Conformal modulus of a generalized quadrilateral

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Background

A conformal modulus of a generalized quadrilateral is a non-negative real number which divides quadrilaterals into conformal equivalence classes. The history of numerical computation of a conformal modulus of a generalized quadrilateral dated back to 1960. The history of a conformal modulus began in 1950 by Lars Ahlfors and Arne Beurling [AB]. The authors introduced the concept of the extremal length which is a reciprocal to the modulus of a curve family. The conformal modulus of a generalized quadrilateral can be derived from the latter.

Modulus of a quadrilateral

There exists many equivalent ways to define the conformal modulus of a generalized quadrilateral. Here we present two different definitions. First we give a geometric definition.

Definition. (Geometric) [Küh]
Let $Q(\Omega; z_1, z_2, z_3, z_4)$ be a generalized quadrilateral. Let the function $w = f(z)$, where $w = u+iv$, be a one-to-one conformal mapping of the domain Ω onto rectangle $0 < u < 1$, $0 < v < M$ such that the vertices z_1, z_2, z_3,and z_4 correspond to the vertices $0, 1, 1+iM$, and iM, respectively. The number M is called the conformal modulus of the generalized quadrilateral $Q(\Omega; z_1, z_2, z_3, z_4)$ and we will denote it by $M(Q; z_1, z_2, z_3, z_4)$.

Another way to define the conformal modulus of a generalized quadrilateral is through the solution of the Dirichlet–Neumann problem, which is closely related to the consept of the capacity in the plane.

Definition. (Dirichlet–Neumann) [Ahl, Thm 4.5]
Let $Q(\Omega; z_1, z_2, z_3, z_4)$ be a generalized quadrilateral such that γ_j, $j = 1, 2, 3, 4$, are the arcs of ∂D between $(z_1.z_2), (z_2.z_3), (z_3, z_4)$, and (z_4, z_1), respectively. Suppose that u is the unique harmonic solution of the Dirichlet–Neumann problem with boundary values $u = 0$ on γ_2, $u = 1$ on γ_4, and $\frac{\partial u}{\partial n} = 0$ on $\gamma_1 \cup \gamma_4$.
Then

$$M(Q; z_1, z_2, z_3, z_4) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

Numerical methods

There are two natural approaches to numerical methods for computation of the conformal modulus of a generalized quadrilateral can be divided into two categories:

1. methods based on the definition of modulus and use of a conformal mapping onto a rectangle or annulus.
2. methods that gives only the modulus and not the conformal mapping.

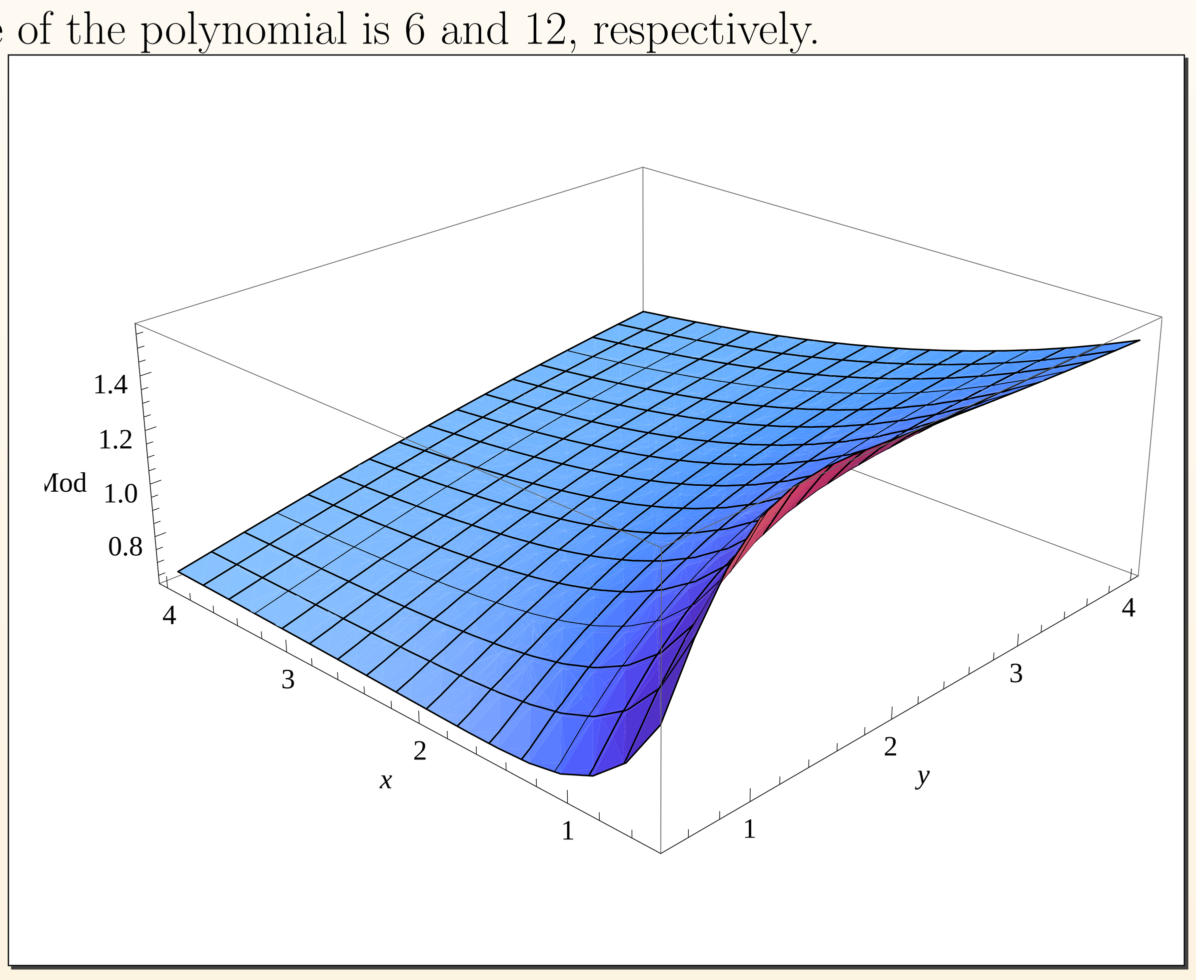
The former methods give the conformal mapping as well and like in the Schwarz–Christoffel mappings, it involves solving a parameter problem. The latter methods usually depends on solving the Dirichlet–Neumann boundary condition problem for the Laplace equation.

