Conjugate Function Method for Numerical Conformal Mappings

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Introduction - Motivation

- Conformal mappings can be applied, e.g., in aerodynamics and study of magnetic fields.
- The cross-section of the cylinder with an orthogonal plane defines a two-dimensional ring domain and the expressions

$$C = \frac{2\pi\varepsilon}{\ln(R/r)},$$

define the capacity of this ring domain and ε is permittivity.

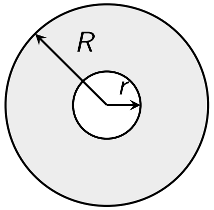

Figure: The cross-section of the cylinder.

Introduction - Numerical Methods

Schwarz-Christoffel Mapping

- Conformal mappings between polygons, circles and half-planes.
- Widely used software
  - SC Toolbox implemented for Matlab by Driscoll [DriTre].
  - Hu's [Hu] algorithm for doubly connected domains.

Conjugate Function Method

- Based on the conjugate harmonic function and properties of quadrilaterals.
- Harmonic functions associated with Dirichlet-Neumann problems can be solved by any suitable methods.

Other methods

- Circle Packing, Zipper algorithm.

Preliminaries - Generalized Quadrilateral

Definition (Generalized Quadrilateral)

A Jordan domain Ω in $\mathbb{C}$ with marked (positively ordered) points $z_1, z_2, z_3, z_4 \in \partial\Omega$ is called a *(generalized) quadrilateral*, and is denoted by $Q := (\Omega; z_1, z_2, z_3, z_4)$.

- Denote the arcs of $\partial\Omega$ between (z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1), by $\gamma_j, j = 1, 2, 3, 4$.
- Quadrilateral $\tilde{Q} = (\Omega; z_2, z_3, z_4, z_1)$ is called the *conjugate quadrilateral* of Q.

Preliminaries – Modulus of Quadrilateral

Definition (Geometric)

Let Q be a quadrilateral. Let the function $f = u + iv$ be a one-to-one conformal mapping of Q onto a rectangle $R_h = \{z \in \mathbb{C} : 0 < \mathrm{Re}\, z < 1, 0 < \mathrm{Im}\, z < h\}$ such that the image of z_1, z_2, z_3, z_4 are $1 + ih, ih, 0, 1$, respectively. Then the number h is called the *(conformal) modulus* of the quadrilateral Q and is denoted by $\mathrm{M}(Q)$.

- Note that the conformal modulus of a quadrilateral is unique.
- Modulus of a quadrilateral is also conformally invariant, i.e., if $f : \Omega \to \Omega'$ then

$$\mathrm{M}(\Omega; z_1, z_2, z_3, z_4) = \mathrm{M}(\Omega'; f(z_1), f(z_2), f(z_3), f(z_4)).$$

- By the geometry [PapSty, pp. 53-54], we have the reciprocal identity:

$$\mathrm{M}(Q) \cdot \mathrm{M}(\tilde{Q}) = 1,$$

where $Q = (\Omega; z_1, z_2, z_3, z_4)$ and $\tilde{Q} = (\Omega; z_2, z_3, z_4, z_1)$.

Preliminaries – Dirichlet-Neumann Problem

- Let Ω be a Jordan domain such that the boundary $\partial\Omega$ consist of finite number of regular Jordan curves and the normal of $\partial\Omega$ is defined for all points, except possibly at finitely many points.
- Let $\Omega = A \cup B$ such that $A \cap B$ is finite. Suppose ψ_A, ψ_B be real-valued continuous functions defined on A, B, respectively.

Then the Dirichlet-Neumann problem is to find a function u satisfying the following conditions:

1. u is continuous and differentiable in $\overline{\Omega}$.
2. Dirichlet condition: $u(t) = \psi_A(t), \qquad \text{for all } \ t \in A.$
3. Neumann condition: If $\partial/\partial n$ denotes differentiation in the direction of the exterior normal, then

$$\frac{\partial}{\partial n} u(t) = \psi_B(t), \qquad \text{for all } \ t \in B.$$

Preliminaries – Modulus of a Quadrilateral

- Consider the following Dirichlet-Neumann problem for Laplace equation

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \gamma_2, \\ u = 1, & \text{on } \gamma_4, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \gamma_1 \cup \gamma_3. \end{cases} \qquad (1)$$

