A conjugate function method for conformal mappings

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Contents

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Motivation

- Corformal mappings can be applied in electrostatistics, aerodynamics, etc.
- Numerical methods are considered since analytical the solution exists only for few domains.
    - Schwarz–Christoffel (SC) toolbox by Driscoll [DriTre].
    - Hu's [Hu] SC algorithm for doubly connected domains.
    - For multiply connected domains, see eg. [Crowdy, CroMar].
    - Finite element methods (FEM) approach, see eg. [HakRasVuo].

Preliminaries – Dirichlet–Neumann Problem

Definition

Let D be a domain in the complex plane whose boundary ∂D consists of a finite number of regular Jordan curves, so that at every point, except possibly at finitely many points, of the boundary a normal is defined. Let $\partial D = A \cup B$ where A, B both are unions of Jordan arcs. Let ψ_A, ψ_B be a real-valued continuous function defined on A, B, respectively. Find a function u satisfying the following conditions:

1. u is continuous and differentiable in $\overline{D}$.
2. $u(t) = \psi_A(t), \qquad \text{for all } \; t \in A.$
3. If $\partial/\partial n$ denotes differentiation in the direction of the exterior normal, then
$$\frac{\partial}{\partial n} u(t) = \psi_B(t), \qquad \text{for all } \; t \in B.$$

The conjugate problem is a problem with reversed boundary conditions. That is, $u = \psi_B$ on A and $\partial u/\partial n = \psi_A$ on B.

Preliminaries – Modulus of a Quadrilateral

- One can express the modulus of a quadrilateral $(D; z_1, z_2, z_3, z_4)$ in terms of the solution of the Dirichlet–Neumann problem as follows.

  Let γ_j, $j = 1, 2, 3, 4$ be the arcs of ∂D between (z_1, z_2), (z_2, z_3), (z_3, z_4), (z_4, z_1), respectively. If u is the (unique) harmonic solution of the Dirichlet-Neumann problem with boundary values of u equal to 0 on γ_2, equal to 1 on γ_4 and with $\partial u / \partial n = 0$ on $\gamma_1 \cup \gamma_3$, then by [Ahlfors, p. 65/Thm 4.5]:

$$\mathrm{M}(D; z_1, z_2, z_3, z_4) = \int_D |\nabla u|^2 \, dx \, dy. \tag{1}$$

- By the geometry [LehVir, p. 15] we have the following reciprocal identity

$$\mathrm{M}(D; z_1, z_2, z_3, z_4) \cdot \mathrm{M}(D; z_2, z_3, z_4, z_1) = 1. \tag{2}$$

Preliminaries – Modulus of a Ring Domain

- Let E and F be two disjoint compact sets in the extended complex plane $\mathbb{C}_\infty$. Then one of the sets E, F is bounded and without loss of generality we may assume that it is E. If both E and F are connected and the set $R = \mathbb{C}_\infty \setminus (E \cup F)$ is connected, then R is called a *ring domain*. In this case R is a doubly connected plane domain. The *capacity* of R is defined by

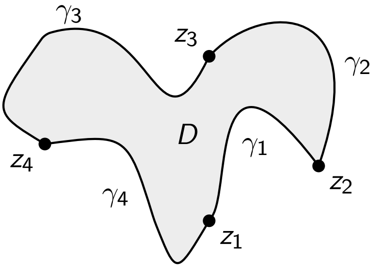
$$\mathrm{cap}\, R = \inf_u \int_R |\nabla u|^2 \, dm,$$

  where the infimum is taken over all nonnegative, piecewise differentiable functions u with compact support in $R \cup E$ such that $u = 1$ on E.
- The harmonic function on R with boundary values 1 on E and 0 on F is the unique function that minimizes the above integral.
- Conformal modulus: $\mathrm{M}(R) = 2\pi / \mathrm{cap}\, R$.

Preliminaries – Model Problem

- Laplace equation with Dirichlet–Neumann boundary conditions

$$\begin{cases} \Delta u = 0, & \text{in } D, \\ u = 0, & \text{on } \gamma_2, \\ u = 1, & \text{on } \gamma_4, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \gamma_1 \cup \gamma_3. \end{cases} \tag{3}$$



Preliminaries – Lemma

Lemma

Let $(D; z_1, z_2, z_3, z_4)$ *be a quadrilateral and let* u *satisfy the equation (3) and let* v *be a conjugate harmonic function of* u*. Then* $f = u + iv$ *is a conformal mapping, and it maps* D *onto a rectangle such that the images of the points* z_1, z_2, z_3, z_4 *are* $1 + ih, ih, 0, 1$*, respectively. The mapping* f *maps the boundary curves* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *onto curves* $\gamma_1', \gamma_2', \gamma_3', \gamma_4'$*, respectively.*