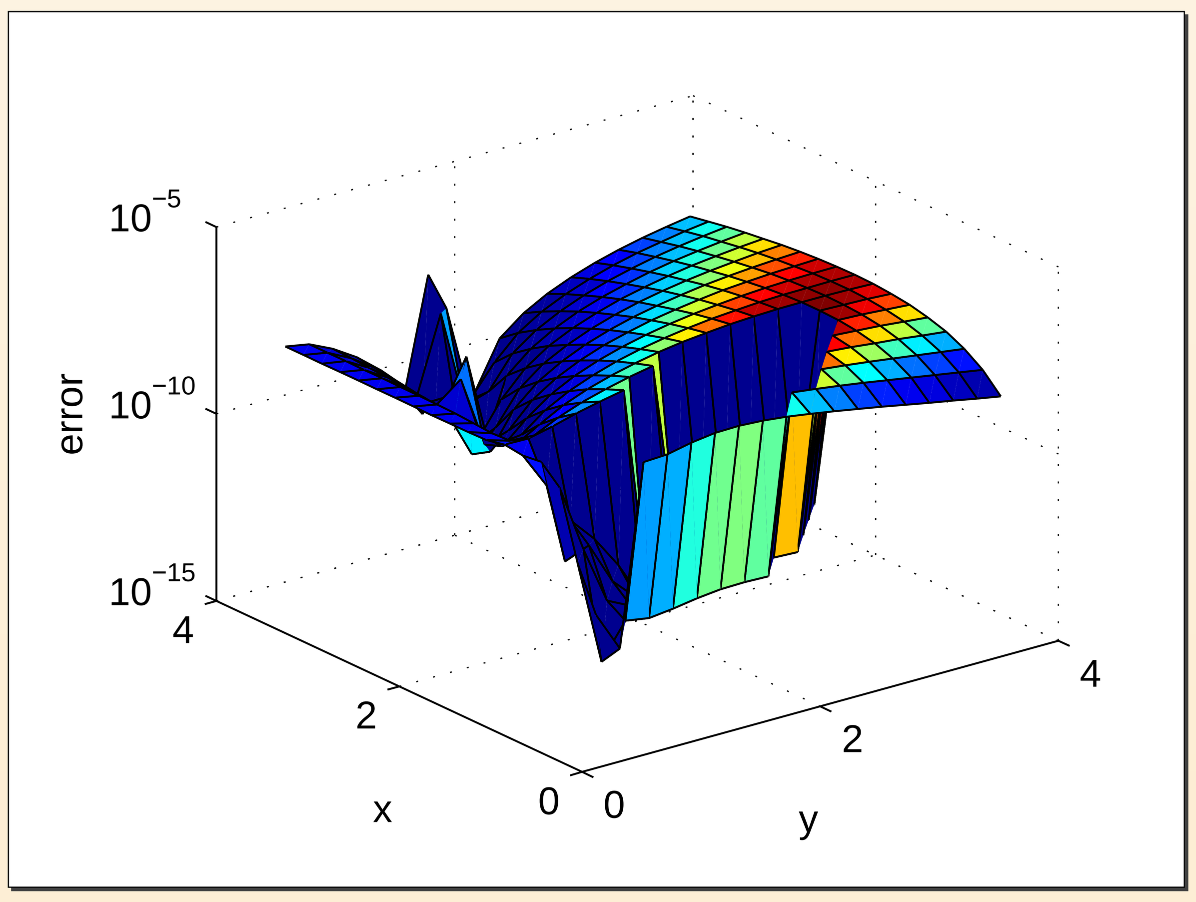
Computational example

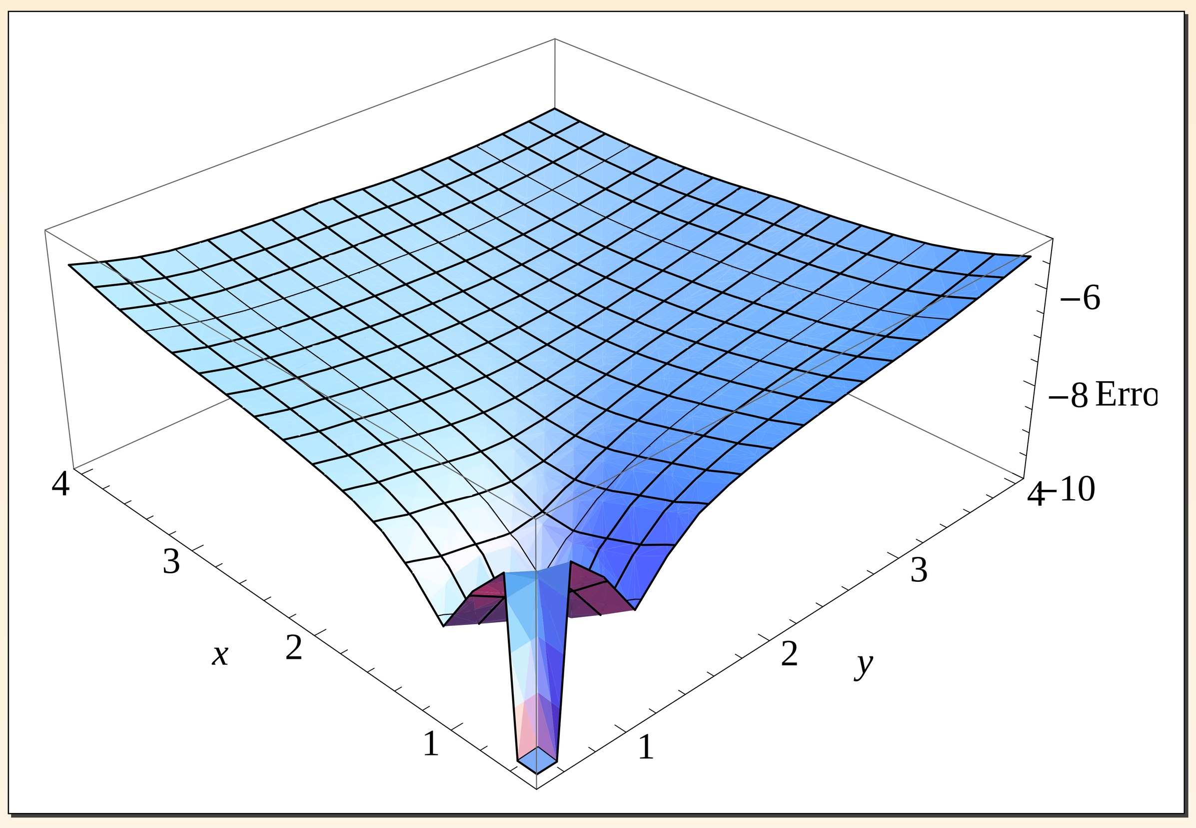
Let us consider a convex quadrilateral $Q(\Omega, 0, 1, x+iy, i)$, where $(x, y) \in [0.5; 3.9] \times [0.5; 3.9]$ and the mesh partition is 0.2. This test is carried out by the Schwarz-Christoffel toolbox [Dri] and Hakula's *hp* version of finite elements methods [HRV]. The latter test is carried out by polynomial degree of 6 and 12. We are using [HVV] as the reference result, since it is proved to be exact. The error of the methods is computed by computing the absolute difference of the result against the reference result. The plotting is done in logarithmic scale.