- If u is the solution to (1). Then by [PapSty, p. 63/Thm 2.3.3]:

$$\mathrm{M}(Q) = \iint_\Omega |\nabla u|^2 \, dx \, dy. \qquad (2)$$

Preliminaries – Laplace problem

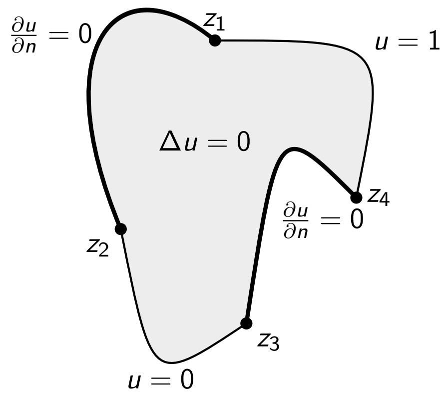

Figure: Laplace equation with Dirichlet-Neuman boundary conditions on a quadrilateral $Q = (\Omega; z_1, z_2, z_3, z_4)$, where Dirichlet and Neumann conditions are mark with thin and thick lines, respectively.

Preliminaries – Modulus of Ring Domain

- Let E and F be two disjoint and connected compact sets in the extended complex plane $\mathbb{C}_\infty$. Then one of the sets E, F is bounded and without loss of generality we may assume that it is E. The set $R = \mathbb{C}_\infty \setminus (E \cup F)$ is connected and it is called a *ring domain*. The *capacity* of R is defined by

$$\mathrm{cap}\, R = \inf_u \iint_R |\nabla u|^2 \, dx \, dy,$$

where the infimum is taken over all non-negative, piecewise differentiable functions u with compact support in $R \cup E$ such that $u = 1$ on E.
- The harmonic function on R with boundary values 1 on E and 0 on F is the unique function that minimizes the above integral.
- Conformal modulus: $\mathrm{M}(R) = 2\pi / \mathrm{cap}\, R$.

Theorem - Formulation

Theorem (Conjugate Function Method)

Let Q be a quadrilateral with modulus h and let u_1 satisfy (1). Suppose that u_2 is the solution to Dirichlet–Neumann boundary value problem associated with the conjugate quadrilateral $\tilde{Q}$ with $u_2(\mathrm{Re}\, z_3, \mathrm{Im}\, z_3) = 0$. Then $f = u_1 + ihu_2$ maps conformally Q onto R_h such that the image of the vertices z_1, z_2, z_3, z_4 are $1 + ih, ih, 0, 1$, respectively. The mapping f maps the boundary curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ onto curves $\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4$.

Proof is based on the following facts:

1. There is a connection between the harmonic conjugate v_1 of u_1 and the solution u_2.
2. There is a conformal mapping $f = u + iv$ that maps Q onto R_h such that vertices z_1, z_2, z_3, z_4 are mapped onto $1 + ih, ih, 0, 1$.

Theorem – Illustration

- Find f such that $f : \Omega \to R_h$.

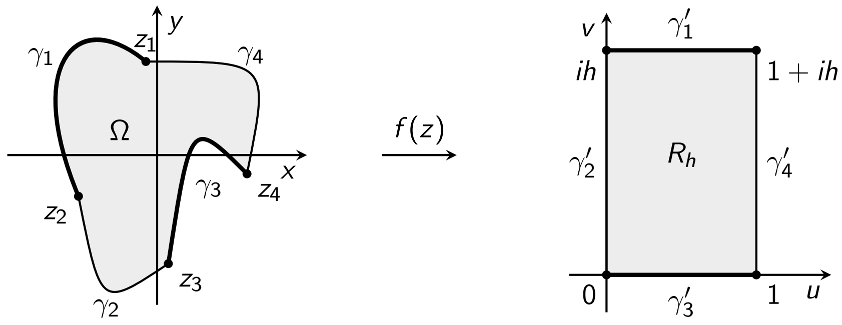

Figure: Conformal mapping of a quadrilateral onto a rectangle. Dirichlet and Neumann boundary conditions are mark with thin and thick lines, respectively.

