

A conjugate function method for conformal mappings

Tri Quach

Department of Mathematics and Systems Analysis
Aalto University School of Science and Technology





Joint work with Harri Hakula and Antti Rasila

HQM 2010 - Chennai, India

16 August 2010



References

-  [Ahlfors] L.V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill Book Co., 1973.
-  [Crowdy] D. Crowdy, *The Schwarz-Christoffel mapping to bounded multiply connected polygonal domains*. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005), no. 2061, 2653–2678.
-  [CroMar] D. Crowdy and J. Marshall, *Conformal mappings between canonical multiply connected domains*. Comput. Methods Funct. Theory 6 (2006), no. 1, 59–76.
-  [DriTre] T.A. Driscoll and L.N. Trefethen, *Schwarz-Christoffel Mapping*. Cambridge Monographs on Applied and Computational Mathematics, 8. Cambridge University Press, Cambridge, 2002.






Contents

- 1 References
- 2 Motivation
- 3 Preliminaries
 - Conformal Modulus
 - Lemma
- 4 Main Theorem
- 5 Error Estimates
- 6 Examples



References

-  [HakRasVuo] H. Hakula, A. Rasila, and M. Vuorinen, *On moduli of rings and quadrilaterals: algorithms and experiments*. arXiv math.NA 0906.1261, 2009. TTK-A575 (2009)
-  [Hu] C. Hu, *Algorithm 785: a software package for computing Schwarz-Christoffel conformal transformation for doubly connected polygonal regions*, ACM Transactions on Mathematical Software (TOMS), v.24 n.3, p.317-333, Sept. 1998
-  [LehVir] O. Lehto and K.I. Virtanen, *Quasiconformal mappings in the plane*, Springer, Berlin, 1973.



Motivation

- Application in
 - electrostatics,
 - aerodynamics, etc.
- Methods
 - Schwarz–Christoffel (SC) toolbox by Driscoll [DriTre].
 - Hu’s [Hu] SC algorithm for doubly connected domains.
 - For multiply connected domains, see eg. [Crowdy, CroMar].

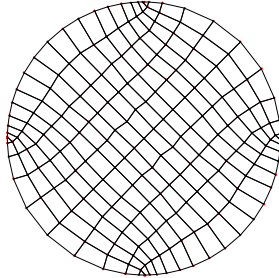


Figure: Conformal mapping from a square onto a disk. (Schwarz, 1869) **A?**

Preliminaries – Modulus of a Quadrilateral

- One can express the modulus of a quadrilateral $(D; z_1, z_2, z_3, z_4)$ in terms of the solution of the Dirichlet–Neumann problem as follows.

Let γ_j , $j = 1, 2, 3, 4$ be the arcs of ∂D between (z_1, z_2) , (z_2, z_3) , (z_3, z_4) , (z_4, z_1) , respectively. If u is the (unique) harmonic solution of the Dirichlet–Neumann problem with boundary values of u equal to 0 on γ_2 , equal to 1 on γ_4 and with $\partial u / \partial n = 0$ on $\gamma_1 \cup \gamma_3$, then by [Ahlfors, p. 65/Thm 4.5]:

$$M(D; z_1, z_2, z_3, z_4) = \int_D |\nabla u|^2 dx dy. \quad (1)$$

- By the geometry [LehVir, p. 15] we have the following reciprocal identity

$$M(D; z_1, z_2, z_3, z_4) \cdot M(D; z_2, z_3, z_4, z_1) = 1. \quad (2)$$

A?

Preliminaries – Dirichlet–Neumann Problem

Definition

Let D be a domain in the complex plane whose boundary ∂D consists of a finite number of regular Jordan curves, so that at every point, except possibly at finitely many points, of the boundary a normal is defined. Let $\partial D = A \cup B$ where A, B both are unions of Jordan arcs. Let ψ_A, ψ_B be a real-valued continuous function defined on A, B , respectively. Find a function u satisfying the following conditions:

- 1 u is continuous and differentiable in \bar{D} .
- 2 $u(t) = \psi_A(t)$, for all $t \in A$.
- 3 If $\partial / \partial n$ denotes differentiation in the direction of the exterior normal, then

$$\frac{\partial}{\partial n} u(t) = \psi_B(t), \quad \text{for all } t \in B.$$

The conjugate problem is a problem with reversed boundary conditions. That is, $u = \psi_B$ on A and $\partial u / \partial n = \psi_A$ on B . **A?**

