

# Conjugate Function Method for Numerical Conformal Mappings

Tri Quach

Institute of Mathematics  
Aalto University School of Science

Joint work with Harri Hakula and Antti Rasila

KAUS 2011 – Gothenburg, Sweden

21 January 2011

## Contents

- 1 References
- 2 Motivation
- 3 Preliminaries
  - Quadrilateral
  - Conformal Modulus of a Quadrilateral
- 4 Theorem
- 5 Algorithm
- 6 Examples
  - Quadrilaterals
  - Ring Domains

## References

- [Ahlfors1] L.V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Book Co., 1973.
- [Ahlfors2] L.V. Ahlfors, *Complex Analysis*, An introduction to the theory of analytic functions of one complex variable, Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.
- [Crowdy] D. Crowdy, *The Schwarz-Christoffel mapping to bounded multiply connected polygonal domains*. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005), no. 2061, 2653–2678.
- [CroMar] D. Crowdy and J. Marshall, *Conformal mappings between canonical multiply connected domains*. Comput. Methods Funct. Theory 6 (2006), no. 1, 59–76.
- [DriTre] T.A. Driscoll and L.N. Trefethen, *Schwarz-Christoffel Mapping*. Cambridge Monographs on Applied and Computational Mathematics, 8. Cambridge University Press, Cambridge, 2002.

## References

- [HakQuaRas] H. Hakula, T. Quach, and A. Rasila, *Conjugate function method for numerical conformal mappings*, Manuscript.
- [HakRasVuo] H. Hakula, A. Rasila, and M. Vuorinen, *On moduli of rings and quadrilaterals: algorithms and experiments*. arXiv math.NA 0906.1261, 2009. TKK-A575 (2009). SIAM J. Sci. Comput. (to appear).
- [Hu] C. Hu, *Algorithm 785: a software package for computing Schwarz-Christoffel conformal transformation for doubly connected polygonal regions*, ACM Transactions on Mathematical Software (TOMS), v.24 n.3, p.317-333, Sept. 1998
- [LehVir] O. Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer, Berlin, 1973.
- [PapSty] N. Papamichael and N.S. Stylianopoulos, *Numerical Conformal Mapping: Domain Decomposition and the Mapping of Quadrilaterals*, World Scientific Publishing Company, 2010.

- Conformal mappings can be applied in electrostatics, aerodynamics, etc.
- Numerical methods are considered since the analytical solution exists only for few domains.
  - Schwarz–Christoffel (SC) toolbox by Driscoll [DriTre].
  - Hu’s [Hu] SC algorithm for doubly connected domains.
  - For multiply connected domains, see eg. [Crowdy, CroMar].
  - Finite element methods (FEM) approach, see eg. [HakRasVuo].

## Definition (Geometric)

Let  $Q$  be a quadrilateral. Let the function  $f = u + iv$  be a one-to-one conformal mapping of the domain  $\Omega$  onto a rectangle  $R_h = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, 0 < \operatorname{Im} z < h\}$  such that the image of  $z_1, z_2, z_3, z_4$  are  $1 + ih, ih, 0, 1$ , respectively. Then the number  $h$  is called the (*conformal*) *modulus* of the quadrilateral  $Q$  and we will denote it by  $M(Q)$ .

- Note that the conformal modulus of a quadrilateral is unique.
- By the geometry [LehVir, p. 15], [PapSty, pp. 53-54], we have the reciprocal identity:

$$M(Q) \cdot M(\tilde{Q}) = 1.$$

## Definition (Generalized Quadrilateral)

A Jordan domain  $\Omega$  in  $\mathbb{C}$  with marked (positively ordered) points  $z_1, z_2, z_3, z_4 \in \partial\Omega$  is called a (*generalized*) *quadrilateral*, and denoted by  $Q := (\Omega; z_1, z_2, z_3, z_4)$ .

- Denote the arcs of  $\partial\Omega$  between  $(z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1)$ , by  $\gamma_j, j = 1, 2, 3, 4$ .
- Quadrilateral  $\tilde{Q} = (\Omega; z_2, z_3, z_4, z_1)$  is called the *conjugate quadrilateral* of  $Q$ .