Proof of the Theorem – Part 1

Lemma

Let Q, h, u, v be as before. If $\tilde{u}$ is the solution to the Dirichlet-Neumann problem associated with the conjugate quadrilateral $\tilde{Q}$. Then $v = h^2\tilde{u}$.

- It is clear that $v, \tilde{u}$ are harmonic. Thus $\tilde{v} := h^2\tilde{u}$ is harmonic.
- Since $\mathrm{M}(\tilde{Q}) = 1/h$, therefore v and $\tilde{v}$ have the same values on γ_1, γ_3.
- On γ_2, γ_4 we have

$$\frac{\partial v}{\partial n} = \langle \nabla v, n \rangle = v_x n_1 + v_y n_2 = u_y n_1 - u_x n_2 = 0,$$

because u is constant on γ_2, γ_4, and therefore $u_x = u_y = 0$.
- Thus v and $\tilde{v}$ also have same values on γ_2, γ_4.
- Then by the uniqueness theorem for harmonic functions we have $v = \tilde{v}$.

□

Proof of the Theorem – Part 2

Lemma

Let u satisfy (1) and suppose that v is the harmonic conjugate function of u with $v(\mathrm{Re}\, z_3, \mathrm{Im}\, z_3) = 0$. Then $f = u + iv$ is a conformal mapping from Ω onto R_h.

- $\mathrm{Re}\, f = u$ and $u = 0$ on γ'_2 and $u = 1$ on γ'_4. (Dirichlet boundary)
- On γ'_1, γ'_3, we use Lemma from previous slide. Since $v = 0$ on γ'_3, we have $v = h$ on γ'_1. (Neumann boundary)
- For univalency, suppose that f is not univalent, i.e., there exists $z_1, z_2 \in \Omega$, $z_1 \neq z_2$ such that $f(z_1) = f(z_2)$.
- Thus $\mathrm{Re}\, f(z_1) = \mathrm{Re}\, f(z_2)$, so z_1, z_2 are on the same equipotential curve C of u.
- Similarly for the imaginary part, thus we have that $z_1 = z_2$.

□

Algorithm – Conjugate Function Method

Algorithm (HakQuaRas)

1. *Solve the Dirichlet-Neumann problem to obtain u_1 and compute the modulus h.*
2. *Solve the Dirichlet-Neumann problem associated with $\tilde{Q}$ to obtain u_2.*
3. *Then $f = u_1 + ihu_2$ is the conformal mapping from Q onto R_h such that the vertices (z_1, z_2, z_3, z_4) are mapped onto the corners $(1 + ih, ih, 0, 1)$.*

Note:

- The solution u can be obtained by any suitable numerical methods.
- In our examples the *hp*-FEM software by H. Hakula, [HakRasVuo], is used.
- The reciprocal identity is used for the error analysis.
- Draw the rectangular grid on R_h and map it onto Ω.

Algorithm – Ring Domain

In the case of ring domains, we use a special cross cut to make the ring simply connected and then use the following algorithm:

Algorithm

1. *Solve the Dirichlet problem on R to obtain u and the modulus $M(R)$.*
2. *Cut R through the steepest descent curve to a quadrilateral where Neumann condition is on the steepest descent curve and Dirichlet boundaries remains as before.*
3. *Use the algorithm for quadrilaterals to obtain the conformal mapping.*

Note that the steepest descent curve is given by the gradient of the potential function u.

Example – Schwarz (1869)

- $(\Omega; z_1, z_2, z_3, z_4)$, where $z_j = e^{i\theta_j}$, $\theta_j = (j-1)\pi/2$.

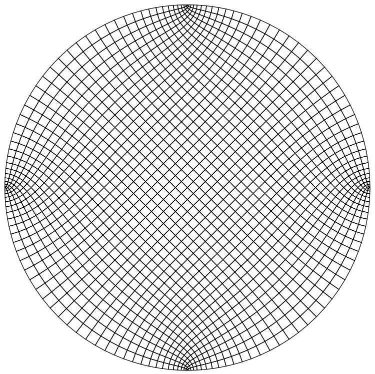

Figure: Error of the conformal mapping is $4.34 \cdot 10^{-14}$.

