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Pricing by hedging and no-arbitrage beyond semimartingales

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Abstract We show that pricing a big class of relevant options by hedging and no-arbitrage can be extended beyond semimartingale models. To this end we construct a subclass of self-financing portfolios that contains hedges for these options, but does not contain arbitrage opportunities, even if the stock price process is a non-semimartingale of some special type. Moreover, we show that the option prices depend essentially only on a path property of the stock price process, viz. on the quadratic variation. We end the paper by giving no-arbitrage results even with stopping times for our model class.

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1 Introduction

The foundation of modern theory of option pricing is twofold. On the one hand pricing by no arbitrage arguments requires that the market model must not allow arbitrage opportunities within the investor's subclass of allowed self-financing strategies. On the other hand one would like to hedge as many contingent claims as possible to determine option prices unambiguously. It is well-known that the classical Black-Scholes model does not permit arbitrage, if one is not allowed to use so-called doubling strategies. In this model this restriction on strategies does not destroy the completeness, i.e. all contingent claims can he hedged, and the initial capital needed for the hedge gives the unique price for the option. Note, however, that, in more general situations, the no-arbitrage principle and the hedging principle may put opposed requirements to the class of the investor's allowed strategies: While it is always possible to restrict the set of strategies in a way that no arbitrage possibilities remain, the set of replicable options might become uninteresting, when one imposes too restrictive conditions on the set of allowed trading strategies. Hence, in order to set up a sensible pricing model one can pose the following question: Given a model, is it possible to find a subclass of self-financing strategies that is arbitrage-free and, at the same time, rich enough to be interesting from the hedging point of view?

In this paper we construct such a class of strategies for possibly non-semimartingale models that have the same quadratic variation as the Black-Scholes model or, more generally, as Brownian models with local volatility structures. To explain the main ideas, let us have a look at the classical Black-Scholes model.

In this model the hedging of simple European options depends on the volatility parameter σ through the Black-Scholes partial differential equation. In particular, the unique hedging price is independent of the drift. The Black-Scholes equation is well known to be a consequence of Itô's formula. Note, however, that the Itô-formula has the same form for any continuous process that has the same pathwise quadratic variation as the Brownian motion. This was observed by Föllmer, and he applied this observation to define stochastic integrals without probability in [7]. By using this pathwise Itô calculus Schoenmakers and Kloeden extended the hedging of simple European options by allowing stochastic trends with zero quadratic variation, see [15]. This type of extended models can, of course, lack the semimartingale property as for example some models simultaneously driven by a Brownian and a fractional Brownian motion do.

We extend the results of Schoenmakers and Kloeden to a certain class of path dependent options, including e.g. Asian and lookback options. This leads to a class of hedging strategies, which depend in a smooth way on the time, on the spot of the stock, and on 'hindsight' factors (see Definition 3 below) which include the running minimum, running maximum and running average of the stock. We call this class of strategies 'allowed'. We believe that, apart from the idealization of continuous readjustment of the portfolio, this class of portfolios is economically meaningful, as it covers the hedges to many practically relevant options. In particular we prove a robustness result, that basically says: If an option (which is a continuous functional of the stock path) can be replicated with an allowed strategy then the hedge and its initial capital, i.e. the hedging price, depend only on the pathwise quadratic variation of the stock-price process.

As mentioned, we allow non-semimartingale models, which are typically ruled out as sensible pricing models by the fundamental theorem of asset pricing. This theorem states that a notion of absence of arbitrage, namely the property of 'no free lunch with vanishing risk', is equivalent to the existence of equivalent local martingale measures, see [4]. However, we show that the aforementioned class of allowed strategies is free of arbitrage for a large class of non-semimartingale models. In particular this result covers the mixed fractional Black-Scholes model, our prime example throughout the paper (see [1,12,20] for related results on mixed fractional Brownian models). We hence contribute to the arbitrage discussion related to fractional Brownian motion which has gained considerable interest in recent years (see also [8] for some discussion on arbitrage results for pure fractional models). Our no-arbitrage result shows that some non-smooth functional behavior is required to construct arbitrage via distributional properties, which is not inherent in hedges of many interesting options. We even show that the no-arbitrage result still holds, if a portfolio is changed abruptly at stopping times from a reasonably large class. In particular, our no-arbitrage results cover many simple predictable strategies. Therefore, by restricting trading to allowed strategies, we put the investor in an environment, where prices for many relevant options can be determined by replication and the no-arbitrage principle. We emphasize that the option prices essentially depend only on the quadratic variation which can be viewed as a path property. Therefore option prices are robust with respect to probabilistic properties. Finally, we notice that, when the model is a non-semimartingale, it will allow for some sort of approximate arbitrage, and we show this explicitly in the context of the mixed fractional Black-Scholes model.

