REGULARITY OF SETS WITH QUASIMINIMAL BOUNDARY SURFACES IN METRIC SPACES

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Abstract. This paper studies regularity of perimeter quasiminimizing sets in metric measure spaces with a doubling measure and a Poincaré inequality. The main result shows that the measure theoretic boundary of a quasiminimizing set coincides with the topological boundary. We also show that such a set has finite Minkowski content and apply the regularity theory to study rectifiability issues related to quasiminimal sets in the strong A_∞-weighted Euclidean case.

1. Introduction

It is now a well-known fact that Euclidean sets with (locally) minimal surfaces have smooth boundary apart from a set of co-dimension 2. This result is due to De Giorgi, see [DG1] and [DG2]. The analogous result for Euclidean quasiminimal surfaces is due to David and Semmes [DS1], who showed that bounded sets with quasiminimal boundary surfaces are uniformly rectifiable and are locally John domains.

The paper [DS1] considered a double obstacle problem in constructing quasiminimal surfaces in Euclidean spaces; A similar problem was considered by Caffarelli and de la Llave in [CL], where the setting is C^2-Riemannian manifolds. In [CL, Theorem 1.1] it is shown that given a Euclidean hyperplane (and the manifold is obtained by a perturbation of the Euclidean metric in a C^2-fashion) there is a quasiminimal surface in the Riemannian metric that lies close to the hyperplane. In [KKST2] a double obstacle problem similar to the one considered by [DS1] was studied in the setting of doubling metric measure spaces supporting a $(1,1)$-Poincaré inequality. It is therefore natural to ask what type of regularity properties do the minimizing sets have away from the boundaries of the obstacles.

In this paper we study the regularity properties of quasiminimal sets or, more precisely, quasiminimal boundary surfaces in the setting of metric measure spaces with a doubling measure that supports a $(1,1)$-Poincaré inequality. We will show, by modifying De Giorgi's technique using a part of the argument of David and Semmes, that such a set

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is porous and satisfies a measure density property. In particular, this implies that the measure theoretic boundary of a quasiminimizing set coincides with the topological boundary. We also show that such a set has finite Minkowski content. In the metric setting the classical definition of rectifiability may not be as widely applicable. For instance, in the setting of Heisenberg groups there are sets of finite perimeter that are not rectifiable [Mag]. Hence the finiteness of the Minkowski content is the best one can hope for in this generality. Since the problem studied in [CL] is a minimization problem and comes with an associated PDE, the techniques used there are essentially of PDE. The problem studied in [DS1] is a quasiminimization problem, and hence we find some of the methods used in this paper to be more easily adaptable to the general metric measure space setting.

In the last two sections of this paper we apply the regularity theory developed in the first part of the paper to study rectifiability issues related to quasiminimal sets in the strong A_∞-weighted Euclidean setting. Observe that when equipped with a strong A_∞-weight, the Euclidean space with Euclidean metric need not satisfy a 1-Poincaré inequality. However, there is a natural metric induced by the strong A_∞-weight, and we show in Section 6 that the Euclidean space equipped with this natural metric and weighted measure satisfy a 1-Poincaré inequality. Hence we are able to use the theory developed in the first part to study rectifiability issues of the boundary of quasiminimal sets in this modified Euclidean space. We consider this application in Section 7 of this paper. It is known that not every strong A_∞-weight is comparable to the Jacobian of a Euclidean quasiconformal mapping; it is therefore not possible to use (unweighted) Euclidean results about regularity of sets with quasiminimal surfaces to study rectifiability issues of boundaries of such sets in the strong A_∞-weighted setting. We were able to apply the theory developed in the general metric setting in the first five sections of this paper to successfully address rectifiability issues in this weighted Euclidean setting.

For related results about isoperimetric sets in the Carnot group setting we refer the interested reader to [LR], where they show that isoperimetric sets (which are necessarily a special class of sets of quasiminimal boundary surfaces) are Ahlfors regular (which also now follows from Corollary 5.3) and are porous. Regularity for Euclidean quasiminimizers that are asymptotic minimizers was studied by Rigot [R], where it was shown that if the asymptotic minimality condition is sufficiently controlled, then the quasiminimal surface is Hölder smooth in big pieces. We point out that our results about the sets with quasiminimal boundary surfaces apply to every boundary point of the set (of course, a modification of such a set on a measure zero subset would still maintain quasiminimality while destroying the regularity at some boundary point; to avoid this trivial modification we ensure, without loss of generality, that each point x of the boundary of the set E satisfies $\mu(B(x,r) \cap E) > 0$ and $\mu(B(x,r) \setminus E) > 0$ for all $r > 0$), and hence our results are weaker than Hölder regularity of the boundary, but are applicable to each boundary point. Hence it might well be that the studies related to rectifiability and weak tangents of locally minimal surfaces in the Carnot group setting would be more approachable using the regularity properties studied in this paper. It is a result of Lu and Wheeden [LW] that Carnot groups are doubling and satisfy a 1-Poincaré inequality, and hence the results of this paper apply in the setting of Carnot groups (and indeed in more general Carnot-Carathéodory spaces, which satisfy local versions of these conditions). Nice surveys about Poincaré inequalities and isoperimetric inequalities in the setting of Carnot groups can also be found in [J] and [Hei].

It was shown in [AKL] that a subset E of a Carnot group, with locally finite perimeter, has vertical weak tangents for $\|D\chi_E\|$-almost every point. Combining this with our results (in particular, the consequence that every boundary point of such a set is in the measure-theoretic boundary), we see that $\mathcal{H}^{Q-1}$-a.e. boundary point of a set of quasiminimal boundary surface in a Carnot group has a vertical weak tangent (Q is the homogeneous dimension of the group). The method of [AKL] uses the group structure; it would be interesting to know whether such results hold for other Carnot-Carathéodory spaces such as the Grushin spaces. Note that existence of weak tangents is weaker than rectifiability. For a different notion of rectifiability in the Carnot group setting see [Mag, Section 3].

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2. Preliminaries

A Borel regular outer measure is *doubling* if there is a constant $C_d > 0$ such that for every ball $B = B(x,R) = \{y \in X : d(y,x) < R\}$ of X we have $0 < \mu(B) < \infty$ with

$$\mu(2B) \le C_d \mu(B),$$

where $\lambda B = B(x, \lambda R)$ for $\lambda > 0$.

For such a measure μ, there is a lower mass bound exponent $Q > 0$; that is, whenever $x \in X$, $0 < r \le R$, and $y \in B(x, R)$, we have

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \ge \frac{1}{C} \left(\frac{r}{R}\right)^Q .$$

Given a function u and a non-negative Borel measurable function g on X, we say that g is an *upper gradient* of u if whenever γ is a rectifiable curve in X (that is, a curve with finite length), we have

$$|u(y) - u(x)| \le \int_\gamma g\, ds, \tag{2.1}$$

where x and y denote end points of γ. Here the above inequality should be interpreted to mean that $\int_\gamma g\, ds = \infty$ whenever at least one of $|u(x)|$ and $|u(y)|$ is infinite; see for example [HeiK]. The collection of all upper gradients, together, play the roles of the modulus of the weak derivative of a Sobolev function in the metric setting.

We say that g is a *p-weak upper gradient* of u if the collection Γ of rectifiable curves for which (2.1) does not hold has zero p-modulus, that is, there is a non-negative Borel function $f \in L^p(X)$ such that for each $\gamma \in \Gamma$, the integral $\int_\gamma f\, ds$ is infinite. See [KoMc] for the fact that p-weak upper gradients belong to the L^p-closure of the convex set of all upper gradients in $L^p(X)$. Indeed, more is true, for it then follows from [KaSha] Lemma 3.1 together with the proof of this lemma given there that the $L^p(X)$-closure of the set of all upper gradients in $L^p(X)$ is precisely the set of all p-weak upper gradients of the given function.

We consider the norm

$$\|u\|_{N^{1,1}(X)} := \|u\|_{L^1(X)} + \inf_g \|g\|_{L^1(X)}$$

with the infimum taken over all upper gradients g of u. The *Newton-Sobolev space* considered in this paper is the space

$$N^{1,1}(X) = \{u : \|u\|_{N^{1,1}(X)} < \infty\}/\sim,$$

where the equivalence relation $\sim$ is given by $u \sim v$ if and only if

$$\|u - v\|_{N^{1,1}(X)} = 0.$$

We say that X supports a $(1,1)$-*Poincaré inequality* if there are constants $C > 0$ and $\lambda \ge 1$ such that whenever u is a function on X with upper gradient g_u and B is a ball in X, we have

$$\int_B |u - u_B|\, d\mu \le C \mathrm{rad}(B) \int_{\lambda B} g_u\, d\mu.$$

Sometimes the inequality above is called the *weak* $(1,1)$-*Poincaré inequality.* The term weak refers to the possibility of $\lambda > 1$. For the sake of brevity, in the rest of the paper we suppress the term weak in this connection.

A function u on X is said to be of *bounded variation*, and denoted $u \in BV(X)$, if $u \in L^1(X)$ and there is a sequence $\{u_n\}_n$ of functions from $N^{1,1}(X)$ such that $u_n \to u$ in $L^1(X)$ and

$$\limsup_{n\to\infty} \|u_n\|_{N^{1,1}(X)} < \infty.$$

The BV norm of such a function u is given by

$$\|u\|_{BV(X)} := \inf_{\{u_n\}_n} \liminf_{n\to\infty} \|u_n\|_{N^{1,1}(X)},$$

where the infimum is taken over all such convergent sequences. The BV energy norm of u is given by

$$\|Du\|(X) := \inf_{\{u_n\}_n} \liminf_{n\to\infty} \Big[\|u_n\|_{N^{1,1}(X)} - \|u_n\|_{L^1(X)} \Big].$$

We say that a Borel set $E \subset X$ is of *finite perimeter* if $\chi_E \in BV(X)$. The *perimeter measure* of the set E is

$$P(E, X) := \|D\chi_E\|(X).$$

See [Mi2] and [A] for more on BV functions and sets of finite perimeter in the metric setting. We point out here that in the Euclidean case with Lebesgue measure the above notion coincides with the classical definition of BV functions; see for example [EG].

For open sets $U \subset X$ and $E \subset X$, we denote by $P(E, U)$ the quantity $P(E \cap U, U)$, where U is considered as a metric subspace of X playing the role of the metric space X in this definition. It was shown in [Mi2] that $U \mapsto P(E, U)$ extends to a Radon measure on X via the Carathéodory extension as follows. For $A \subset X$, we set

$$P(E, A) := \inf\{P(E, U) : A \subset U,\, U \text{ open}\}.$$

Observe that for general sets A the quantity $P(E, A)$ is *not* the same as computing the perimeter measure of $E \cap A$ in the metric subspace A. A similar construction also gives $\|Du\|(A)$ for BV functions u. The *coarea formula*

$$\|Du\|(A) = \int_{-\infty}^{\infty} P(\{x \in X : u(x) > t\}, A)\, dt$$

was also proven in [Mi2].