- Consider the following Laplace equation

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \gamma_2, \\ u = 1, & \text{on } \gamma_4, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \gamma_1 \cup \gamma_3. \end{cases} \quad (1)$$

- If  $u$  is the solution to (1). Then by [Ahlfors1, p. 65/Thm 4.5], [PapSty, p. 63/Thm 2.3.3]:

$$M(Q) = \iint_{\Omega} |\nabla u|^2 dx dy. \quad (2)$$

## Preliminaries – Modulus of a Ring Domain

- Let  $E$  and  $F$  be two disjoint compact sets in the extended complex plane  $\mathbb{C}_\infty$ . Then one of the sets  $E, F$  is bounded and without loss of generality we may assume that it is  $E$ . If both  $E$  and  $F$  are connected and the set  $R = \mathbb{C}_\infty \setminus (E \cup F)$  is connected, then  $R$  is called a *ring domain*. In this case  $R$  is a doubly connected plane domain. The *capacity* of  $R$  is defined by

$$\text{cap } R = \inf_u \iint_R |\nabla u|^2 dx dy,$$

where the infimum is taken over all nonnegative, piecewise differentiable functions  $u$  with compact support in  $R \cup E$  such that  $u = 1$  on  $E$ .

- The harmonic function on  $R$  with boundary values 1 on  $E$  and 0 on  $F$  is the unique function that minimizes the above integral.
- Conformal modulus:  $M(R) = 2\pi/\text{cap } R$ .

## Theorem - Formulation

### Theorem (Conjugate Function Method)

Let  $Q$  be a quadrilateral with modulus  $h$  and let  $u_1$  satisfy (1). Suppose that  $u_2$  is the solution to Dirichlet–Neumann boundary value problem associated with the conjugate quadrilateral  $\tilde{Q}$ . Then  $f = u_1 + ihu_2$  is the conformal mapping that maps  $\Omega$  onto a rectangle  $R_h$  such that the image of the points  $z_1, z_2, z_3, z_4$  are  $1 + ih, ih, 0, 1$ , respectively. The mapping  $f$  maps the boundary curves  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  onto curves  $\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4$ , respectively.

Proof is based on the following facts:

- There is a connection between the harmonic conjugate  $v_1$  of  $u_1$  and the solution  $u_2$ .
- There is a conformal mapping  $f = u + iv$  that maps  $\Omega$  onto  $R_h$ .

## Theorem – Illustration of the Problem

- Find  $f$  such that  $f : \Omega \rightarrow R_h$ .

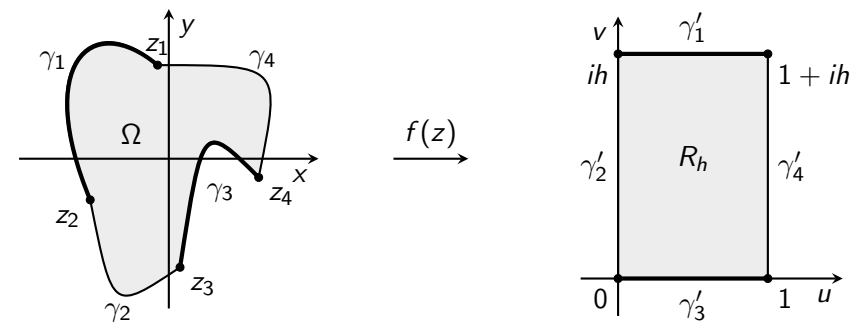


Figure: Dirichlet-Neumann boundary value problem. Dirichlet and Neumann boundary conditions are marked with thin and thick lines, respectively.

## Proof of the Theorem – Part 1

### Lemma

Let  $Q, h, u, v$  be as before. If  $\tilde{u}$  is the solution to the Dirichlet-Neumann problem associated with the conjugate quadrilateral  $\tilde{Q}$ . Then  $v = h^2\tilde{u}$ .

- It is clear that  $v, \tilde{u}$  are harmonic. Thus  $\tilde{v} := h^2\tilde{u}$  is harmonic.
- Since  $M(\tilde{Q}) = 1/h$ , therefore  $v$  and  $\tilde{v}$  have the same values on  $\gamma_1, \gamma_3$ .
- On  $\gamma_2, \gamma_4$  we have

$$\frac{\partial v}{\partial n} = \langle \nabla v, n \rangle = v_x n_1 + v_y n_2 = u_y n_1 - u_x n_2 = 0,$$

because  $u$  is constant on  $\gamma_2, \gamma_4$ , and therefore  $u_x = u_y = 0$ .