The paper is organized as follows: In Section 2 we review some facts on forward integration and pathwise quadratic variation. Model classes in dependence of the local quadratic variation are introduced in Section 3. In Section 4 we present a no-arbitrage result for 'smooth' strategies, while robust replication is studied in Section 5. In Section 6 we discuss absence of arbitrage opportunities with stopping times and exemplify approximate arbitrage. Section 7 concludes.

2 Simple review of forward integration

We consider processes that are not semimartingales. So, the classical stochastic integration theory is not at our disposal. However, there is an economically meaningful notion of integral, viz. the *forward integral*, that can be applied for non-semimartingales. Actually, there are many slightly different versions of the forward integral. In this paper we use a simplistic approach introduced by [7]; see also [16] for more information how this approach can be used in finance. For different kinds of forward integrals we refer to [11], [14] and [20].

Let $T > 0$ be fixed throughout the paper and let $\pi_n = \{0 = t_{n,0} < \cdots < t_{n,K(n)} = T\}$ be such partitions of $[0,T]$ that

$$\mathrm{mesh}(\pi_n) := \max_{t_{n,k}\in\pi_n} |t_{n,k} - t_{n,k-1}| \to 0$$

as $n \to \infty$. Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ is a filtered probability space satisfying the usual conditions of completeness and right continuity of the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Later we cannot assume that our processes are properly integrable over the entire interval $[0,T]$. Thus we define the integrals over the subintervals $[0,t]$, $t < T$. The integral over the interval $[0,T]$ will be then interpreted as an improper forward integral.

Definition 1 *Let $t < T$ and let $X = (X_s)_{s\in[0,T]}$ be a continuous process. The* forward integral *of a process $Y = (Y_s)_{s\in[0,T]}$ with respect to X* along the sequence

$(\pi_n)_{n=1}^{\infty}$ *is*

$$\int_0^t Y_s \, \mathrm{d}X_s := \lim_{n\to\infty} \sum_{\substack{t_{n,k}\in\pi_n \\ t_{n,k}\le t}} Y_{t_{n,k-1}} \left(X_{t_{n,k}} - X_{t_{n,k-1}}\right),$$

where the limit is assumed to exist $\mathbf{P}$*-a.s. The forward integral over the whole interval* $[0,T]$ *is the improper forward integral*

$$\int_0^T Y_s \, \mathrm{d}X_s := \lim_{t\uparrow T} \int_0^t Y_s \, \mathrm{d}X_s,$$

where the limit is again understood in the $\mathbf{P}$*-a.s. sense.*

Definition 2 *A process* $X = (X_t)_{t\in[0,T]}$ *is a* quadratic variation process along the sequence $(\pi_n)_{n=1}^{\infty}$ *if for all* $t \le T$ *the limit*

$$\langle X\rangle_t := \sum_{\substack{t_{n,k}\in\pi_n \\ t_{n,k}\le t}} \left(X_{t_{n,k}} - X_{t_{n,k-1}}\right)^2$$

exists $\mathbf{P}$*-a.s., and is continuous in* t.

Example 1 (i) For standard Brownian motion W we have $\mathrm{d}\langle W\rangle_t = \mathrm{d}t$ if the sequence (π_n) is refining. This follows from the Borel-Cantelli lemma.

(ii) If Z is a continuous process with zero quadratic variation along (π_n) and X is a continuous quadratic variation process along (π_n) then $\mathrm{d}\langle X+Z\rangle_t = \mathrm{d}\langle X\rangle_t$. This follows from the Cauchy-Schwartz inequality.

(iii) If X is a quadratic variation process along (π_n) and $f \in \mathcal{C}^1(\mathbb{R})$ then $Y = f\circ X$ is also a quadratic variation process along (π_n). Indeed,

$$\mathrm{d}\langle Y\rangle_t = f'(X_t)\, \mathrm{d}\langle X\rangle_t$$

(cf. [7] p. 148).

In what follows the sequence (π_n) will be fixed and omitted in the text.

The following Itô formula for the forward integral is a simple generalization of the theorem that can be found in [7] p. 144. The proof is based on a second order multidimensional Taylor expansion. Actually, it is basically the same as in the semimartingale case.