The restricted spherical Hausdorff content of codimension one on X is

$$\mathcal{H}_R(E) = \inf\Big\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\, r_i \le R \Big\},$$

where $0 < R < \infty$. The Hausdorff measure of codimension one of $E \subset X$ is

$$\mathcal{H}(E) = \lim_{R\to 0} \mathcal{H}(E).$$

A combination of [AMP, Theorems 4.4 and 4.6] gives the equivalence of the perimeter measure and the Hausdorff measure of codimension one for sets of finite perimeter. The measure theoretic boundary of E, denoted by $\partial^* E$, is the set of points $x \in X$, where both E and its complement have positive density, i.e.

$$\limsup_{r\to 0} \frac{\mu(E\cap B(x,r))}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0.$$

In this paper, we assume that μ is a doubling Borel measure with lower mass bound exponent $Q > 1$ and that X is complete and supports a $(1,1)$-Poincaré inequality. Note that we can increase the value of Q as we like, and so assuming $Q > 1$ is not a serious restriction, and is assumed merely for book-keeping. We point out here that if X supports a $(1,1)$-Poincaré inequality, then whenever $u \in BV(X)$ and B is a ball in X, we have

$$\int_B |u - u_B|\, d\mu \le C\mathrm{rad}(B)\, \|Du\|(\lambda B).$$

When considering the function $f = \chi_E$ for a set $E \subset X$, the above inequality implies the *relative isoperimetric inequality*

$$\min\{\mu(B\cap E), \mu(B\setminus E)\} \le C\mathrm{rad}(B)\, P(E, \lambda B).$$

In this paper C will denote constants whose precise values are not needed, and so the value of C might differ even within the same line.

The integral average of a function $u \in L^1(A)$ over a μ-measurable set A with finite and positive measure is denoted by

$$u_A = \fint_A u\, d\mu = \frac{1}{\mu(A)} \int_A u\, d\mu.$$

It is well known that the Poincaré inequality implies a *Sobolev-Poincaré inequality* if the measure is doubling. Indeed, by [HaKo] we have

$$\Big(\fint_B |u - u_B|^t\, d\mu\Big)^{1/t} \le C\mathrm{rad}(B) \fint_{2\lambda B} g_u\, d\mu.$$

with $t = Q/(Q-1)$ for all $u \in N^{1,1}(X)$. Note that if $u \in N^{1,1}(2\lambda B_0)$ for some ball $2\lambda B_0$, then for each $\varepsilon > 0$ we consider the ball $(1-\varepsilon)B_0$ and a Lipschitz function η_ε which is 1 on $(1-\varepsilon)2\lambda B_0$, supported in $2\lambda B_0$, and apply the Sobolev-Poincaré inequality to $\eta_\varepsilon u \in N^{1,1}(X)$ on the ball $(1-\varepsilon)B_0$. Note that $g_{\eta_\varepsilon u} = \|Du\|$ on $(1-\varepsilon)2\lambda B_0$. Finally, letting $\varepsilon \to 0$, we see that the Sobolev-Poincaré inequality holds for all functions in $N^{1,1}(2\lambda B)$ as well.

By the definition of BV functions, we can approximate a BV function u in the L^1-sense by a sequence of $N^{1,1}$-functions $\{u_n\}_n$ such that

$$\lim_{n\to\infty} \int_{\lambda B} g_{u_n}\, d\mu = \|Du\|(\lambda B).$$

Now an application of the Sobolev-Poincaré inequality to the functions u_n , $n = 1, 2, \ldots,$ gives the *Sobolev inequality*

$$\Big(\fint_B |u - u_B|^t \, d\mu\Big)^{1/t} \le C\mathrm{rad}(B)\frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)}$$

for all $u \in BV(X)$. Note that if $\{u_n\}_n$ is a sequence of functions in $N^{1,1}(2\lambda B)$ converging in $L^1(2\lambda B)$ to u with

$$\limsup_{n\to\infty} \int_{2\lambda B} g_{u_n} \, d\mu < \infty,$$

then by the Sobolev-Poincaré inequality mentioned above, the sequence $\{u_n - (u_n)_B\}_n$ is bounded in $L^t(B)$. By passing to a subsequence if necessary, we may assume in addition that $u_k \to u$ pointwise μ-a.e. in $2\lambda B$ as well. Hence by the uniform convexity of $L^t(B)$ and by Mazur's lemma (and replacing $\{u_n\}_n$ by a convex combination subsequence if necessary), we see that

$$\lim_{n\to\infty} \fint_B |u_n - (u_n)_B|^t \, d\mu = \fint_B |u - u_B|^t \, d\mu.$$

Taking infimum over all such sequences $\{u_n\}_n$ yields the above Sobolev-Poincaré inequality for all $u \in BV(X)$.

Lemma 2.2. *Let $u \in BV(X)$ and $A = \{x \in B : |u(x)| > 0\}$. If $\mu(A) \le \gamma\mu(B)$ for some $0 < \gamma < 1$, then*

$$\Big(\fint_B |u|^t \, d\mu\Big)^{1/t} \le \frac{C}{1-\gamma^{1-1/t}}\mathrm{rad}(B)\frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)}$$

with $t = Q/(Q-1)$.

Proof. By Minkowski's inequality and the above-mentioned Sobolev inequality,

$$\begin{aligned}\Big(\fint_B |u|^t \, d\mu\Big)^{1/t} &\le \Big(\fint_B |u - u_B|^t \, d\mu\Big)^{1/t} + |u_B| \\ &\le C\mathrm{rad}(B)\frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)} + |u_B|.\end{aligned} \tag{2.3}$$

By the assumption on u and Hölder's inequality,

$$
\begin{aligned}
|u_B| &\le \fint_B |u|\,d\mu = \frac{1}{\mu(B)} \int_A |u|\,d\mu \\
&\le \frac{1}{\mu(B)}\Big(\int_A |u|^t\,d\mu\Big)^{1/t}\mu(A)^{1-1/t} \\
&= \frac{1}{\mu(B)}\Big(\int_B |u|^t\,d\mu\Big)^{1/t}\mu(A)^{1-1/t} \\
&= \Big(\frac{\mu(A)}{\mu(B)}\Big)^{1-1/t}\Big(\fint_B |u|^t\,d\mu\Big)^{1/t} \\
&\le \gamma^{1-1/t}\Big(\fint_B |u|^t\,d\mu\Big)^{1/t}.
\end{aligned}
$$

So by (2.3),

$$
(1-\gamma^{1-1/t})\Big(\fint_B |u|^t\,d\mu\Big)^{1/t} \le C\,\mathrm{rad}(B)\frac{\|Du\|(2\lambda B)}{\mu(2\lambda B)},
$$

from which the lemma follows. □

Corollary 2.4. *If $u \in BV(X)$ such that $u = 0$ in $X \setminus B$ and $X \setminus 2B$ is non-empty, then*

$$
\Big(\fint_B |u|^t\,d\mu\Big)^{1/t} \le C\mathrm{rad}(B)\frac{\|Du\|(\overline{B})}{\mu(B)}.
$$

Proof. Since $X\setminus 2B$ is non-empty, and because by the Poincaré inequality X is path-connected, it follows that there is a point $y \in 2B \setminus B$ such that $d(y,x) = 3r/2$ where $B = B(x,r)$. Therefore by the doubling property of the measure μ, we have

$$
\mu(2B \setminus B) \ge \mu(B)/C \ge \mu(2B)/C^2
$$

for some constant $C > 1$. Because $A = \{z \in 2B : |u(z)| > 0\}$ is a subset of B, it follows that

$$
\frac{\mu(A)}{\mu(2B)} \le \frac{\mu(B)}{\mu(2B)} = \frac{\mu(2B)-\mu(2B\setminus B)}{\mu(2B)} \le 1 - C^{-2} < 1.
$$

We can take $\gamma = 1 - C^{-2}$ in Lemma 2.2 to obtain the desired inequality. □

3. Quasiminimizing surfaces and quasiminimizers

Definition 3.1. Let $E \subset X$ be a Borel set of finite perimeter and $\Omega \subset X$ be an open set. We say that E is a K-quasiminimal set, or has a K-*quasiminimal boundary surface*, in Ω if for all open $U \Subset \Omega$ and for all Borel sets $F, G \Subset U$,

$$P(E,U) \le K\, P((E \cup F) \setminus G, U).$$

We say that a function $u \in BV(\Omega)$ is a K-*quasiminimizer* if for all $\varphi \in BV(\Omega)$ with support in $U \Subset \Omega$,

$$\|Du\|(U) \le K\, \|D(u+\varphi)\|(U).$$

Lemma 3.2. *If E is a K-quasiminimal set in Ω, then $u = \chi_E|_\Omega$ is a K-quasiminimizer in Ω.*

Proof. Since E is of finite perimeter, it follows that $u \in BV(\Omega)$. Also, if $\varphi \in BV(\Omega)$ with compact support in $U \Subset \Omega$, then for $0 < t < 1$, when $x \in X \setminus U$ we have $(u+\varphi)(x) > t$ if and only if $x \in E$, and consequently

$$P(\{u+\varphi > t\}, U) \ge K^{-1}\, P(E,U),$$

and so by the coarea formula,

$$\begin{aligned}
\|Du\|(U) = P(E,U) &= \int_0^1 P(E,U)\, dt \\
&\le K \int_0^1 P(\{u+\varphi > t\}, U)\, dt \\
&\le K \int_{\mathbb{R}} P(\{u+\varphi > t\}, U)\, dt = K\, \|D(u+\varphi)\|(U),
\end{aligned}$$

which shows that u is a K-quasiminimizer. $\square$

4. Density

The main result of this section is Theorem 4.2, where we prove a uniform measure density estimate for quasiminimal sets. To prove the main result, we need the following lemma. For a proof of this lemma, we refer to [Gia, Lemma 5.1].

Lemma 4.1. *Let $R > 0$ and $f : (0, R] \to [0, 1)$ be a bounded function. Suppose that there exist some $\alpha > 0$, $0 \le \theta < 1$, and $\gamma \ge 0$ such that for all $0 < \rho < r \le R < \infty$ we have*

$$f(\rho) \le \gamma (r-\rho)^{-\alpha} + \theta f(r).$$

Then there is a constant $c = c(\alpha, \theta)$ so that for all $0 < \rho < r \le R$,

$$f(\rho) \le c\gamma (r-\rho)^{-\alpha}.$$

The next result implies that every boundary point of a set of quasiminimal surface belongs to the measure theoretic boundary.

Theorem 4.2. *If E is a quasiminimal set in Ω, then by modifying E on a set of measure zero if necessary, there exists $\gamma_0 > 0$ such that for all $x \in \Omega \cap \partial E$,*

$$\frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \geq \gamma_0 \quad \textit{and} \quad \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \geq \gamma_0$$

whenever $0 < r < \operatorname{diam}(X)/3$ is such that $B(x, 2r) \subset \Omega$. The density constant γ_0 depends solely on the doubling constant, the constants associated with the Poincaré inequality, and the quasiminimality constant K.