Preliminaries – Modulus of a Ring Domain

- Let E and F be two disjoint compact sets in the extended complex plane \mathbb{C}_∞ . Then one of the sets E, F is bounded and without loss of generality we may assume that it is E . If both E and F are connected and the set $R = \mathbb{C}_\infty \setminus (E \cup F)$ is connected, then R is called a *ring domain*. In this case R is a doubly connected plane domain. The *capacity* of R is defined by

$$\text{cap } R = \inf_u \int_R |\nabla u|^2 dm,$$

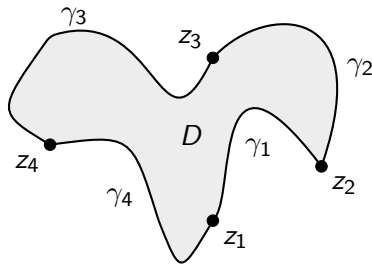
where the infimum is taken over all nonnegative, piecewise differentiable functions u with compact support in $R \cup E$ such that $u = 1$ on E .

- The harmonic function on R with boundary values 1 on E and 0 on F is the unique function that minimizes the above integral.
- Conformal modulus: $M(R) = 2\pi / \text{cap } R$. **A?**

Preliminaries – Model Problem

- Laplace equation with Dirichlet–Neumann boundary conditions

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = 0, & \text{on } \gamma_2, \\ u = 1, & \text{on } \gamma_4, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \gamma_1 \cup \gamma_3. \end{cases} \quad (3)$$



A?

Preliminaries – Lemma

Lemma

Let $(D; z_1, z_2, z_3, z_4)$ be a quadrilateral and let u satisfy the equation (3) with modulus h . Let v be the harmonic conjugate function of u . Then h^2v is the solution to the conjugate Dirichlet–Neumann boundary value problem.

Lemma

Let $(D; z_1, z_2, z_3, z_4)$ be a quadrilateral and let u satisfy the equation (3) and let v be a conjugate harmonic function of u . Then $f = u + iv$ is a conformal mapping, and it maps D onto a rectangle such that the images of the points z_1, z_2, z_3, z_4 are $1 + ih, ih, 0, 1$, respectively. The mapping f maps the boundary curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ onto curves $\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4$, respectively.

A?

Preliminaries – Illustration

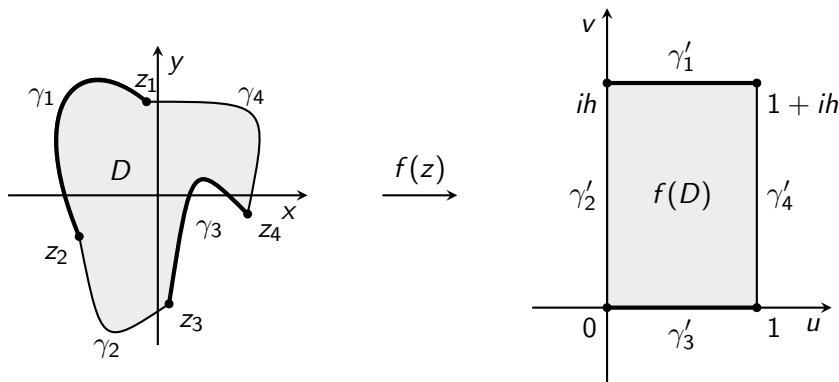


Figure: Dirichlet–Neumann boundary value problem. Dirichlet and Neumann boundary conditions are marked with thin and thick lines, respectively.

A?

Main Theorem

Theorem (Conjugate Function Method)

Let $(D; z_1, z_2, z_3, z_4)$ be a quadrilateral with modulus h and let u_1 satisfy, as before, the Laplace equation with Dirichlet–Neumann boundary conditions. Let u_2 satisfy the conjugate Dirichlet–Neumann boundary value problem. Then there exists a conformal mapping $f = u_1 + ihu_2$ which maps D onto a rectangle such that the images of the points z_1, z_2, z_3, z_4 are $1 + ih, ih, 0, 1$, respectively. The mapping f maps the boundary curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ onto curves $\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4$, respectively.