Lemma 1 (Itô formula) *Let* X *be a continuous quadratic variation process,* $Y^1,\dots,Y^m$ *continuous bounded variation processes and suppose* $f \in \mathcal{C}^{1,2,1}([0,T)\times\mathbb{R}\times\mathbb{R}^m)$. *Let* $0 \le s \le t < T$. *Then*

$$\begin{aligned} f(t,X_t,Y_t^1,\dots,Y_t^m) = {} & f(s,X_s,Y_s^1,\dots,Y_s^m) \\ & + \int_s^t \frac{\partial}{\partial t} f(u,X_u,Y_u^1,\dots,Y_u^m)\,\mathrm{d}u \\ & + \int_s^t \frac{\partial}{\partial x} f(u,X_u,Y_u^1,\dots,Y_u^m)\,\mathrm{d}X_u \end{aligned}$$

$$+\frac{1}{2}\int_s^t \frac{\partial^2}{\partial x^2} f(u, X_u, Y_u^1, \ldots, Y_u^m)\,\mathrm{d}\langle X\rangle_u$$
$$+\sum_{n=1}^m \int_s^t \frac{\partial}{\partial y_n} f(u, X_u, Y_u^1, \ldots, Y_u^m)\,\mathrm{d}Y_u^n.$$

In particular, this formula implies the forward integral on the right hand side exists and has a continuous modification.

Remark 1 In the remainder of the paper we choose continuous modifications of forward integrals, whenever possible.

Remark 2 The forward integral with non-semimartingale integrator does not satisfy a dominated convergence theorem. Therefore, we have to impose some continuity assumptions on the integrands (cf. Definition 3 of hindsight factors in Section 4). The lack of dominated convergence theorem may cause some sort of approximate arbitrage, see Section 6.

3 Model classes

We now introduce model classes in dependence of the quadratic variation. A discounted *market model* is a five-tuple $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$ such that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ is a filtered probability space satisfying the usual conditions and $S = (S_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-progressively measurable quadratic variation process with continuous paths starting at $s_0 > 0$.

Suppose a continuously differentiable function $\sigma : \mathbb{R} \to \mathbb{R}$ with linear growth is given. The corresponding model class $\mathcal{M}_\sigma$ will be defined via the quadratic variation property

$$\mathrm{d}\langle S\rangle_t = \sigma^2(S_t)\,\mathrm{d}t \qquad \mathbf{P}-\text{a.s.} \tag{1}$$

and a non-degeneracy property. In order to formulate the latter property let f_σ denote the unique solution of the ordinary differential equation

$$f'(x) = \sigma(f(x)), \quad f(0) = s_0.$$

Since σ is continuously differentiable, f_σ belongs to $\mathcal{C}^2(\mathbb{R})$ and

$$f_\sigma''(x) = f_\sigma'(x)\sigma'(f_\sigma(x)). \tag{2}$$

Define the space

$$\mathcal{C}_{\sigma,s_0} := \{f_\sigma \circ \theta\ ;\ \theta \in \mathcal{C}([0,T]),\ \theta(0) = 0\}. \tag{3}$$

We assume that the following small ball condition is satisfied: Given $\eta \in \mathcal{C}_{\sigma,s_0}$ and $\varepsilon > 0$

$$\mathbf{P}\Big(\|S-\eta\|_\infty < \varepsilon\Big) > 0, \tag{4}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on the interval $[0,T]$. Summarizing the foregoing, the model class $\mathcal{M}_\sigma$ is defined to contain those discounted market models which satisfy (1), (4) and $\mathbf{P}(S \in \mathcal{C}_{\sigma,s_0}) = 1$.

We illustrate this definition by an example.

Example 2 Suppose $\sigma(x) = \sigma x$ for some constant $\sigma > 0$. Obviously,

$$f_\sigma(x) = s_0 e^{\sigma x}.$$

Hence, condition (4) means that the support of the stochastic process S is the space of nonnegative continuous functions starting in s_0. In particular, the standard Black-Scholes model belongs to this class $\mathcal{M}_\sigma$. We will consider its risk-neutral version (i.e. with zero drift) as reference model in $\mathcal{M}_\sigma$, because of its martingale property. A prominent non-semimartingale model in the class $\mathcal{M}_\sigma$ is the mixed fractional Black-Scholes model, where the Brownian motion of the standard Black-Scholes model is replaced by a sum of a Brownian motion and an independent fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, \frac{3}{4}]$. Recall that a fractional Brownian motion Z is a centred stationary increment Gaussian process with variance $\mathbf{E}(Z_t^2) = t^{2H}$ for some $H \in (0,1)$, and for $H \in (\frac{1}{2}, 1)$ it has zero quadratic variation. Another way of characterizing the fractional Brownian motion is to say that it is the unique (up to a multiplicative constant) centred H-self-similar Gaussian process with stationary increments. It is known from [2] that the sum of independent Brownian and fractional Brownian motion is a semimartingale if and only if $H \in (\frac{3}{4}, 1)$.