Proof. We can modify E on a set of measure zero so that $\mu(B(x,r) \cap E) > 0$ for all $x \in E$ and $r > 0$, and $\mu(B(x,r) \setminus E) > 0$ for all $x \in X \setminus E$ and $r > 0$. This is done by removing points $x \in E$ for which there is a positive number r_x such that $\mu(B(x, r_x) \cap E) = 0$ (and in doing so, note that we remove the ball $B(x, r_x)$ from E as well since all points in this ball also satisfy this condition) and adding into E points y for which there is a positive number r_y such that $\mu(B(y, r_y) \setminus E) = 0$ (and in doing so, note that we include the ball $B(y, r_y)$ back into E). By Lebesgue differentiation theorem, such a modification is done only on a set of μ-measure zero. This implies that for all $x \in \partial E$ and $r > 0$, we have

$$\mu(B(x,r) \cap E) > 0 \quad \text{and} \quad \mu(B(x,r) \setminus E) > 0.$$

Let $u = \chi_E$, and for $z \in \Omega$ let $0 < R < \operatorname{diam}(X)/3$ be such that $B(z, 2R) \subset \Omega$. For $0 < r < R$ let η be a $2/(R-r)$-Lipschitz continuous function such that $\eta = 1$ on $B(z,r)$ and η has compact support in $B(z,R)$, with $0 \leq \eta \leq 1$ on X. Set

$$v = u - \eta u = (1 - \eta)u.$$

Then $v = u$ on $X \setminus B(z,R)$, and so by the quasiminimality property of u and the product rule

$$\begin{aligned} \|Du\|(B(z,r)) &\leq \|Du\|(B(z,R)) \leq K\|Dv\|(B(z,R)) \\ &\leq K \left(\|Du\|(B(z,R) \setminus B(z,r)) + \frac{C}{R-r} \int_{B(z,R)} u \, d\mu \right). \end{aligned}$$

Observe that η is a bounded Lipschitz function and so the product rule is valid. By setting $\theta = K/(K+1) < 1$, we see that

$$\|Du\|(B(z,r)) \leq \theta \|Du\|(B(z,R)) + \frac{C}{R-r} \int_{B(z,R)} u \, d\mu.$$

Hence by Lemma 4.1, there is a constant $C > 0$, which is independent of z, R and E, such that

$$\|Du\|(B(z,r)) \leq \frac{C}{R-r} \int_{B(z,R)} u \, d\mu = \frac{C}{R-r} \, \mu(B(z,R) \cap E).$$

For $r = 3R/4$, from the above we get

$$\|Du\|(B(z, 3R/4)) \le \frac{C}{R}\,\mu(B(z, R) \cap E). \tag{4.3}$$

Let ν be a C/R–Lipschitz function such that $0 \le \nu \le 1$ on X, $\nu = 1$ on $B(z, R/2)$, and $\nu = 0$ on $X \setminus B(z, 3R/4)$. Setting $\varphi = \nu u$, the product rule implies that

$$\|D\varphi\|(B(z, 3R/4)) \le \|Du\|(B(z, 3R/4)) + \frac{C}{R}\mu(B(z, R) \cap E).$$

So by (4.3), we arrive at

$$\|D\varphi\|(B(z, 3R/4)) \le \frac{C}{R}\mu(B(z, R) \cap E). \tag{4.4}$$

Notice that $\varphi^t = \varphi = \chi_E$ in $B(z, R/2)$ and therefore by Corollary 2.4 and (4.4), we obtain

$$\begin{aligned}\left(\frac{\mu(B(z, R/2) \cap E)}{\mu(B(z, R/2))}\right)^{1-1/Q} =& \Big(\fint_{B(z,R/2)} \varphi^t \, d\mu \Big)^{1/t} \\ \le& CR \frac{\|D\varphi\|(B(z, 3R/4))}{\mu(B(z, 3R/4))} \\ \le& C\, \frac{\mu(B(z, R) \cap E)}{\mu(B(z, R))}.\end{aligned} \tag{4.5}$$

Up to now we have been using an adaptation of a part of the De Giorgi machinery. To complete the proof we adapt the proof of [DS1, Lemma 3.30]. Recall that by our assumption, if $x \in \Omega \cap \overline{E}$ and $r > 0$ then $\mu(B(x, r) \cap E) > 0$. For $x \in \Omega \cap \overline{E}$ and $z \in B(x, R/4)$, by the doubling property of μ, we have

$$\frac{\mu(B(z, R/2) \cap E)}{\mu(B(z, R/2))} \le C_d \frac{\mu(B(x, R) \cap E)}{\mu(B(x, R))}. \tag{4.6}$$

Let $\gamma_0 = 1/(C^Q C_d) > 0$, where C is as in (4.5). Suppose that

$$\frac{\mu(B(x, R) \cap E)}{\mu(B(x, R))} = \gamma < \gamma_0.$$

For positive integers j we set $B_j = B(z, R/2^j)$. Then by a repeated application of (4.5), with $t = Q/(Q-1) > 1$, we obtain

$$\begin{aligned}
\frac{\mu(B_j \cap E)}{\mu(B_j)} &\le \left(C\,\frac{\mu(B_{j-1} \cap E)}{\mu(B_{j-1})} \right)^{Q/(Q-1)} \\
&\le C^{Q/(Q-1)} \left(C\,\frac{\mu(B_{j-2} \cap E)}{\mu(B_{j-2})} \right)^{(Q/(Q-1))^2} \\
&\le C^{t+t^2+\dots+t^{j-1}} \left(\frac{\mu(B_1 \cap E)}{\mu(B_1)} \right)^{t^{j-1}} \\
&\le C^{Q\,t^{j-1}} (C_d \gamma)^{t^{j-1}} = \left(C^Q C_d \gamma \right)^{t^{j-1}},
\end{aligned}$$

where we also used (4.6). Since $C^Q C_d \gamma < 1$, it follows that for all $z \in B(x, R/4)$,

$$\liminf_{r \to 0} \frac{\mu(B(z,r) \cap E)}{\mu(B(z,r))} = 0,$$

and the Lebesgue differentiation theorem now implies that $\mu(B(x, R/4) \cap E) = 0$, resulting in a contradiction. Consequently, we have

$$\frac{\mu(B(x,R) \cap E)}{\mu(B(x,R))} \ge \gamma_0.$$

A similar argument for $X \setminus E$ also gives

$$\frac{\mu(B(x,R) \setminus E)}{\mu(B(x,R))} \ge \gamma_0.$$

This completes the proof. □

5. Porosity

By a result of David and Semmes [DS1], sets with quasiminimal surfaces in the complement of two disjoint cubes in the Euclidean space are uniform domains whose complements are also uniform (and indeed, are isoperimetric sets). Whether quasiminimal surfaces must enclose uniform domains is still open in the general metric setting, but now that we know that such sets have each boundary point as a point of density for both the set and its complement, we next show that these sets are uniformly locally porous. For us, the porosity is a reasonable weakening of the uniform domain condition.

By Theorem 4.2, without loss of generality we may assume that every point $x \in \Omega \cap \partial E$ has the property that

$$\frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \ge \gamma_0 \quad \text{and} \quad \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \ge \gamma_0$$

whenever $0 < r < \operatorname{diam}(X)/3$ is such that $B(x, 2r) \subset \Omega$.

Lemma 5.1. *Let E be a quasiminimal set in Ω and $x \in \Omega \cap \partial E$. Then there exist positive real numbers $r_0 < \operatorname{diam}(X)/3$ and $C > 0$ such that*

$$C^{-1}\,\frac{\mu(B(x,r))}{r} \le P(E, B(x,r)) \le C\,\frac{\mu(B(x,r))}{r},$$

whenever $0 < r < r_0$ is such that $B(x, 2r) \subset \Omega$. The constant C is independent of x and r.

Proof. The inequality on the left-hand side follows immediately from the density property of both E and $X \setminus E$ together with the relative isoperimetric inequality, so it suffices to prove the inequality on the right-hand side.

By the results in [KKST1, Lemma 6.2], we have that for all $0 < r < r_0$ there exists $r < \rho < 2r$ (indeed, a positive 1-dimensional measure amount of them) such that

$$P(B(x,\rho), X) \approx \frac{\mu(B(x,\rho))}{\rho}$$

and we can also choose such ρ so that $P(E, S(x,\rho)) = 0$, where

$$S(x,\rho) = \{z \in X : d(z,x) = \rho\}$$

is the sphere centered at x with radius ρ. Fix $\epsilon > 0$. Then $B(x,r) \subset \overline{B(x,\rho)} \subset B(x,\rho+\epsilon)$, and so by the quasiminimizer property of E we have

$$\begin{aligned} P(E, B(x,r)) &\le P(E, B(x,\rho+\epsilon)) \le K\,P(E \cup B(x,\rho), B(x,\rho+\epsilon)) \\ &\le K\,[P(B(x,\rho), B(x,\rho+\epsilon)) + P(E, B(x,\rho+\epsilon) \setminus B(x,\rho-\epsilon))] \\ &= K\,[P(B(x,\rho), X) + P(E, B(x,\rho+\epsilon) \setminus B(x,\rho-\epsilon))]\,. \end{aligned}$$

Since $P(E, S(x,\rho)) = 0$, we have that

$$\lim_{\epsilon \to 0} P(E, B(x,\rho+\epsilon) \setminus B(x,\rho-\epsilon)) = 0.$$

It follows from the choice of ρ and the doubling property of μ that

$$\begin{aligned} P(E, B(x,r)) &\le K\,P(B(x,\rho), X) \\ &\approx K\frac{\mu(B(x,\rho))}{\rho} \approx CK\frac{\mu(B(x,r))}{r}. \end{aligned}$$

□

Theorem 5.2. *If E is a quasiminimal set in Ω, then E and $X \setminus E$ are locally porous in Ω; that is, for every $x \in \Omega \cap \partial E$ there exists a positive $r_x < \operatorname{diam}(X)/3$ and $C \ge 1$ such that whenever $0 < r < r_x$, there are points $y \in B(x,r)$ and $z \in B(x,r)$ such that*

$$B(y, r/C) \subset E \cap \Omega \quad \text{and} \quad B(z, r/C) \subset X \setminus E.$$

The constant C is independent of x, r. Furthermore, r_x depends on x only so far as to have $B(x, 10r_x) \subset \Omega$.

Proof. Let r_0 be as in Lemma 5.1. Fix $x \in \Omega \cap \partial E$. For $0 < r < \operatorname{diam}(X)/3$ such that such that $r < r_0$ and $B(x, 4r) \subset \Omega$, let $0 < \rho \le r$ be such that for all $y \in B(x,r) \cap E$ the ball $B(y, \rho)$ intersects $X \setminus E$. Note that $\rho = r$ would satisfy this requirement. If there is some ρ with $r/20\lambda < \rho < r/10\lambda$ such that the above condition fails, then there is some $y \in B(x,r) \cap E$ such that $B(y, r/20\lambda) \subset E$, and the porosity requirement is satisfied at the scale r. If not, we can choose $\rho < r/10\lambda$ so that for every $y \in B(x,r) \cap E$, the set $B(y,\rho) \setminus E$ is non-empty. In this case, we can cover $B(x,r) \cap E$ by a family of balls $\{B(y_i, 10\lambda\rho)\}_i$, such that the collection $\{B(y_i, 2\lambda\rho)\}_i$ is pairwise disjoint. Then by the doubling property of μ together with the density property of the previous section,

$$\begin{aligned}\gamma_0\, \mu(B(x,r)) &\le \mu(B(x,r) \cap E)\\ &\le \sum_i \mu(B(y_i, 10\lambda\rho)) \le C \sum_i \mu(B(y_i,\rho)).\end{aligned}$$