Preliminaries – Illustration

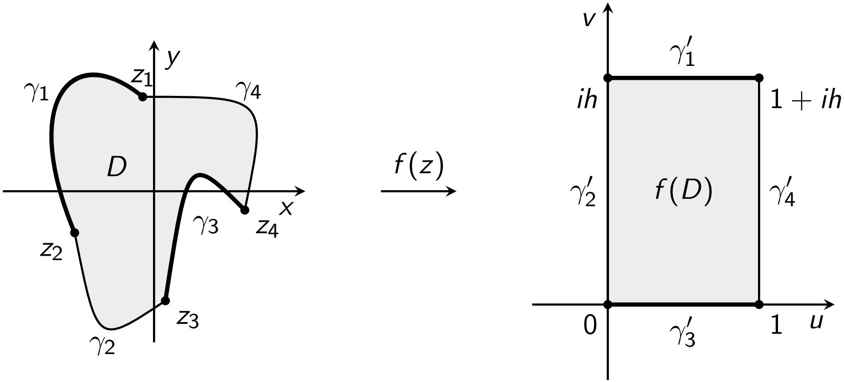


Figure: Dirichlet–Neumann boundary value problem. Dirichlet and Neumann boundary conditions are mark with thin and thick lines, respectively.

Main Theorem

Theorem (Conjugate Function Method)

Let $(D; z_1, z_2, z_3, z_4)$ *be a quadrilateral which modulus* h *and let* u_1 *satisfy, as before, the Laplace equation with Dirichlet–Neumann boundary conditions. Let* u_2 *satisfy the conjugate Dirichlet–Neumann boundary value problem. Then there exists a conformal mapping* $f = u_1 + ihu_2$ *which maps* D *onto a rectangle such that the images of the points* z_1, z_2, z_3, z_4 *are* $1 + ih, ih, 0, 1$*, respectively. The mapping* f *maps the boundary curves* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *onto curves* $\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4$*, respectively.*

Proof of the Main Theorem

- **Proof:** By the above lemma there exists a conformal mapping $f_1 = u_1 + iv_1$ that maps D onto a rectangle with the given properties.
- On the other hand by conjugating boundary conditions, we have a conformal mapping $f_2 = u_2 + iv_2$ that maps D onto a rectangle such that z_1, z_2, z_3, z_4 maps to $1 + i/h, i/h, 0, 1$, respectively.
- Then by multiplying (scaling and rotating) the latter case by ih and translating the rectangle such z_2 maps to the origin, we have mapped the points z_1, z_2, z_3, z_4 to $ih, 0, 1, 1 + ih$, respectively. Thus we have a mapping $z \mapsto ihf_2 + 1 = ihu_2 - hv_2 + 1$.
- The rectangle is exactly the same as $f_1(D)$ with conjugated boundariy conditions. This implies $f_1 = ihf_2 + 1$, and $v_1 = hu_2$.
- Since scaling, rotating and translating are conformally invariant and with the fact that the solution of the Laplace equation is unique yields the proof.

Error Estimates for Conformal Mappings

- **Error:** Two distinctive components: conformal mappings and the point-wise error from level curves.
- **Quadrilateral**: For an error estimates, first we compute the modulus h_1 and conjugate modulus h_2. Then by reciprocal identity it should hold that $h_1 h_2 = 1$, and we use the absolute value of the difference as the error estimate

$$\mathrm{err} = |h_1 h_2 - 1|.$$

- **Ring domain**: First we compute the modulus h. Then we split the domain through the curve of the deepest descent. Thus we obtain a quadrilateral such that the Neumann condition lies on the splitting curve and Dirichlet condition are as before. Then we compute modulus h_1 and conjugate modulus h_2 and use following quantity as an error estimate

$$\mathrm{err} = \max\{|hh_2 - 1|, |h_1 h_2 - 1|\}.$$

Algorithm

The algorithm to compute the conformal mapping and level curves.

