-st}}{-s} + \frac{1}{s}\int_a^b e^{-st}\,dt$

$$= \Big|_a^b \left\{ \frac{t e^{-st}}{-s} - \frac{1}{s^2} e^{-st} \right\}$$

$$= -\Big|_a^b \frac{st e^{-st} + e^{-st}}{s^2}$$

$$\hookrightarrow \mathcal{L}(f) = -\Big|_0^1 \frac{st e^{-st} + e^{-st}}{s^2}$$

$$+ \Big|_1^2 \frac{st e^{-st} + e^{-st}}{s^2}$$

$$= -\frac{1}{s^2}\left(s e^{-s} + e^{-s} - 1\right) + \frac{1}{s^2}\left(2s e^{-2s} + e^{-2s} - s e^{-s} - e^{-s}\right)$$

$$= \frac{-2s e^{-s} - 2e^{-s} + 2s e^{-2s} + e^{-2s} + 1}{s^2}$$

$$= \frac{2(e^{-2s} - e^{-s})}{s} + \frac{e^{-2s} - 2e^{-s} + 1}{s^2}$$

A3. $f(t) = e^{-t}(a_0 + a_1 t + \dots a_n t^n)$

Ydeiseki: $\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}}$,

$$\mathcal{L}(e^{at} f(t)) = F(s-a),$$

$$F(s) = \mathcal{L}(f)$$

$$\Rightarrow \mathcal{L}\left(\sum_{k=0}^{n} a_k t^k\right) = \sum_{k=0}^{n} a_k \frac{k!}{s^{k+1}}$$

$$\mathcal{L}(f) = \sum_{k=0}^{n} \frac{a_k k!}{(s+1)^{k+1}}$$

$$= \frac{a_0}{s+1} + \frac{a_1}{(s+1)^2} + \frac{2a_2}{(s+1)^3} + \dots + \frac{k!\,a_k}{(s+1)^{k+1}}$$

L1. $\dfrac{\pi}{(s+\pi)^2} = \pi \cdot F(s+\pi)$, miss: $F(s) = \dfrac{1}{s^2}$.

„shifting theorem": $F(s-a) = e^{at} f(t)$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{\pi}{(s+\pi)^2}\right\} = \pi e^{-\pi t} \underbrace{\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)}_{t}$$

$$= \underline{\underline{\pi t e^{-\pi t}}}$$

L2

$$f(t) = t\cos 5t$$

$$f'(t) = \cos 5t + 5t(-\sin 5t)$$

$$f''(t) = -5\sin t - 5\sin 5t - 25t\cos t$$

$$= -10\sin 5t - 25 f(t)$$

2. derivaatan $\mathcal{L}$-muunnos:

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

Nyt $\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\cdot 0 - 1$

$\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2}$

$$= -10\cdot\mathcal{L}(\sin 5t) - 25\mathcal{L}(f)$$

$$= -\frac{50}{s^2+25} - 25\mathcal{L}(f)$$

$$\Rightarrow \mathcal{L}(f)(s^2+25) = -\frac{50}{s^2+25} + 1$$

$$\mathcal{L}(f) = \frac{1}{s^2+25} - \frac{50}{(s^2+25)^2}$$

$$= \underline{\underline{\frac{s^2-25}{(s^2+25)^2}}}$$

1-L3 $\quad y' + \frac{1}{2}y = 17\sin 2t \quad , \quad y(0) = -1 \quad \| \mathcal{L}(\cdot)$

$$sY(s) - y(0) + \frac{1}{2}Y(s) = 17 \cdot \frac{2}{s^2+4}$$

$$Y(s)\left(s + \frac{1}{2}\right) = \frac{34}{s^2+4} - 1$$

$$Y(s) = \underbrace{\frac{34}{(s^2+4)(s+\frac{1}{2})}}_{\text{osamurtohajotelma}} - \frac{1}{s+\frac{1}{2}}$$

$$\Leftrightarrow \frac{34}{(s+\frac{1}{2})(s^2+4)} = \frac{As+B}{s^2+4} + \frac{C}{s+\frac{1}{2}}$$

$$A\left(s^2 + \tfrac{1}{2}s\right) + B\left(s + \tfrac{1}{2}\right) + C\left(s^2+4\right) = 34$$

$$\Rightarrow \quad A + C = 0$$

$$\frac{A}{2} + B = 0 \quad \Rightarrow A = -2B$$

$$\frac{B}{2} + 4C = 34 \quad \Rightarrow C = 2B = -A$$

$$\Leftrightarrow \frac{1}{2}B + 8B = 34 \quad \Rightarrow B = 4$$

$$C = 8$$

$$A = -8$$

$$\Rightarrow Y(s) = \frac{-8s+4}{s^2+4} + \frac{8-1}{s+1/2}$$

$$= -8\frac{s}{s^2+4} + 2 \cdot \frac{2}{s^2+4} + \frac{7}{s+1/2}$$

$$\Rightarrow \underline{\underline{y(t) = -8\cos 2t + 2\sin 2t + 7e^{-\frac{1}{2}t}}}$$