Note that by the density results of the previous section,

$$\mu(B(y_i, 2\rho) \cap E) \ge C\, \mu(B(y_i, 2\rho))$$

and

$$\mu(B(y_i, 2\rho) \setminus E) \ge C\, \mu(B(y_i, 2\rho)).$$

Hence by the relative isoperimetric inequality,

$$P(E, B(y_i, 2\lambda\rho)) \ge \frac{1}{C} \frac{\mu(B(y_i,\rho))}{\rho}.$$

By the pairwise disjointness property, we have

$$\begin{aligned}P(E, B(x, 2r)) &\ge \sum_i P(E, B(y_i, 2\lambda\rho))\\ &\ge \sum_i \frac{1}{C} \frac{\mu(B(y_i,\rho))}{\rho} \ge \frac{1}{C} \frac{\mu(B(x,r))}{\rho}.\end{aligned}$$

By Lemma 5.1, we now have

$$\frac{1}{C} \frac{\mu(B(x,r))}{\rho} \le C \frac{\mu(B(x,r))}{r},$$

and consequently $\rho \ge r/C$. This means that there is a point $y \in B(x,r) \cap E$ such that $B(y, r/2C) \subset E$, thus proving the porosity of E. A similar argument with $X \setminus E$, which also is a quasiminimal set since E is a quasiminimal set, gives the porosity of $X \setminus E$ in Ω. □

The following corollary is a consequence of the porosity property proved above. Note that in the Euclidean setting, if a set satisfies the conclusion of the following corollary, then it is uniformly rectifiable; see for example the discussion in [DS1]. Indeed, David and Semmes use this fact together with the notion of tangent hyperplanes to prove that E then has to be locally a John domain. Recall that a domain E is a *local John domain* if there exist constants $C, \delta > 0$ such that whenever $x_0 \in \partial E$ and $0 < r < \delta$, for all points $x \in B(x_0, r) \cap E$ there is a point $y \in E \cap B(x_0, Cr)$ with $\delta_E(y) \geq r/C$ and a curve $\gamma \subset E$, called a John curve, connecting x to y satisfying

$$\ell(\gamma_{x,z}) \leq C\, \delta_E(z)$$

for all $z \in \gamma$; here $\gamma_{x,z}$ denotes a subcurve of γ with end points x and z.

As a consequence of the following corollary together with the results from [LT, Theorem 4.1], the *Assouad dimension* of ∂E is at most $Q-1$, and by [LT, Theorem 4.2], the Assouad dimension of ∂E is $Q-1$ if the measure μ is Ahlfors Q-regular, that is,

$$\mu(B) \approx \operatorname{rad}(B)^Q.$$

In [LT] the supremum of all such possible α is called the *Aikawa co-dimension* of ∂E. We also refer to [LT] for the definition of the *Minkowski content of codimension* α. Let $\delta_E(x)$ denote $\operatorname{dist}(x, X \setminus E)$.

Corollary 5.3. *If E is a quasiminimal set in a domain Ω, then $\Omega \cap \partial E$ has finite Minkowski content of codimension α for $0 < \alpha < 1$, and*

$$\int_{B(x_0,r)\cap E} \frac{1}{\delta_E(y)^\alpha}\, d\mu(y) \leq C\, \frac{\mu(B(x_0, r))}{r^\alpha}$$

for all $x_0 \in \partial E$ and $r > 0$ such that $r < \operatorname{diam}(X)/3$ and $B(x_0, 10\lambda r) \subset \Omega$. Furthermore, if $\alpha \geq 1$ then

$$\int_{B(x_0,r)\cap E} \frac{1}{\delta_E(y)^\alpha}\, d\mu(y) = \infty.$$

Proof. By the Cavalieri principle and Theorem 4.2, we see that

$$\begin{aligned}
&\int_{B(x_0,r)\cap E} \frac{1}{\delta_E(y)^\alpha}\, d\mu(y) \\
&= \int_0^\infty \mu\left(\{y \in B(x_0, r) \cap E : \delta_E(y)^{-\alpha} > t\}\right) dt \\
&\approx \int_0^\infty \mu\left(\{y \in B(x_0, r) \cap E : \delta_E(y) < \tau\}\right) \frac{d\tau}{\tau^{1+\alpha}} \\
&\approx \int_0^r \mu\left(\{y \in B(x_0, r) \cap E : \delta_E(y) < \tau\}\right) \frac{d\tau}{\tau^{1+\alpha}} \\
&\qquad + \int_r^\infty \frac{\mu(E \cap B(x_0, r))}{\tau^{1+\alpha}}\, d\tau \\
&\approx \int_0^r \frac{\mu(E_\tau^+ \cap B(x_0, r))}{\tau^{1+\alpha}}\, d\tau + C\, \frac{\mu(B(x_0, r))}{r^\alpha}.
\end{aligned}$$

Here

$$E_\tau^+ = \bigcup_{x \in \partial E} B(x, \tau) \cap E.$$

To compute the measure of $E_\tau^+ \cap B(x_0, r)$, we can cover $E_\tau^+ \cap B(x_0, r)$ by countably many balls $5\lambda B_j$ with radius $5\lambda\tau$, such that λB_j are pairwise disjoint. We also ensure that $5\lambda B_j$ has its center located in $B(x_0, r) \cap \partial E$. Now we have by the relative isoperimetric inequality and the porosity of E and $\Omega \setminus E$ that

$$\mu(B_j) \le C\tau\, P(E, \lambda B_j).$$

Thus by the doubling property of μ we conclude that

$$\begin{aligned}\mu(E_\tau^+ \cap B(x_0, r)) &\le \sum_j \mu(5\lambda B_j) \le C \sum_j \mu(B_j)\\ &\le C\tau \sum_j P(E, \lambda B_j) \le C\tau\, P(E, B(x_0, 2\lambda r)).\end{aligned}$$

By Lemma 5.1, we know that

$$P(E, B(x_0, 2\lambda r)) \approx \frac{\mu(B(x_0, r))}{r}.$$

Hence we can conclude that

$$\int_0^r \frac{\mu(E_\tau^+ \cap B(x_0, r))}{\tau^{1+\alpha}}\, d\tau \le C\, \frac{\mu(B(x_0, r))}{r} \int_0^r \frac{d\tau}{\tau^\alpha},$$

and so

$$\int_{B(x_0,r)\cap E} \frac{1}{\delta_E(y)^\alpha}\, d\mu(y) \le C\, \frac{\mu(B(x_0, r))}{r^\alpha}.$$

To see the second part of the claim, we can use Lemma 5.1 from which we get that $\mu(B_j) \approx \tau\, P(E, B_j)$. For $\tau < 4r/(30\lambda)$, we know that whenever λB_j contains a point in $\partial E \cap B(x_0, r/(5\lambda))$, it follows that $5\lambda B_j \subset B(x_0, r)$. Since the collection $\{5\lambda B_j\}_j$ covers $E_\tau^+ \cap B(x_0, r)$, the balls $\{5\lambda B_j\}_j$ for which $5\lambda B_j$ is contained in $B(x_0, r)$ cover $\partial E \cap B(x_0, r/(5\lambda))$. From this we conclude that

$$\begin{aligned}\mu(E_\tau^+ \cap B(x_0, r)) &\ge \frac{1}{C} \sum_{\{j:5\lambda B_j\subset B(x_0,r)\}} \mu(B_j)\\ &\ge \frac{1}{C} \sum_{\{j:5\lambda B_j\subset B(x_0,r)\}} \mu(5\lambda B_j)\\ &\ge \frac{\tau}{C} \sum_{\{j:5\lambda B_j\subset B(x_0,r)\}} P(E, 5\lambda B_j)\\ &\ge \frac{\tau}{C} P\Big(E, \bigcup_{\{j:5\lambda B_j\subset B(x_0,r)\}} 5\lambda B_j\Big)\\ &\ge \frac{\tau}{C} P(E, B(x_0, r/(5\lambda))).\end{aligned}$$

This implies that

$$\int_{B(x_0,r)\cap E} \frac{1}{\delta_E(y)^\alpha}\, d\mu(y) \ge \frac{1}{C} P(E, B(x_0, r/(5\lambda))) \int_0^{4r/(30\lambda)} \frac{d\tau}{\tau^\alpha} = \infty$$

when $\alpha \geq 1$. $\square$

Remark 5.4. The above proof also indicates that

$$C^{-1}\tau\, P(E, B(x_0, r/(5\lambda))) \leq \mu(E^+_\tau \cap B(x_0, r)) \leq C\tau\, P(E, B(x_0, 2\lambda r))$$

whenever $x_0 \in \partial E$ and $B(x_0, 10\lambda r) \subset \Omega$ with $r < \operatorname{diam}(X)/3$.

Because of the porosity (Theorem 5.2), given every $x \in \partial E$, for sufficiently small $r > 0$ we have $y_r \in \operatorname{int}(E)$ with $d(y_r, x) < r$. Similarly we also have $z_r \in \operatorname{int}(\Omega \setminus E)$ such that $d(z_r, x) < r$. Therefore $\partial E \subset \partial\operatorname{int}(E)$. Because we always have $\partial\operatorname{int}(E) \subset \partial E$, we have $\overline{E} = \overline{\operatorname{int}(E)}$, and so we can replace E with its interior $\operatorname{int}(E)$. Since $\mathcal{H}$ is a σ-finite measure on $\partial E = \partial_* E$, it follows that $\mu(\partial E) = 0$. Hence if $\operatorname{int}(E) \subset F \subset \overline{E}$, then the perimeter measure of F and the perimeter measure of E are the same, and so F is also a quasiminimal set.

We conclude this section with the following open question: if E is a domain of locally quasiminimal surface, then is it true that E is a local John domain? In the Euclidean setting this question was answered in the affirmative by David and Semmes [DS1]. The crucial part of the proof of [DS1] is to show that the boundary of a set of quasiminimal surface lies locally close to a hyperplane; in the setting of metric measure spaces one does not have such a structure, and the challenge is to construct an alternative approach.

6. Support of Poincaré inequality in $(\mathbb{R}^n, d, \mu)$.

A non-negative measurable function ω on $\mathbb{R}^n$ is a weight on $\mathbb{R}^n$ if ω is positive almost everywhere. A weight ω is a *strong* A_∞*-weight* on $\mathbb{R}^n$ if there is a metric d on $\mathbb{R}^n$ and a constant $C \geq 1$ such that, with the measure μ on $\mathbb{R}^n$ defined by the density condition

$$d\mu(x) = \omega(x)\, d\mathcal{L}^n(x),$$

whenever $x, y \in \mathbb{R}^n$ and

$$B_{x,y} = B((x+y)/2, |x-y|/2)$$

is the smallest Euclidean ball in $\mathbb{R}^n$ containing x and y in the closure, then

$$\frac{1}{C}\mu(B_{x,y})^{1/n} \leq d(x,y) \leq C\mu(B_{x,y})^{1/n}.$$

Since strong A_∞-weights are A_∞-weights, ω is a Muckenhoupt A_p-weight for some p. It follows that μ is a doubling measure with respect to the Euclidean metric. Hence, we have a constant $C \geq 1$ such that whenever $x, y \in \mathbb{R}^n$,

$$\frac{1}{C}\mu(B(x, |x-y|)) \leq d(x,y)^n \leq C\mu(B(x, |x-y|)). \tag{6.1}$$

For properties of strong A_∞-weights, we refer the interested reader to [DS2].

The metric space we consider here is $(\mathbb{R}^n, d, \mu)$. Balls in this metric are denoted with the superscript d in order to distinguish them from the Euclidean balls. So

$$B^d(x,r) = \{y \in \mathbb{R}^n \,:\, d(x,y) < r\},$$

while

$$B(x,r) = \{y \in \mathbb{R}^n \,:\, |x-y| < r\}.$$

We note that the topology generated by the metric d is the same one as the Euclidean topology.