- Thus  $v$  and  $\tilde{v}$  also have same values on  $\gamma_2, \gamma_4$ .
- Then by the uniqueness theorem for harmonic functions [Ahlfors2, p. 166] we have  $v = \tilde{v}$ .

□

- Let  $u$  satisfy (1) and suppose that  $v$  is the harmonic conjugate function of  $u$ . Then  $f = u + iv$  is analytic.
- $\operatorname{Re} f = u$  and  $u = 0$  on  $\gamma'_2$  and  $u = 1$  on  $\gamma'_4$ . (Dirichlet boundary)
- On  $\gamma'_1, \gamma'_3$ , we use Lemma from previous slide. Since  $v = 0$  on  $\gamma'_3$ , we have  $v = h$  on  $\gamma'_1$ . (Neumann boundary)
- For univalence, suppose that  $f$  is not univalent, i.e., there exists  $z_1, z_2 \in \Omega$ ,  $z_1 \neq z_2$  such that  $f(z_1) = f(z_2)$ .
- Thus  $\operatorname{Re} f(z_1) = \operatorname{Re} f(z_2)$ , so  $z_1, z_2$  are on the same equipotential curve  $C$  of  $u$ .
- Similarly for the imaginary part, we have that  $z_1 = z_2$ .

□

## Algorithm (HakQuaRas)

- 1 Solve the Dirichlet-Neumann problem to obtain  $u_1$  and compute the modulus  $h$ .
- 2 Solve the conjugate problem for  $u_2$ .
- 3 Then the conformal mapping is given by  $f = u_1 + ihu_2$ .

Note:

- The solution  $u$  can be obtained by any standard numerical methods. In our examples the  $hp$ -FEM software by H. Hakula, [HakRasVuo], is used.
- The reciprocal identity is used for the error analysis.
- Draw the rectangular grid on  $R_h$  and map it onto  $\Omega$ .

## Example – Analytic

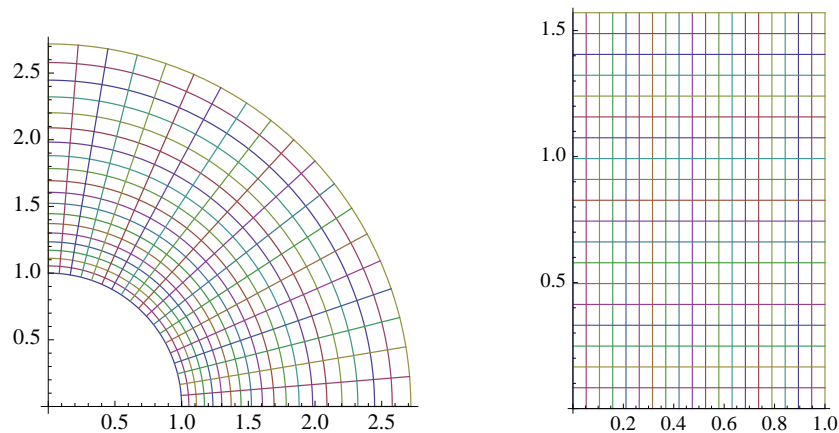


Figure: Conformal mapping from a quadrilateral onto a rectangle.

## Example – Schwarz (1869)

- $(\Omega; z_1, z_2, z_3, z_4)$ , where  $z_j = e^{i\theta_j}$ ,  $\theta_j = (j - 1)\pi/2$ .

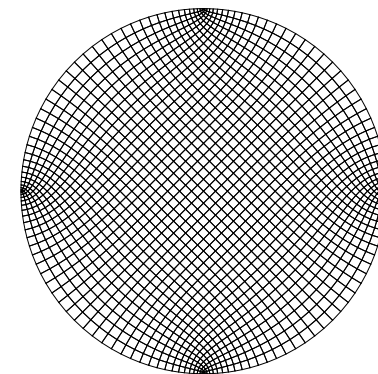


Figure: Error of the conformal mapping is  $10^{-13}$ .

## Example – Flower

- $\Omega$  is the domain bounded by the curve  $r(\theta) = 0.8 + t \cos(n\theta)$ ,  $0 \leq \theta \leq 2\pi$ ,  $n = 6$  and  $t = 0.1$ .
- $(\Omega; z_1, z_2, z_3, z_4)$ , where  $z_j = r(\theta_j)$ ,  $\theta_j = (j - 1)\pi/2$ .