L4 Integraalin $\mathcal{L}$-muunnos:

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s).$$

Nyt $F(s) = \dfrac{10}{s^3 - \pi s^2} = 10 \cdot \dfrac{1}{s} \cdot \dfrac{1}{s(s-\pi)}$

$$\mathcal{L}^{-1}(F) = 10\int_0^t \mathcal{L}^{-1}\left(\frac{1}{s}\cdot\frac{1}{s-\pi}\right)(\tau)\,d\tau$$

Osamurtoille: $\dfrac{1}{s}\cdot\dfrac{1}{s-\pi} = \dfrac{A}{s-\pi} + \dfrac{B}{s}$

$\Rightarrow As + B(s-\pi) = 1$

$A = -B$

$B = -1/\pi \Rightarrow A = 1/\pi$

$$\mathcal{L}\left(\frac{1}{s(s-\pi)}\right) = \mathcal{L}\left(\frac{1}{\pi}\cdot\frac{1}{s-\pi} - \frac{1}{\pi}\cdot\frac{1}{s}\right)$$

$$= \frac{1}{\pi}\left(e^{\pi t} - 1\right)$$

$$\Rightarrow \mathcal{L}^{-1}(F) = \frac{10}{\pi}\int_0^t e^{\pi\tau} - 1\,d\tau$$

$$= \frac{10}{\pi}\left(\Big/_0^t \frac{e^{\pi t}}{\pi} - t\right) = \underline{\underline{\frac{10}{\pi^2}\left(e^{\pi t} - \pi t - 1\right)}} = f(t)$$

L5 a) $f(t) = \begin{cases} t^2, & t > 3 \\ 0, & t \le 3 \end{cases}$

$= t^2 u(t-3)$, missä

$u(t)$ on yksikköaskel:

$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}$.

"Toinen siirtolause":

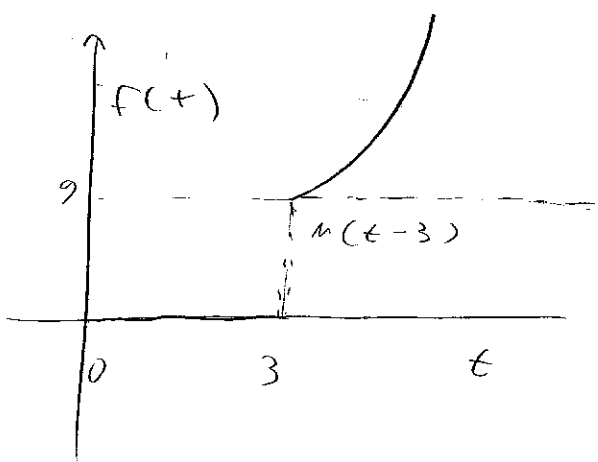
$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$.

Nyt $f(t) = (t+3-3)^2 u(t-3)$

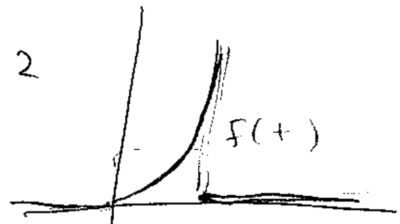
$\mathcal{L}(f) = e^{-3s}\mathcal{L}\{(t+3)^2\} = e^{-3s}\mathcal{L}\{t^2+6t+9\}$

$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$

$= \underline{\underline{\dfrac{e^{-3s}(9s^2+6s+2)}{s^3}}}$



L5b) $f(t) = \begin{cases} \sinh t, & 0 < t < 2 \\ 0 & \text{muuten} \end{cases}$



$$= \sinh t \,(u(t) - u(t-2))$$

$$= \sinh(t) u(t) - \sinh(t+2-2)\, u(t-2)$$

$$\mathcal{L}(f) = \mathcal{L}(\sinh(t)) - e^{-2s} \mathcal{L}(\sinh(t+2))$$

$$= \frac{1}{s^2-1} - e^{-2s} \mathcal{L}(\sinh t \cosh 2 + \cosh t \sinh 2)$$

$$= \frac{1}{s^2-1} - e^{-2s}\left(\cosh 2 \cdot \frac{1}{s^2-1} + \sinh 2 \cdot \frac{s}{s^2-1}\right)$$

$$= \underline{\underline{\frac{1 - e^{-2s}(\cosh 2 + \sinh(2) s)}{s^2 - 1}}}$$

<u>L6</u>

$$F(s) = \frac{e^{-2\pi s} - e^{-8\pi s}}{s^2+1}$$

$$= e^{2\pi s} \cdot \frac{1}{s^2+1} - e^{8\pi s} \cdot \frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}(F) = f(t) = u(t-2\pi)\sin(t-2\pi)$$
$$- u(t-8\pi)\sin(t-8\pi)$$

$$= \sin t \,(u(t-2\pi) - u(t-8\pi))$$

$$= \underline{\underline{\begin{cases} \sin t, & 2\pi < t < 8\pi \\ 0, & \text{muuten.} \end{cases}}}$$