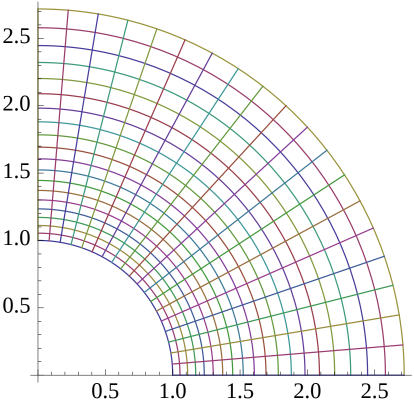
1. Solve the potential function u and modulus h. (doubly connected)
2. Solve the DN–problem to obtain u_1 and compute the modulus h_1.
3. Solve the conjugate DN–problem for u_2 and h_2.
4. Compute the error,

$$\mathrm{err} = \begin{cases} |h_1 h_2 - 1|, & \text{(simply connected)}, \\ \max\{|hh_2 - 1|, |h_1 h_2 - 1|\}, & \text{(doubly connected)}. \end{cases}$$

5. Form the conformal mapping $f = u_1 + ih_1 u_2$.
6. Fix the level curves Γ in the rectangle and solve (x, y) from the equation

$$f(x, y) = u + iv, \quad (u, v) \in \Gamma.$$

Example – Analytic



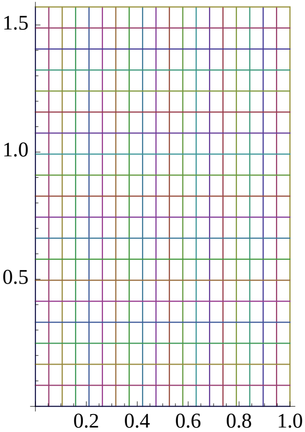


Figure: Conformal mapping from a quadrilateral onto a rectangle.

Example – Schwarz (1869)

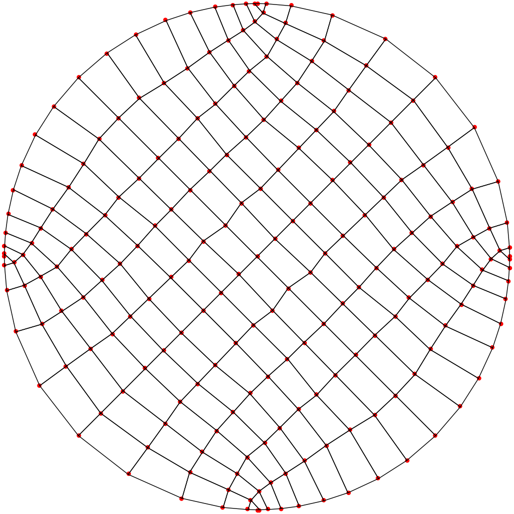


Figure: Error of the conformal mapping, $\mathrm{err} = 10^{-13}$.

Example – Quadrilateral

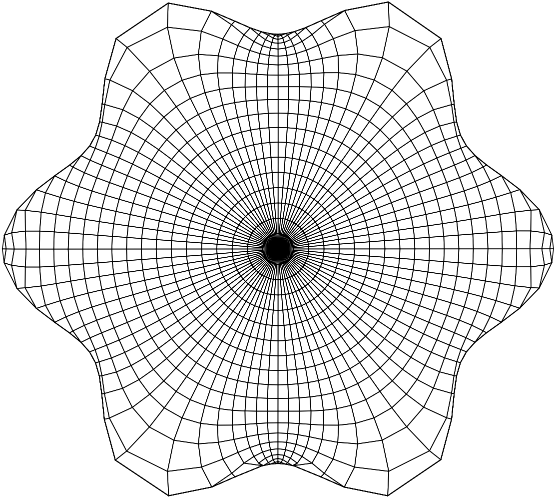


Figure: Error of the conformal mapping, $\mathrm{err} = 10^{-11}$.

Example – Ring Domain

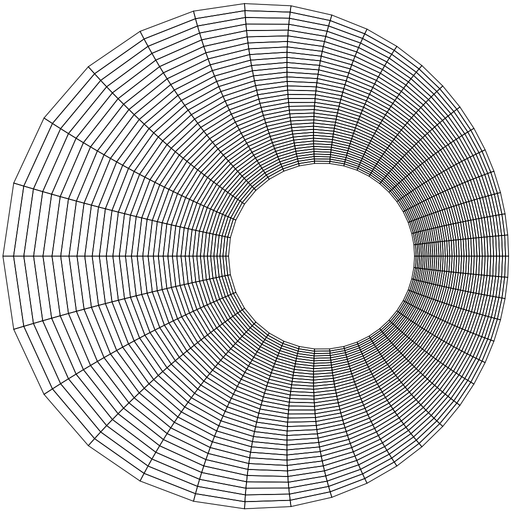


Figure: Error of the conformal mapping, $\mathrm{err} = 10^{-14}$.