In this section we show that when the measure μ on $\mathbb{R}^n$ is given by a strong A_∞-weight, then the space $(\mathbb{R}^n, d, \mu)$ is an Ahlfors regular space supporting a $(1,1)$-Poincaré inequality. We do not know whether every strong A_∞-weight supports a (1,1)-Poincaré inequality with respect to the Euclidean metric (though it does support a (1,p)-Poincaré inequality for some $1 \le p < \infty$), and so in general the weighted Euclidean space $(\mathbb{R}^n, |\cdot|, \mu)$ perhaps may not support a (1,1)-Poincaré inequality. For more discussion on Poincaré inequalities satisfied by strong A_∞-weights, see [Bj]. The next result states that $(\mathbb{R}^n, d, \mu)$ is Ahlfors n-regular.

Lemma 6.2. *There is a constant $C \ge 1$ such that whenever $x \in \mathbb{R}^n$ and $r > 0$, we have*

$$\frac{1}{C} r^n \le \mu(B^d(x,r)) \le C r^n.$$

Proof. Let $y \in \partial B^d(x,r)$ be such that

$$|x-y| = \sup\{|x-z| : z \in \partial B^d(x,r)\}.$$

Note that as $\overline{B}^d(x,r)$ is compact, such y exists. Then $B^d(x,r) \subset B(x, |x-y|)$, and so by (6.1), we have

$$\mu(B^d(x,r)) \le \mu(B(x,|x-y|)) \le C d(x,y)^n = C r^n.$$

Next, let $z \in \partial B^d(x,r)$ be such that

$$|x-z| = \inf\{|x-z| \,:\, z \in \partial B^d(x,r)\}.$$

Then $B(x,|x-z|) \subset B^d(x,r)$, and so again by (6.1) we have

$$\mu(B^d(x,r)) \ge \mu(B(x,|x-z|)) \ge \frac{1}{C} d(x,z)^n = \frac{r^n}{C},$$

completing the proof. □

We point out that ds represents the arc length measure with respect to the Euclidean metric in this section.

Lemma 6.3. *There is a Borel set $F \subset \mathbb{R}^n$ with $|F| = 0$ such that whenever γ is a curve in $\mathbb{R}^n$ which is rectifiable with respect to the Euclidean metric, it is rectifiable with respect to the metric d if $\int_\gamma (\infty\chi_F + \omega^{1/n})\, ds$*

is finite. In this case we also have that the length of γ *with respect to the metric* d*, denoted* $\ell_d(\gamma)$*, satisfies*

$$\ell_d(\gamma) \approx \int_\gamma \omega^{1/n}\, ds.$$

Proof. Fix $x \in \mathbb{R}^n$. Then for $y \in \mathbb{R}^n$, by (6.1) we have

$$\begin{aligned}\frac{d(x,y)}{|x-y|} &\approx \frac{\mu(B(x,|x-y|))^{1/n}}{|x-y|} \\ &\approx \frac{1}{|B(x,|x-y|)|^{1/n}} \left(\int_{B(x,|x-y|)} \omega(z)\, dz\right)^{1/n} \\ &= \left(\frac{1}{|B(x,|x-y|)|} \int_{B(x,|x-y|)} \omega(z)\, dz\right)^{1/n}.\end{aligned}$$

Denote

$$\underline{\rho}(x) = \liminf_{y\to x} \frac{d(x,y)}{|x-y|} \quad \text{and} \quad \overline{\rho}(x) = \limsup_{y\to x} \frac{d(x,y)}{|x-y|}.$$

Since $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (the integrals being taken with respect to the Lebesgue measure), we see by the Lebesgue differentiation theorem that for almost every $x \in \mathbb{R}^n$,

$$\begin{aligned}\omega(x)^{1/n} &= \lim_{y\to x} \left(\frac{1}{|B(x,|x-y|)|} \int_{B(x,|x-y|)} \omega(z)\, dz\right)^{1/n} \\ &\le C\underline{\rho}(x) \le C\overline{\rho}(x) \\ &\le C^2 \lim_{y\to x} \left(\frac{1}{|B(x,|x-y|)|} \int_{B(x,|x-y|)} \omega(z)\, dz\right)^{1/n} = C^2\omega(x)^{1/n}.\end{aligned}$$

Let F be the set of all non-Lebesgue points of ω; then $\mu(F) = |F| = 0$. Let γ be a Euclidean rectifiable curve with $\int_\gamma (\infty\chi_F + \omega^{1/n})\, ds < \infty$. Then $\mathcal{H}^1(\gamma^{-1}(\gamma\cap F)) = 0$, and in addition we have

$$\int_\gamma \underline{\rho}\, ds \le \ell_d(\gamma) \le \int_\gamma \overline{\rho}\, ds,$$

where $\ell_d(\gamma)$ is the length of γ in the metric d. It follows that

$$\int_\gamma \omega^{1/n}\, ds \le C\ell_d(\gamma) \le C^2 \int_\gamma \omega^{1/n}\, ds. \qquad \square$$

Lemma 6.4. *If* u *is Lipschitz continuous with respect to the metric* d*, then* $u \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $h \in \mathbb{R}$, and let e_j, $j = 1, \dots, n$, denote the standard orthonormal basis for $\mathbb{R}^n$. Since u is Lipschitz with respect to the metric d, we see by (6.1) that

$$|u(x+he_j) - u(x)| \le Cd(x, x+he_j) \le C\mu(B(x,|h|))^{1/n}.$$

Thus, for $a \in \mathbb{R}^n$, $R > 0$ and $|h| \le R$, by Fubini's theorem we see that

$$\begin{aligned}
\int_{B(a,R)} |u(x+he_j) - u(x)|^n \, dx &\le C \int_{B(a,R)} \mu(B(x,|h|)) \, dx \\
&= C \int_{B(a,R)} \int_{B(a,2R)} \chi_{B(x,|h|)}(y) \, d\mu(y) \, dx \\
&= C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{B(a,R)}(x) \, \chi_{B(x,|h|)}(y) \, d\mu(y) \, dx \\
&= C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{B(a,R)}(x) \, \chi_{B(y,|h|)}(x) \, dx \, d\mu(y) \\
&\le C|h|^n \int_{B(a,2R)} d\mu(y) = C|h|^n \, \mu(B(a,2R)).
\end{aligned}$$

So whenever $\Omega \Subset \mathbb{R}^n$ is an open set and $\Omega' \Subset \Omega$, we can cover Ω' by a countable collection $\{B_i\}$ of balls of radius $R = \operatorname{dist}(\Omega', \mathbb{R}^n \setminus \Omega)/2$ centered at points in Ω' and with bounded overlap of the balls $\{2B_i\}$, to obtain that

$$\begin{aligned}
\int_{\Omega'} \frac{|u(x+he_j) - u(x)|^n}{|h|^n} \, dx &\le \sum_i \int_{B_i} \frac{|u(x+he_j) - u(x)|^n}{|h|^n} \, dx \\
&\le C \sum_i \mu(2B_i) \ \le C\mu(\Omega) < \infty.
\end{aligned}$$

So by [GT, Lemma 7.24], we see that $u \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$. $\square$

By Lemma 6.4 we know that if u is Lipschitz continuous with respect to the metric d, then it is in the Euclidean Sobolev class $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$; it then follows that $|\nabla u|$ is a 1-weak upper gradient (in the Euclidean metric) of u, see [BB].

In order to prove that $(\mathbb{R}^n, d, \mu)$ supports a $(1,1)$-Poincaré inequality, it suffices to prove the inequality for Lipschitz functions with respect to d and their *continuous* upper gradients in the metric d; see for example [Ke]. Let u be a Lipschitz function with continuous upper gradient g in $(\mathbb{R}^n, d, \mu)$.

Lemma 6.5. *There is a constant $C > 0$ which is independent of u and g such that*

$$|\nabla u(x)| \le C\omega(x)^{1/n} g(x)$$

for almost every $x \in \mathbb{R}^n$.

Proof. In this proof, Mod_1 denotes the 1-modulus with respect to the Euclidean metric and Lebesgue measure. As in the proof of Lemma 6.4, we let e_j, $j = 1, \dots, n$, denote the canonical orthonormal basis of $\mathbb{R}^n$. For each $j = 1, \dots, n$, we consider the collection Γ_j of all line segments parallel to the direction of e_j. From [V, Section 7.2, page 21], we know that whenever $\Gamma \subset \Gamma_j$ satisfies $\mathrm{Mod}_1(\Gamma) = 0$, we have

$$\Big|\bigcup_{\gamma\in\Gamma} \cup_{z\in\gamma} z\Big| = 0.$$

Here the 1-modulus is taken with respect to the Euclidean metric.

Let Γ_a denote the collection of all compact line segments in Γ_j for which

$$\int_\gamma (\infty\chi_F + \omega^{1/n})\, ds = \infty$$

(with F as in Lemma 6.3). Since by Hölder's inequality we know that $\omega^{1/n} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, it follows that $\mathrm{Mod}_1(\Gamma_a) = 0$; see for example [KoMc]. Observe that the 1-modulus is taken with respect to the Euclidean metric.

Since u is Lipschitz continuous with respect to d and the topology induced by d and the Euclidean topology coincide, we see that u is continuous in the Euclidean space $\mathbb{R}^n$. By Lemma 6.4 we also know that $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. It follows from the discussion in [V] that u is absolutely continuous on Mod_1-almost every compact Euclidean rectifiable curve in $\mathbb{R}^n$. Let Γ_b denote the collection of all line segments γ in Γ_j along which $(u, |\nabla u|)$ does not support the upper gradient inequality (2.1); that is, there is some sub-segment β of γ for which the inequality (2.1) fails. Since $|\nabla u|$ is a 1-weak upper gradient of u with respect to the Euclidean metric and Lebesgue measure on $\mathbb{R}^n$, it follows that $\mathrm{Mod}_1(\Gamma_b) = 0$. Furthermore, let Γ_c denote the collection of all segments $\gamma \in \Gamma_j$ for which $\int_\gamma |\nabla u|\, ds$ is infinite. Then $\mathrm{Mod}_1(\Gamma_c) = 0$.

Because g is an upper gradient of u in the metric d, by Lemma 6.3 we know that whenever $\gamma \in \Gamma_j \setminus \Gamma_a$, for all sub-segments β of γ we have

$$|u(x_\beta) - u(y_\beta)| \le C \int_\beta \omega^{1/n} g\, ds.$$

Here x_β and y_β denote the two end points of β. It follows that if $\gamma \notin \Gamma_b \cup \Gamma_c$ as well, then for $\mathcal{H}^1$-almost every point $x \in \beta$,

$$|\partial_j u(x)| \le C\omega^{1/n}(x) g(x).$$

Note that $\mathrm{Mod}_p(\Gamma_a \cup \Gamma_b \cup \Gamma_c) = 0$. Hence by the use of [V] again, we see that for almost every $x \in \mathbb{R}^n$ we have

$$|\partial_j u(x)| \le C\omega^{1/n}(x) g(x).$$

Now the conclusion follows by summing up over $j = 1, \ldots, n$. □

We next compare Euclidean balls with balls in the metric d.