Proof: By above lemmas there exists a conformal mapping $f_1 = u_1 + iv_1$ that maps D onto a rectangle with the given properties.

A?

Proof of the Main Theorem

- On the other hand by reversing the boundary conditions, we have a conformal mapping $f_2 = u_2 + iv_2$ that maps D onto a rectangle such that z_1, z_2, z_3, z_4 are $1 + i/h, i/h, 0, 1$, respectively.
- Then by multiplying (scaling and rotating) the latter case by ih and translating the rectangle such z_2 maps to the origin, we have mapped the points z_1, z_2, z_3, z_4 are $ih, 0, 1, 1 + ih$, respectively. Thus we have a mapping $z \mapsto ihf_2 + 1 = ihu_2 - hv_2 + 1$.
- The rectangle is exactly the same as $f_1(D)$ with reversed boundary conditions. This implies $f_1 = ihf_2 + 1$, and $v_1 = hu_2$.
- Since scaling, rotating and translating are conformal invariant and with the fact that the solution of the Laplace equation is unique yields the proof. \square

A?

Error Estimates for Conformal Mappings

- **Quadrilateral:** For an error estimates, first we compute the modulus h_1 and conjugate modulus h_2 . Then by reciprocal identity it should hold that $h_1 h_2 = 1$, and we use the absolute value of the difference as the error estimate

$$\text{err} = |h_1 h_2 - 1|.$$

- **Ring domain:** First we compute the modulus h . Then we split the domain through the curve of the deepest descent. Thus we obtain a quadrilateral such that the Neumann condition lies on the splitting curve and Dirichlet condition are as before. Then we compute modulus h_1 and conjugate modulus h_2 and use following quantity as an error estimate

$$\max\{|hh_2 - 1|, |h_1 h_2 - 1|\}.$$

A?

Example – Analytic

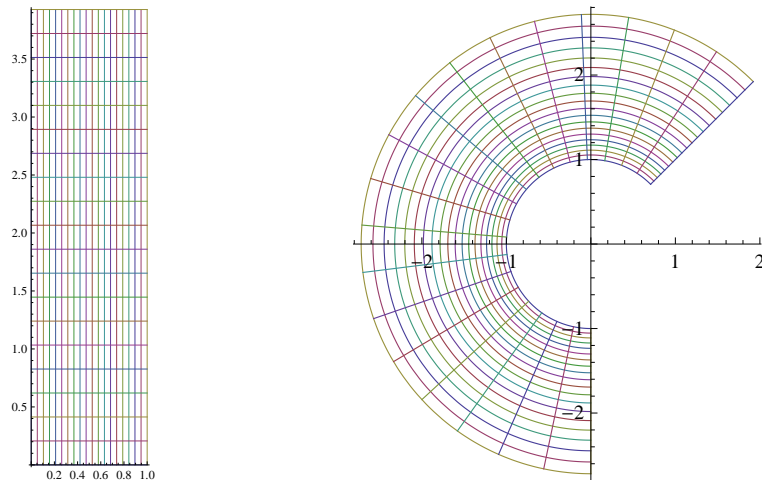
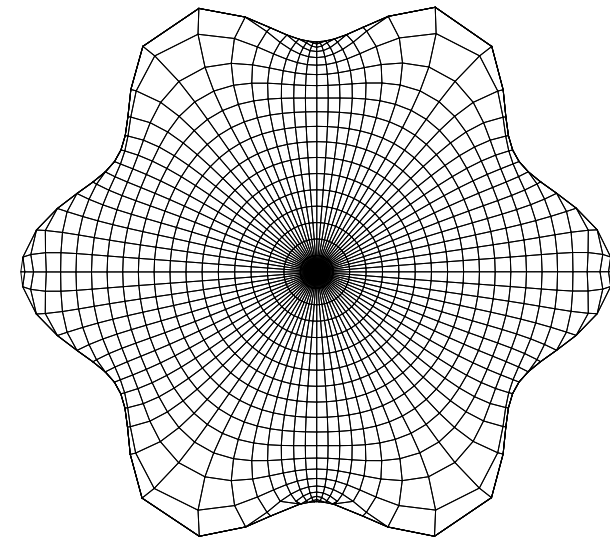


Figure: Conformal mapping from a rectangle onto a quadrilateral.

A?

Example – Quadrilateral



A?

Example – Ring Domain