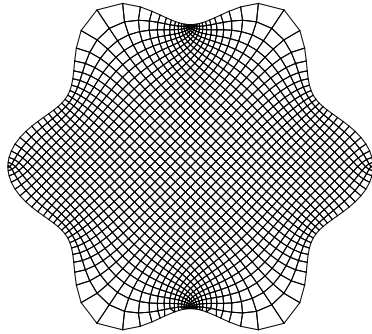


Figure: Error of the conformal mapping is  $10^{-11}$ .

## Example – Circular Quadrilateral

- Consider a quadrilateral whose sides are circular arcs of intersecting orthogonal circles.

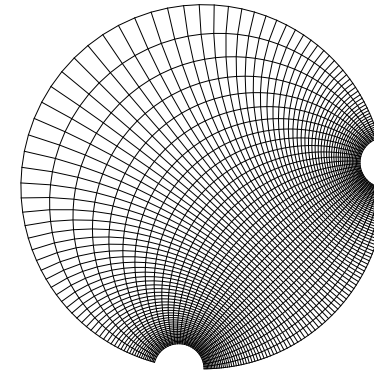


Figure: Circular quadrilateral and the visualization of the pre-image of the rectangular grid.

## Example – Cross in Square

- Let  $G_{ab} = \{(x, y) : |x| \leq a, |y| \leq b\} \cup \{(x, y) : |x| \leq b, |y| \leq a\}$ , and  $G_c = \{(x, y) : |x| < c, |y| < c\}$ , where  $a < c$  and  $b < c$ .
- Cross in Square:  $R = G_c \setminus G_{ab}$ .

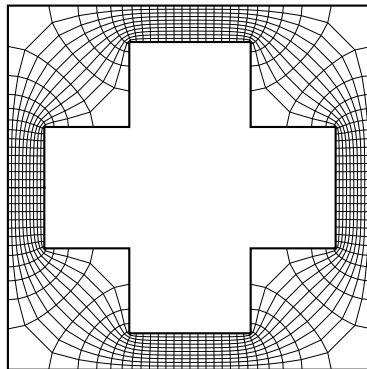


Figure: The ring domain  $G_c \setminus G_{ab}$ , where  $a = 0.5$ ,  $b = 1.2$ ,  $c = 1.5$ , with the pre-image of the annular grid.

## Example – Flower in Square

- Flower and the square are as before.

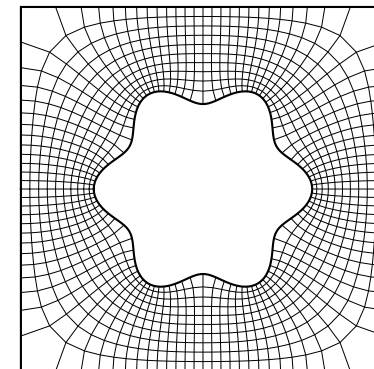


Figure: Flower in a square domain with equipotential curves.

## Example – Droplet in Square

- Droplet is a Bezier curve.

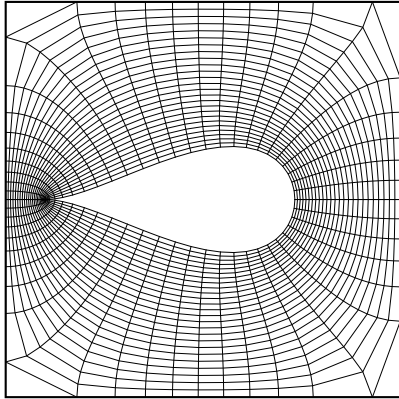


Figure: Droplet in square. The reciprocal error of the conformal mapping is  $10^{-9}$ .

## Example – Circle in L-block

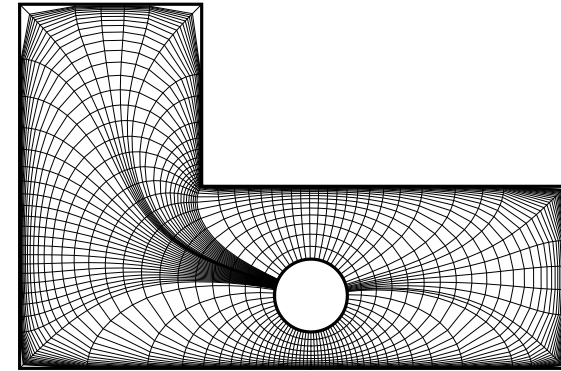


Figure: Circle in a L-block. Equipotential grid is not regular.