Lemma 6.6. *There is a constant $C > 0$ such that whenever $x \in \mathbb{R}^n$ and $r > 0$, there exist positive numbers λ_x^r and τ_x^r such that*

$$B(x, \lambda_x^r r) \subset B^d(x, r) \subset B(x, C\lambda_x^r r) \tag{6.7}$$

and

$$B^d(x, \tau_x^r r) \subset B(x,r) \subset B^d(x, C\tau_x^r r). \tag{6.8}$$

Proof. Since μ is a doubling measure on the Euclidean space $\mathbb{R}^n$ and $\mathbb{R}^n$ is uniformly perfect, there is a constant $Q_1 > 0$ such that whenever $0 < r < R$,

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \le C \left(\frac{r}{R}\right)^{Q_1}. \tag{6.9}$$

Let $z_x^r, y_x^r \in \partial B^d(x,r)$ such that for $y \in \partial B^d(x,r)$ we have $|x - z_x^r| \le |x-y| \le |x - y_x^r|$. Then set

$$\lambda_x^r = \frac{|x - z_x^r|}{r}.$$

We have

$$B(x, |x-z_x^r|) = B(x, \lambda_x^r r) \subset B^d(x,r) \subset B(x, |x - y_x^r|).$$

Because of the upper mass bound (6.9), by the twice-repeated use of (6.1),

$$\frac{1}{C} \le \frac{\mu(B(x,|x-z_x^r|))}{\mu(B(x,|x-y_x^r|))} \le C \left(\frac{|x-z_x^r|}{|x-y_x^r|}\right)^{Q_1},$$

and so it follows that $|x - y_x^r| \le C|x - z_x^r|$, whence we obtain that $B(x, |x-y_x^r|) \subset B(x, C|x - z_x^r|)$, and this proves (6.7).

To prove (6.8), we consider $w_x^r \in \partial B(x,r)$ such that $d(x, w_x^r) \le d(x,y)$ whenever $y \in \partial B(x,r)$, and set

$$\tau_x^r = \frac{d(x, w_x^r)}{r}.$$

As in the previous argument, we consider also $a_x^r \in \partial B(x,r)$ such that $d(x, a_x^r) \ge d(x,y)$ for all $y \in \partial B(x,r)$, and obtain by the use of Lemma 6.2 that

$$\frac{\mu(B^d(x, d(x,w_x^r)))}{\mu(B^d(x, d(x,a_x^r)))} \approx \left(\frac{d(x,w_x^r)}{d(x,a_x^r)}\right)^n,$$

and from (6.1) we also see that

$$\mu(B^d(x, d(x,w_x^r))) \approx d(x,w_x^r)^n \approx \mu(B(x, |x-w_x^r|)) = \mu(B(x,r)).$$

A similar argument as above also shows that

$$\mu(B^d(x, d(x,a_x^r))) \approx \mu(B(x,r)).$$

It follows that

$$\left(\frac{d(x,w_x^r)}{d(x,a_x^r)}\right)^n \ge \frac{1}{C},$$

that is, $d(x, a_x^r) \le C\, d(x, w_x^r)$. From this (6.8) follows. $\square$

Now we are ready to prove the main result of this section. The paper [DS2] proved that $(\mathbb{R}^n, |\cdot|, \omega dx)$, where $|\cdot|$ the Euclidean metric, satisfies a *two-weighted*version of a (1,1)-Poincaré inequality; see the proof of the following proposition. We will prove here that the metric measure space $(\mathbb{R}^n, d, \omega dx)$ also satisfies a (1,1)-Poincaré inequality. As far as we know, the validity of (1,1)-Poincaré inequality with respect to the metric d has not appeared in the literature so far.

Proposition 6.10. *The metric measure space $(\mathbb{R}^n, d, \mu)$ is an Ahlfors n-regular space supporting a $(1,1)$-Poincaré inequality.*

Proof. As pointed out by [Ke], it suffices to prove the inequality for functions u that are Lipschitz continuous on $(\mathbb{R}^n, d)$ with continuous upper gradient g. By [DS2, Inequality (1.10)], we know that when $x \in \mathbb{R}^n$ and $r > 0$, the following two-weighted version of a (1,1)-Poincaré inequality on Euclidean balls holds:

$$\begin{aligned}\int_{B(x,C\lambda_x^r r)} \fint_{B(x,C\lambda_x^r r)} &|u(x)-u(y)|\, d\mu(x)\, d\mu(y) \\ &\le C\mu(B(x,C\lambda_x^r r))^{1/n} \int_{B(x,2C\lambda_x^r r)} \omega(x)^{-1/n} |\nabla u(x)|\, d\mu(x).\end{aligned}$$

By the doubling property of μ on the Euclidean $\mathbb{R}^n$, Lemma 6.5, Lemma 6.2, and Lemma 6.6, we see that

$$\begin{aligned}\int_{B(x,C\lambda_x^r r)} \fint_{B(x,C\lambda_x^r r)} &|u(x)-u(y)|\, d\mu(x)\, d\mu(y) \\ &\le C\mu(B(x,C\lambda_x^r r))^{1/n} \int_{B(x,2C\lambda_x^r r)} g(x)\, d\mu(x) \\ &\le C\mu(B(x,\lambda_x^r r))^{1/n} \int_{B(x,2C\lambda_x^r r)} g(x)\, d\mu(x) \\ &\le C\mu(B^d(x,r))^{1/n} \int_{B(x,2C\lambda_x^r r)} g(x)\, d\mu(x) \\ &\le Cr \int_{B(x,2C\lambda_x^r r)} g(x)\, d\mu(x).\end{aligned}$$

Hence,

$$\begin{aligned}\inf_{c\in\mathbb{R}} \int_{B^d(x,r)} |u-c|\,d\mu &\le \int_{B^d(x,r)} |u-u_{B(x,C\lambda_x^r r)}|\,d\mu\\ &\le \int_{B(x,C\lambda_x^r r)} |u-u_{B(x,C\lambda_x^r r)}|\,d\mu\\ &\le C\int_{B(x,C\lambda_x^r r)} \fint_{B(x,C\lambda_x^r r)} |u(x)-u(y)|\,d\mu(x)\,d\mu(y)\\ &\le Cr\int_{B(x,2C\lambda_x^r r)} g(x)\,d\mu(x).\end{aligned}$$

By Lemma 6.6 we have

$$B(x,2C\lambda_x^r r)\subset B^d(x,2C^2\tau_x^{2C\lambda_x^r r}\,\lambda_x^r r).$$

Note that $B^d(x,\tau_x^{2C\lambda_x^r r}\,2C\lambda_x^r\,r)$ is the largest metric ball centered at x that fits inside the Euclidean ball $B(x,2C\lambda_x^r r)$. Let $\rho>0$ be such that $B^d(x,\rho)$ is the largest metric ball centered at x and contained in the Euclidean ball $B(x,\lambda_x^r r)$, and let $y_1\in\partial B^d(x,\rho)\cap\partial B(x,\lambda_x^r r)$, and correspondingly let

$$y_2\in\partial B^d(x,\tau_x^{2C\lambda_x^r r}\,2C\lambda_x^r r)\cap\partial B(x,2C\lambda_x^r r).$$

Then by (6.1),

$$\rho=d(x,y_1)\approx\mu(B(x,\lambda_x^r r))^{1/n}$$

and again by (6.1),

$$\tau_x^{2C\lambda_x^r r}\,2C\lambda_x^r r=d(x,y_2)\approx\mu(B(x,2C\lambda_x^r r))^{1/n}.$$

By the doubling property of μ in the Euclidean space $\mathbb{R}^n$, we see that

$$\mu(B(x,\lambda_x^r r))\approx\mu(B(x,2C\lambda_x^r r)).$$

It follows that $\rho\approx\tau_x^{2C\lambda_x^r r}\,2C\lambda_x^r r$. On the other hand, since $B^d(x,\rho)$ is the largest metric ball centered at x and fitting inside the Euclidean ball $B(x,\lambda_x^r r)$, and by the construction of λ_x^r from Lemma 6.6 we know that $B(x,\lambda_x^r r)\subset B^d(x,r)$, we can conclude that $\rho\le r$. Hence

$$\tau_x^{2C\lambda_x^r r}2C\lambda_x^r r\le C\rho\le Cr.$$

Thus we have

$$B(x,2C\lambda_x^r r)\subset B^d(x,2C^2\tau_x^{2C\lambda_x^r r}\,\lambda_x^r r)\subset B^d(x,C_2 r),$$

from which we conclude that

$$\begin{aligned}\inf_{c\in\mathbb{R}}\int_{B^d(x,r)}|u-c|\,d\mu &\le Cr\int_{B(x,2C\lambda_x^r r)} g(x)\,d\mu(x)\\ &\le Cr\int_{B^d(x,C_2 r)} g\,d\mu,\end{aligned}$$

which is equivalent to the $(1,1)$-Poincaré inequality on $(\mathbb{R}^n,d,\mu)$ because the constants C, C_2 are independent of x,r,u,g. $\square$

7. Rectifiability of quasiminimal surfaces in Euclidean spaces with strong A_∞ weights

In this section we apply the results from the earlier part of this paper (Sections 4 and 5) to the setting of weighted Euclidean spaces where the weight ω is a strong A_∞-weight.

If ω is an A_1-weight, then the space $(\mathbb{R}^n, |\cdot|, \mu)$ is a doubling metric measure space supporting a $(1,1)$-Poincaré inequality. However, not all strong A_∞-weights are A_1-weights, but as shown in the previous section, the metric measure space $(\mathbb{R}^n, d, \mu)$ is also an Ahlfors n-regular space supporting a $(1,1)$-Poincaré inequality. In this section we will prove that a set $E \subset \mathbb{R}^n$ that has a locally quasiminimal boundary surface in $(\mathbb{R}^n, d, \mu)$ will have a rectifiable boundary. Here of course, the notion of rectifiability is in terms of the Euclidean metric. As in [Mat, page 204, Definition 15.3], we say that a set $A \subset \mathbb{R}^n$ is m-rectifiable if there is a countable collection $\{f_i\}$ of Euclidean Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\mathcal{H}^m_{\mathrm{Euc}}(A \setminus \bigcup_i f_i(\mathbb{R}^m)) = 0.$$

In this section, we consider the issue of whether the boundary ∂E of the set with quasiminimal boundary surface in $(\mathbb{R}^n, d, \mu)$ is $(n-1)$-rectifiable in the above sense (with respect to the Euclidean metric). We also recall that a set $K \subset \mathbb{R}^n$ is purely m-unrectifiable if whenever $A \subset \mathbb{R}^n$ is m-rectifiable, we have $\mathcal{H}^m_{\mathrm{Euc}}(K \cap A) = 0$.

Let $E \subset \Omega \subset \mathbb{R}^n$ be a set of finite perimeter with locally quasiminimal boundary surface with respect to the metric d and measure μ. Then the results obtained in the previous sections of this note apply to E. So we may assume that $E = \mathrm{int}(\overline{E})$.

If $n = 1$, then the fact that we can choose $E = \mathrm{int}(\overline{E})$ tells us that E is a pairwise disjoint union of countably many open intervals in $\mathbb{R}$. Thus it is immediate that E is Euclidean rectifiable in $\mathbb{R}$. Therefore, in the rest of the section we will assume that $n \geq 2$.