Example – Flower

- Ω is the domain bounded by the curve $r(\theta) = 0.8 + t\cos(n\theta)$, $0 \le \theta \le 2\pi$, $n = 6$ and $t = 0.1$.
- $(\Omega; z_1, z_2, z_3, z_4)$, where $z_j = r(\theta_j)$, $\theta_j = (j-1)\pi/2$.

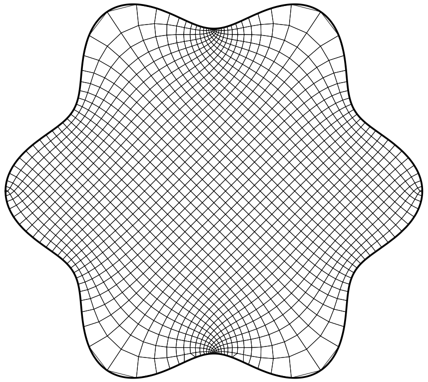

Figure: The reciprocal error of the conformal mappings is $3.74 \cdot 10^{-11}$.

Example – Circular Quadrilateral

- Consider a quadrilateral $(Q_A; e^{i\pi/12}, e^{i17\pi/12}, e^{i3\pi/2}, 1)$ whose sides are circular arcs of intersecting orthogonal circles.

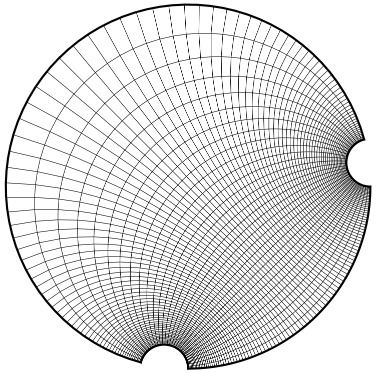

Figure: The reciprocal error of the conformal mappings is $1.68 \cdot 10^{-13}$. The modulus $\mathrm{M}(Q_A) = 0.63058735108476$ and $\mathrm{M}(\tilde{Q}_A) = 1.58582311915985$.

Example – Asteroidal Cusp

- Asteroidal cusp is a domain Ω given by a $G_c = \{(x, y) : |x| < c, |y| < c\}$, where $c = 1$ and the left-handside vertical boundary line-segment is replaced by the following curve $r(t) = \cos^3 t + i \sin^3 t,\ \ t \in [-\pi/2, \pi/2]$. The vertices are chosen as $z_1 = 1 - i, z_2 = 1 + i, z_3 = -1 + i, z_4 = -1 - i$.

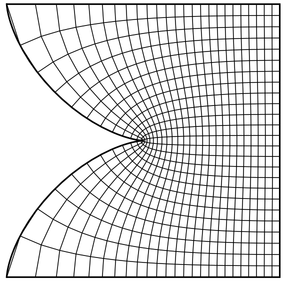

Figure: The reciprocal error of the conformal mappings is of the order 10^{-10}. The modulus $\mathrm{M}(Q) = 0.684354$ and $\mathrm{M}(\tilde{Q}) = 1.46123$.

Example – Disk in Pentagon

- Let Ω be a regular pentagon centered at the origin and having short radius (apothem) equal to 1.
- Let $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| \le r\}$, where $r = 0.4$.
- Consider a ring domain $R = \Omega \backslash \mathbb{D}(r)$.

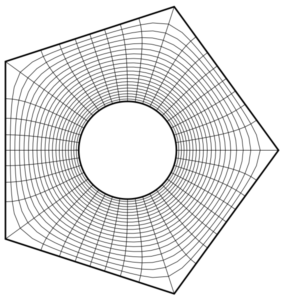

Figure: The vertices of the pentagon are of the form $e^{2\pi ik/5}$, where $k = 0, 1, 2, 3, 4$, and the modulus $\mathrm{M}(R) = 0.96742460017$. The exponential modulus $e^{\mathrm{M}(R)}$ have been studied in papers [BetSamVuo, PapWar].

Example – Cross in Square

- Let
  $$G_{ab} = \{(x, y) : |x| \le a, |y| \le b\} \cup \{(x, y) : |x| \le b, |y| \le a\},$$
  $$G_c = \{(x, y) : |x| < c, |y| < c\}, \text{ where } a < c \text{ and } b < c.$$
- Cross in Square: $R = G_c \backslash G_{ab}$, where $a = 0.5, b = 1.2, c = 1.5$.