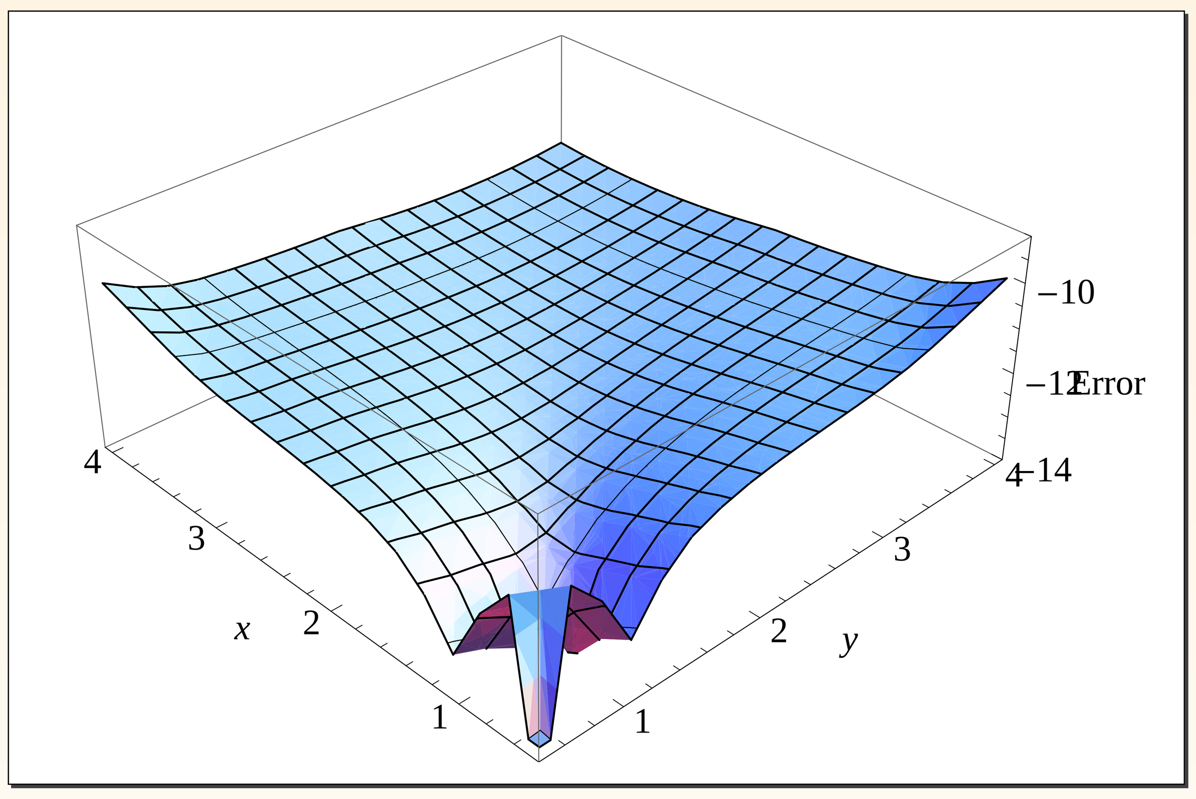
First we have a reproduction of the moduli surface from [HVV]. Then we have plotted the logarithmic error of the Schwarz–Christoffel toolbox, Hakula's *hp* version of finite element methods with polynomial degree of 6 and 12, respectively.

The value z-axis of the error graphs tells the accuracy of the methods at the corresponding point. Schwarz-Christoffel toolbox gives us $8-14$ correct digits as can be seen in the second top figure. While *hp*-FEM routine gives us $5-8$ and $10-13$ correct digits when the degree of the polynomial is 6 and 12, respectively.









References

[Ahl] L.V. Ahlfors. *Complex Analysis.* An introduction to the theory of analytic functions of one complex variable, Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.

[AB] L.V. Ahlfors, A. Beurling. *Conformal invariants and function-theoretic null-sets.* Acta Math. 83, (1950). 101–129.

[Dri] T.A. Driscoll. *Schwarz-Christoffel toolbox for MATLAB.* http://www.math.udel.edu/~driscoll/software/SC/

[HRV] H. Hakula, A. Rasila, and M. Vuorinen. *On moduli of rings and quadrilaterals: algorithms and experiments.* manuscript.

[HVV] V. Heikkala, M.K. Vamanamurthy and M. Vuorinen. *Generalized elliptic integrals.* Comput. Methods Funct. Theory 9 (2009), no. 1, 75–109.

[Küh] R. Kühnau. *The conformal module of quadrilaterals and of rings.* Handbook of complex analysis: geometric function theory. Vol. 2, 99–129, Elsevier, Amsterdam, 2005.