Lemma 7.1. *Let $\Lambda > 0$ and*

$$A_\Lambda = \left\{ x \in \partial E : \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n} \geq \Lambda \right\}.$$

Then

$$\mathcal{H}^{n-1}_{\mathrm{Euc}}(A_\Lambda) \leq \frac{C}{\Lambda^{(n-1)/n}} P(E, \Omega).$$

Note that in the above limes supremum condition, if we replace $B(x,r)$ with $B^d(x,r)$, then by Lemma 6.2 we have $A_\Lambda = \emptyset$ whenever $\Lambda > C$.

Proof. Fix $0 < \delta < \Lambda$; then by the condition imposed on A_Λ, for every $x \in A_\Lambda$ we can find $0 < r_x < \delta/5$ such that

$$\mu(B(x, r_x)) \geq (\Lambda - \delta) r_x^n.$$

Note also that if $x \in A_\Lambda$ and $r > 0$ such that

$$\mu(B(x,r)) \geq (\Lambda - \delta) r^n,$$

then for any $y \in \partial B(x,r)$ we have that

$$(\Lambda - \delta) r^n \leq \mu(B(x,r)) = \mu(B(x,|x-y|)) \leq C d(x,y)^n,$$

and so we have $d(x,y) \geq C\,(\Lambda-\delta)^{1/n}\, r$. It follows that

$$B^d(x, \tfrac{(\Lambda-\delta)^{1/n}}{C}\, r) \subset B(x,r).$$

The family

$$B^d(x, \tfrac{(\Lambda-\delta)^{1/n}}{5C}\, r_x), \quad x \in A_\Lambda,$$

forms a cover of A_Λ, and hence we can find a pairwise disjoint countable subfamily

$$\left\{ B^d(x_i, \tfrac{(\Lambda-\delta)^{1/n}}{5C}\, r_i) \right\}_i$$

such that

$$A_\Lambda \subset \bigcup_i B^d(x_i, \tfrac{(\Lambda-\delta)^{1/n}}{C} r_i).$$

Hence by Lemma 6.2 and the fact that the balls $\{B(x_i, r_i)\}_i$ therefore also form a cover of A_Λ by Euclidean balls,

$$\begin{aligned}
\mathcal{H}^{n-1}_{\mathrm{Euc},\delta}(A_\Lambda) &\leq \sum_i (r_i)^{n-1} = \sum_i \frac{r_i^n}{r_i} \\
&\leq \frac{C}{\Lambda - \delta} \sum_i \frac{\mu(B^d(x_i, \frac{(\Lambda-\delta)^{1/n}}{C} r_i))}{r_i} \\
&\leq \frac{C}{\Lambda - \delta} \sum_i \frac{\mu(B^d(x_i, \frac{(\Lambda-\delta)^{1/n}}{5C} r_i))}{r_i} \\
&\leq \frac{C}{(\Lambda-\delta)^{1-\frac{1}{n}}} \sum_i \frac{\mu(B^d(x_i, \frac{(\Lambda-\delta)^{1/n}}{5C} r_i))}{\frac{(\Lambda-\delta)^{1/n}}{5C} r_i}.
\end{aligned}$$

By Lemma 5.1, we now have

$$\mathcal{H}^{n-1}_{\mathrm{Euc},\delta}(A_\Lambda) \leq \frac{C}{(\Lambda-\delta)^{1-\frac{1}{n}}} \sum_i P\left(E, B^d(x_i, \tfrac{(\Lambda-\delta)^{1/n}}{5C}\, r_i)\right).$$

Since the balls are pairwise disjoint, we see that

$$\mathcal{H}^{n-1}_{\mathrm{Euc},\delta}(A_\Lambda) \leq \frac{C}{(\Lambda-\delta)^{1-1/n}}\, P(E,\Omega).$$

Letting $\delta \to 0$ completes the proof. □

Lemma 7.2. *For $\mathcal{H}^{n-1}_{\mathrm{Euc}}$-almost every $x \in \partial E$ we have*

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n} < \infty.$$

Proof. By Lemma 7.1, we know that

$$\mathcal{H}^{n-1}_{\mathrm{Euc}}(A_\Lambda) \leq C\,\Lambda^{-(n-1)/n} P(E,\Omega).$$

Since $n \geq 2$, and the set of all points $x \in \partial E$ for which

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n} = \infty$$

is the set $\bigcap_{\Lambda>0} A_\Lambda$, we see that the claim of the lemma holds true. □

We set F_0 to be the collection of all points $x \in \partial E$ for which

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n} = 0.$$

Let $Z = \bigcap_{\Lambda>0} A_\Lambda$; from the above discussion we know that $\mathcal{H}^{n-1}_{\mathrm{Euc}}(Z)$ is zero.

Lemma 7.3. *We have that $F_\infty = \bigcup_{k\in\mathbb{N}}(A_{1/k}\setminus Z)$ is σ-finite with respect to the measure $\mathcal{H}^{n-1}_{\mathrm{Euc}}$, and*

$$\partial E = Z \cup F_0 \cup F_\infty.$$

Proof. By Lemma 7.1, we know that $\mathcal{H}^{n-1}_{\mathrm{Euc}}(A_{1/k}) < \infty$. Thus we see that F_∞ is σ-finite with respect to the measure $\mathcal{H}^{n-1}_{\mathrm{Euc}}$. □

Lemma 7.4. *Either $\mathcal{H}(F_0) = 0$ or $\mathcal{H}^{n-1}_{\mathrm{Euc}}(F_0) = \infty$. Furthermore, with*

$$K_\epsilon = \left\{ x \in \partial E \,:\, \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n} < \epsilon \right\},$$

we have

$$\mathcal{H}\left(K_\epsilon\right) \leq C\,\epsilon^{(n-1)/n}\,\mathcal{H}^{n-1}_{\mathrm{Euc}}\left(K_\epsilon\right).$$

Note that here $\mathcal{H}$ is the codimension 1 Hausdorff measure with respect to the metric d and measure μ, while $\mathcal{H}^{n-1}_{\mathrm{Euc}}$ is the $(n-1)$-dimensional Hausdorff measure with respect to the Lebesgue measure and Euclidean metric.

Proof. Suppose $\mathcal{H}(F_0) > 0$. We will show that then $\mathcal{H}^{n-1}_{\mathrm{Euc}}(F_0) = \infty$. To this end, fix $\epsilon > 0$. For each $x \in F_0$ there is a positive number δ_x such that whenever $0 < r < \delta_x$ we have $\mu(B(x,r)) \leq 2\epsilon r^n$. For each $j \in \mathbb{N}$ let

$$F_j = \{x \in F_0 : \delta_x \geq 1/j\}.$$

Note that for large j we have F_j non-empty since $F_0 = \bigcup_j F_j$ is non-empty. Furthermore, because $\mathcal{H}(F_0) > 0$ we have that $\mathcal{H}(F_j) > 0$ for sufficiently large j. Let $0 < \delta < 1/j$, and for each $x \in F_j$, whenever $r < \delta$, we have that $\mu(B(x,r)) \leq 2\epsilon r^n$. For $y \in \partial B(x,r)$, we see by (6.1) that

$$2\epsilon\, r^n \geq \mu(B(x,r)) = \mu(B(x,|x-y|)) \geq \frac{1}{C}\, d(x,y)^n.$$

It follows that $d(x,y) \leq C\epsilon^{1/n} r$ whenever $y \in \partial B(x,r)$. Therefore $B(x,r) \subset B^d(x, C\epsilon^{1/n} r)$. Now we choose a countable cover of F_j by balls $B(x_i, r_i)$ with $r_i < \delta$ and

$$\mathcal{H}^{n-1}_{\text{Euc}}(F_j) + \delta \geq \sum_i r_i^{n-1}.$$

But then by Lemma 6.2 we have

$$\begin{aligned}
\mathcal{H}^{n-1}_{\text{Euc}}(F_j) + \delta &\geq \frac{1}{C\epsilon} \sum_i \frac{\mu(B^d(x_i, C\epsilon^{1/n} r_i))}{r_i} \\
&\geq \frac{1}{C\epsilon^{1-1/n}} \sum_i \frac{\mu(B^d(x_i, C\epsilon^{1/n} r_i))}{C\epsilon^{1/n} r_i} \\
&\geq \frac{1}{C\epsilon^{1-1/n}} \mathcal{H}_{C\epsilon^{1/n}\delta}(F_j).
\end{aligned}$$

Now letting $\delta \to 0$, we can conclude that

$$\mathcal{H}^{n-1}_{\text{Euc}}(F_0) \geq \mathcal{H}^{n-1}_{\text{Euc}}(F_j) \geq \frac{1}{C\epsilon^{1-1/n}} \mathcal{H}(F_j).$$

Because E is of finite perimeter and E satisfies the density conditions discussed in the previous sections, we know by the result in [AMP] that $\mathcal{H}(F_0) \leq \mathcal{H}(\partial E) < \infty$. Indeed, if a set K has finite perimeter, then the codimension one Hausdorff measure of the measure theoretic boundary of K is finite and is comparable to the perimeter measure of K, see [AMP]. By the density conditions of the set E discussed in Theorem 4.2, the boundary of E is the measure-theoretic boundary of E, and so $\mathcal{H}(\partial E)$ is finite. Hence $\mathcal{H}(F_0)$ is also finite. Also, if $j_1 > j_2$ then $F_{j_1} \supset F_{j_2}$. Therefore we have

$$\mathcal{H}(F_0) = \lim_{j \to \infty} \mathcal{H}(F_j).$$

Therefore

$$\mathcal{H}^{n-1}_{\text{Euc}}(F_0) \geq \frac{1}{C\epsilon^{(n-1)/n}} \mathcal{H}(F_0),$$

and now the desired conclusion that $\mathcal{H}^{n-1}_{\text{Euc}}(F_0) = \infty$ follows by taking $\epsilon \to 0$ and using the fact that $n \geq 2$. The proof of the second claim of the lemma follows from an argument similar to the first part of the proof above. □

Recall that

$$F_0 := \left\{ x \in \partial E : \limsup_{r \to 0} \tfrac{\mu(B(x,r))}{r^n} = 0 \right\}.$$

Theorem 7.5. *With*

$$D_\infty := \left\{ x \in \partial E : \limsup_{r \to 0} \frac{\mathcal{H}^{n-1}_{\text{Euc}}(\partial E \cap B(x,r))}{r^{n-1}} = \infty \right\},$$

the set $\partial E \setminus (F_0 \cup D_\infty)$ is $(n-1)$-rectifiable.