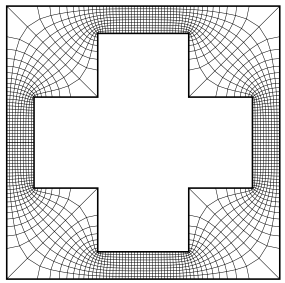

Figure: Cross in square: $R = G_c \backslash G_{ab}$, where $a = 0.5, b = 1.2, c = 1.5$ with equipotential curves. The reciprocal error of the conformal mapping is of the order 10^{-6}. The modulus $\mathrm{M}(R) = 0.2862861$. Modulus of R for different a, b, c have been studied in papers [BetSamVuo, HakRasVuo, PapKok].

Example – Flower in Square

- Ω is the domain bounded by the curve $r(\theta) = 0.8 + t\cos(n\theta)$, $0 \leq \theta \leq 2\pi$, $n = 6$ and $t = 0.1$.
- $G_c = \{(x,y) : |x| < c,\ |y| < c\}$, where $c = 1.5$.
- $R = G_c \backslash \Omega$.

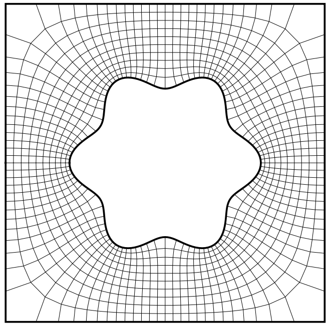

Figure: The reciprocal error of the conformal mapping is of the order 10^{-8}. The modulus $\mathrm{M}(R) = 0.666955462$.

Example – Droplet in Square

- Droplet Ω is bounded by a Bezier curve:

$$r(t) = \frac{1}{640}\left(45t^6 + 75t^4 - 525t^2 + 469\right) + \frac{15}{32}t\left(t^2 - 1\right)^2 i,\ t \in [-1,1].$$

- $R = G_c \backslash \Omega$.

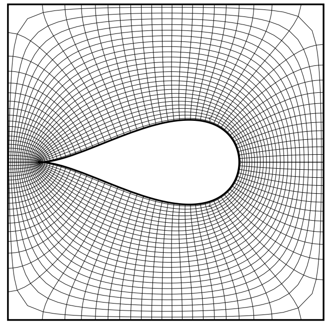

Figure: The reciprocal error of the conformal mapping is of the order 10^{-10}. The modulus $\mathrm{M}(R) = 0.89797750989$.

Example – Circle in L-block (1/2)

- Let

  $L_1 = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < a,\ 0 < \mathrm{Im}(z) < b\}$,
  $L_2 = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < d,\ 0 < \mathrm{Im}(z) < c\}$,
  where $0 < d < a$, $0 < b < c$.

- Then $L(a,b,c,d) = L_1 \cup L_2$ is called an L-domain.
- Suppose that $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.
- We consider $R = L(a,b,c,d) \backslash \mathbb{D}(z_0, r)$, where

  $(a,b,c,d) = (3,1,2,1)$,
  $z_0 = 8/5 + 2i/5$,
  $r = 1/5$.

Example – Circle in L-block (2/2)

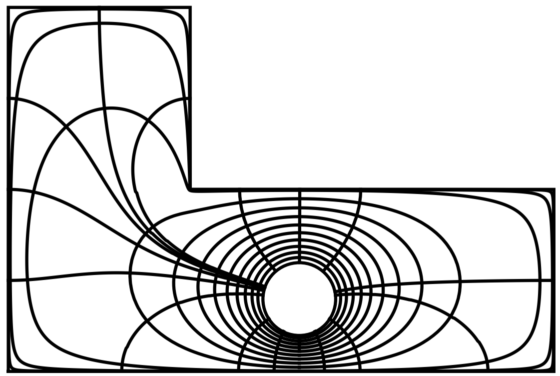

Figure: Circle in a L-block. Equipotential grid is non-uniform. The reciprocal error of the conformal mapping is of the order 10^{-6}. The modulus $\mathrm{M}(R) = 1.0935085$.