Proof. Suppose not. Then combining Lemma 7.1 with [Mat, Theorem 15.6], we know that there is a purely $(n-1)$-unrectifiable set $K \subset \partial E \backslash (F_0 \cup D_\infty)$ with $\mathcal{H}^{n-1}_{\text{Euc}}(K) > 0$. Since $K \cap D_\infty$ is empty, it follows that $\mathcal{H}^{n-1}_{\text{Euc}}(\partial E \cap B)$ is finite for balls B centered at points in K with sufficiently small radii. Fix such a ball B. Then the restriction of $\mathcal{H}^{n-1}_{\text{Euc}}$ to $\partial E \cap B$ is a Radon measure. Thus there is a density point x_0 of K with respect to the restriction of the measure $\mathcal{H}^{n-1}_{\text{Euc}}$ to ∂E. By Lemma 7.2 we can assume without loss of generality that

$$Q := \limsup_{r \to 0} \frac{\mu(B(x_0, r))}{r^n} < \infty.$$

Since $x_0 \notin F_0$,

$$\infty > Q = \limsup_{r \to 0} \frac{\mu(B(x_0, r))}{r^n} > 0.$$

Furthermore, because $x_0 \notin D_\infty$, we have

$$M = \limsup_{r \to 0} \frac{\mathcal{H}^{n-1}_{\text{Euc}}(\partial E \cap B(x_0, r))}{r^{n-1}} < \infty.$$

Let ϵ be some small number to be determined later; by the choice of x_0, for sufficiently small $r_0 > 0$, for all $0 < r < r_0$ by the fact that x_0 is a density point of K with respect to the restriction of $\mathcal{H}^{n-1}_{\text{Euc}}$ to ∂E , we have

$$\frac{\mathcal{H}^{n-1}_{\text{Euc}}((\partial E \backslash K) \cap B(x_0, r))}{\mathcal{H}^{n-1}_{\text{Euc}}(\partial E \cap B(x_0, r))} < \frac{\epsilon}{2M + \epsilon}. \tag{7.6}$$

Since $x_0 \notin D_\infty$, by the definition of M, for sufficiently small r we also have

$$\mathcal{H}^{n-1}_{\text{Euc}}(\partial E \cap B(x_0, r)) \leq (M + \epsilon) r^{n-1}.$$

Therefore, by (7.6), for sufficiently small $r > 0$,

$$\mathcal{H}^{n-1}_{\text{Euc}}((\partial E \backslash K) \cap B(x_0, r)) < \epsilon r^{n-1}. \tag{7.7}$$

We can find a small positive number $r > 0$ that satisfies the above requirements and in addition satisfies

$$2Q r^n \geq \mu(B(x_0, r)) \geq \frac{Q}{2} r^n.$$

By inequality (6.1),

$$B^d(x_0, C^{-1} \mu(B(x_0, r))^{1/n}) \subset B(x_0, r) \subset B^d(x_0, C \mu(B(x_0, r))^{1/n}).$$

So in particular, by the definition of Q and the choice of r, we have that $B^d(x_0, c Q^{1/n} r) \subset B(x_0, r)$ for $c = C^{-1}\, 2^{-1/n}$.

Now by Theorem 5.2, we can find $y_0 \in E$ and $y_1 \in E^c$ such that

$$B^d(y_0, c_1 Q^{1/n} r) \subset E \cap B^d(x_0, c Q^{1/n} r) \subset B(x_0, r)$$

and

$$B^d(y_1, c_1 Q^{1/n} r) \subset B^d(x_0, c Q^{1/n} r) \setminus E \subset B(x_0, r).$$

For $i = 0, 1$ let

$$\gamma_i = \inf \left\{ h > 0 : B^d(y_i, c_1 Q^{1/n} r) \subset B(y_i, h) \right\},$$

that is, $B(y_i, \gamma_i)$ is the smallest Euclidean ball containing the metric ball $B^d(y_i, c_1 Q^{1/n} r)$. Note that by Lemma 6.2,

$$\mu(B(y_i, \gamma_i)) \geq \mu(B^d(y_i, c_1 Q^{1/n} r)) \geq c_2 Q\, r^n.$$

Because $B^d(y_i, c_1 Q^{1/n} r) \subset B(x_0, r)$, it follows that $B^d(y_i, c_1 Q^{1/n} r) \subset B(y_i, 2r)$, and so $\gamma_i \leq 2r$.

By the choice of r, we also have that $\mu(B(x_0, r)) \leq 2Qr^n$. Since $B(y_i, 2r) \subset B(x_0, 3r)$, we have by the doubling property of μ with respect to the Euclidean metric that $\mu(B(y_i, 2r)) \leq CQr^n$. So by (6.9),

$$\frac{c_2 Q r^n}{C Q r^n} \leq \frac{\mu(B(y_i, \gamma_i))}{\mu(B(y_i, 2r))} \leq C \left(\frac{\gamma_i}{2r} \right)^{Q_1},$$

and so $\gamma_i \geq cr$ with c independent of r, y_i. As with C, the symbol c here will denote a constant that is independent of the relevant quantities, but whose value might change even within the same line. By the right inclusion of (6.7) and the definition of γ_i, we know that

$$\gamma_i \leq C\, \lambda_{y_i}^{c_1 Q^{1/n} r}\, c_1 Q^{1/n} r = C\, c_1\, Q^{1/n}\, \lambda_{y_i}^{c_1 Q^{1/n} r}\, r,$$

that is,

$$\lambda_{y_i}^{c_1 Q^{1/n} r} \geq \frac{\gamma_i}{C c_1\, Q^{1/n} r} \geq \frac{cr}{C c_1\, Q^{1/n} r}.$$

Now an application of the left inclusion of (6.7),

$$B(y_i, \lambda_{y_i}^{c_1 Q^{1/n} r}\, c_1\, Q^{1/n} r) \subset B^d(y_i, c_1\, Q^{1/n} r).$$

An application of the previous inequality above now gives

$$B(y_i, cr/C) \subset B^d(y_i, c_1 Q^{1/n} r).$$

Since c, C are independent of y_i, r, and in this paper we do not keep track of specific values of the constants, we denote c/C by c from now on, and so get

$$B(y_i, cr) \subset B^d(y_i, c_1 Q^{1/n} r).$$

As K is purely unrectifiable, by the Besicovich-Federer Projection theorem, Theorem 18.1(2) of [Mat], for $i = 0, 1$ there must exist points $\widetilde{y}_i \in B(y_i, c\, r/4)$ such that for $v = (\widetilde{y}_1 - \widetilde{y}_2)/|\widetilde{y}_1 - \widetilde{y}_2|$ we have

$$\mathcal{H}^{n-1}_{\mathrm{Euc}}(P_{v^\perp}(K)) = 0.$$

Here $P_{v^\perp}$ is the projection to the $(n-1)$-dimensional hyperplane orthogonal to the vector v and passing through the point $\widetilde{y}_0$.

Let $\xi_1 = v$; then we can find unit vectors $\xi_2, \xi_3, \ldots, \xi_n$ such that $\{\xi_1, \xi_2, \ldots, \xi_n\}$ forms an orthonormal basis for the vector space $\mathbb{R}^n$. For any $z \in \mathbb{R}^n$ and $\beta > 0$ let $Q_\beta(z)$ denote the cube whose faces are normal to the vectors $\{\xi_1, \xi_2, \dots, \xi_n\}$, with Euclidean side length β, and center located at z. Note

$$Q_{\frac{c\,r}{8\,n^{1/2}}}(\widetilde{y}_0) \subset E \quad \text{and} \quad Q_{\frac{c\,r}{8\,n^{1/2}}}(\widetilde{y}_1) \subset \mathbb{R}^n \setminus E.$$

Consider the following cross-section of the cube $Q_{\frac{c\,r}{8\,n^{1/2}}}(\widetilde{y}_0)$,

$$\Pi := Q_{\frac{c\,r}{8\,n^{1/2}}}(\widetilde{y}_0) \cap \left(v^\perp + \widetilde{y}_0\right),$$

where $v^\perp$ is the $(n-1)$-dimensional hyperplane orthogonal to the vector v. For each $z \in \Pi$ let

$$e_z = P_{v^\perp}^{-1}(P_{v^\perp}(z)) \cap Q_{\frac{c\,r}{8\,n^{1/2}}}(\widetilde{y}_1) \cap \left(v^\perp + \widetilde{y}_1\right),$$

that is, e_z is the point in the region

$$Q_{\frac{cr}{8\,n^{1/2}}}(\widetilde{y}_1) \cap \left(v^\perp + \widetilde{y}_1\right) = \Pi + \widetilde{y}_1 - \widetilde{y}_0$$

corresponding to $z \in \Pi$ such that $z - e_z = \widetilde{y}_0 - \widetilde{y}_1$.

Let

$$\Pi' = \left\{ z \in \Pi \,:\, Q_{\frac{cr}{8n^{1/2}}} \cap P_{v^\perp}^{-1}(P_{v^\perp}(z)) \cap K = \emptyset \right\},$$

that is, Π' is the collection of all points $z \in \Pi$ such that the line segment $[z, e_z]$ connecting z to e_z does *not* intersect the purely unrectifiable set K. By the choice of $\widetilde{y}_i$, $i = 0, 1$, we know that

$$\mathcal{H}^{n-1}_{\mathrm{Euc}}(\Pi \setminus \Pi') = 0,$$

since the points $z \in \Pi$ that are not in Π' belong to $P_{v^\perp} K$. However, for any $z \in \Pi'$ we know that $z \in E$ and $e_z \in E^c$ so we must have the line segment $[z, e_z]$ intersecting $\partial E \backslash K$. So for each $z \in \Pi'$ we can pick a point $b_z \in (\partial E \backslash K) \cap [z, e_z]$. On the other hand, as orthogonal projections do not increase the measure $\mathcal{H}^{n-1}_{\mathrm{Euc}}$,

$$\begin{aligned}
&\mathcal{H}^{n-1}_{\mathrm{Euc}}((\partial E \setminus K) \cap B(x_0, r)) \\
&\qquad \geq \mathcal{H}^{n-1}_{\mathrm{Euc}}\left(\bigcup_{z \in \Pi'} b_z\right) \geq \mathcal{H}^{n-1}_{\mathrm{Euc}}\left(P_{v^\perp}\left(\bigcup_{z \in \Pi'} b_z\right)\right) \\
&\qquad = \mathcal{H}^{n-1}_{\mathrm{Euc}}(\Pi') = \left(\frac{c\,r}{8\,\sqrt{n}}\right)^{n-1} = \left(\frac{c}{8\,\sqrt{n}}\right)^{n-1} r^{n-1}.
\end{aligned}$$

This contradicts (7.7) when we choose $0 < \epsilon < (c/(8\,\sqrt{n}))^{n-1}$. □

Corollary 7.8. *Suppose that there is a positive number α such that $\omega(x) \geq \alpha$ for $\mathcal{L}^n$-almost every x in a neighborhood of $\overline{E}$. Then ∂E has $\mathcal{H}^{n-1}_{\mathrm{Euc}}$ finite measure and is $(n-1)$-rectifiable.*

Proof. By Lemma 7.3, we have $\partial E = Z \cup F_\infty \cup F_0$, with $\mathcal{H}^{n-1}_{\mathrm{Euc}}(Z) = 0$ and $\mathcal{H}^{n-1}_{\mathrm{Euc}}$ being σ-finite on F_∞. It follows from the assumption $\omega \geq \alpha$ almost everywhere that for all $x \in \partial E$ we have

$$\liminf_{r \to 0} \frac{\mu(B(x, r))}{r^n} \geq C_n\,\alpha > 0,$$

that is, F_0 is empty.

Now we look at

$$F_\infty = \bigcup_{k\in\mathbb{N}} A_{1/k} \setminus Z.$$

Because of the assumption that $\omega \geq \alpha$ almost everywhere, we know that $F_\infty = A_{1/k_0} \setminus Z$ where $k_0 \in \mathbb{N}$ is large enough so that $1/k_0 < C_n\, \alpha$. So by Lemma 7.1 we have that

$$\mathcal{H}^{n-1}_{\mathrm{Euc}}(\partial E) = \mathcal{H}^{n-1}_{\mathrm{Euc}}(F_\infty) < \infty.$$

Thus an application of [Mat, Theorem 6.2] gives $\mathcal{H}^{n-1}_{\mathrm{Euc}}(D_\infty) = 0$, and so ∂E is rectifiable by Theorem 7.5. $\square$

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