Next we construct a reference model which plays the role of the risk-neutral Black-Scholes model for general $\mathcal{M}_\sigma$. Suppose $(\tilde{\Omega}, \tilde{\mathcal{F}}, \bar{\mathbf{P}})$ is the canonical Wiener space on the time interval $[0,T]$, $W_t(\omega) = \omega(t)$ the coordinate process Brownian motion and $\tilde{\mathcal{F}}_t$ the filtration generated by W. We impose the following standing assumption:

(H) The process

$$M_t = \exp\left\{-\frac{1}{2}\int_0^t \sigma'(f_\sigma(W_r))\mathrm{d}W_r - \frac{1}{8}\int_0^t \left(\sigma'(f_\sigma(W_r))\right)^2 \mathrm{d}r\right\}$$

is well defined (i.e. the integrals exist) and is a martingale under $\bar{\mathbf{P}}$.

Under hypothesis **(H)** we can define a probability measure $\tilde{\mathbf{P}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}_T)$ by

$$\tilde{\mathbf{P}}(A) = \int_A M_T \,\mathrm{d}\bar{\mathbf{P}}, \quad A \in \tilde{\mathcal{F}}_T.$$

Then $(\tilde{W}_t)_{t\in[0,T]}$ given by

$$\tilde{W}_t = W_t + \frac{1}{2}\int_0^t \sigma'(f_\sigma(W_r))\,\mathrm{d}r$$

is a Brownian motion under $\tilde{\mathbf{P}}$ by the Girsanov theorem.

Define a discounted stock price by $\tilde{S}_t = f_\sigma(W_t)$. We obtain:

Lemma 2 $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}}) \in \mathcal{M}_\sigma$ *and* $\tilde{S}$ *is a martingale. We call* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}})$ *the* reference model.

Proof The small ball property is trivially satisfied under $\bar{\mathbf{P}}$ and hence under $\tilde{\mathbf{P}}$. Moreover, Itô's formula and (2) yield,

$$\begin{aligned}\tilde{S}_t &= s_0 + \int_0^t f'_\sigma(W_r)\,\mathrm{d}W_r + \frac{1}{2}\int_0^t f'_\sigma(W_r)\sigma'(f_\sigma(W_r))\,\mathrm{d}r \\ &= s_0 + \int_0^t f'_\sigma(W_r)\,\mathrm{d}\tilde{W}_r \\ &= s_0 + \int_0^t \sigma(\tilde{S}_r)\,\mathrm{d}\tilde{W}_r.\end{aligned}$$

Thus, (1) is satisfied. Moreover $\tilde{S}$ is a martingale under $\tilde{\mathbf{P}}$ since σ is of linear growth.

We now give an important example where condition **(H)** is satisfied.

Example 3 Suppose $\sigma(x) = x\tilde{\sigma}(x)$ with $\tilde{\sigma} \in \mathcal{C}^1(\mathbb{R})$ bounded and $x\tilde{\sigma}'(x)$ bounded. Both boundedness conditions are met, when $\tilde{\sigma}$ is constant for $|x|$ sufficiently large. Then $\sigma'(x) = \tilde{\sigma}(x) + x\tilde{\sigma}'(x)$ is bounded and consequently **(H)** follows from Novikov's condition. In this situation the reference model is a risk neutral generalized Black-Scholes model with stochastic volatility depending on the spot,

$$\tilde{S}_t = s_0 + \int_0^t \tilde{\sigma}(\tilde{S}_r)\tilde{S}_r\,\mathrm{d}\tilde{W}_r.$$

These so-called local volatility models were suggested by [6] in order to capture the implied volatility smile.

We conclude this section with an example on how to construct further models in $\mathcal{M}_\sigma$.

Example 4 Suppose W is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ and Y is a continuous process with zero quadratic variation independent of W which satisfies the small ball condition

$$\mathbf{P}\Big(\|Y\|_\infty < \varepsilon\Big) > 0.$$

For instance, Y could be a fractional Brownian motion with Hurst parameter bigger than a half. Define $S_t = f_\sigma(W_t + Y_t)$. Then $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$ belongs to $\mathcal{M}_\sigma$. Indeed, the quadratic variation of S is easily calculated by Example 1 and from the small ball property of Y around zero and the independence one obtains (4). Observe that by Itô's formula

$$S_t = s_0 + \int_0^t \sigma(S_r)\,\mathrm{d}(W+Y)_r + \frac{1}{2}\int_0^t f'_\sigma(W_r + Y_r)\sigma'(f_\sigma(W_r + Y_r))\,\mathrm{d}r.$$

More general drifts can be introduced by performing a Girsanov change of measure on the Brownian motion only. (Note, the law of Y remains unchanged by the Girsanov transformation due to independence.)

Remark 3 The introduction of time dependent local quadratic variation functions $\sigma(t,x)$ does not cause any difficulties, but makes the presentation more cumbersome.

4 No-arbitrage with smooth strategies

We shall now derive a no-arbitrage result for strategies that depend in a deterministic and smooth way on time, the spot price S_t and some additional economically relevant factors such as e.g. the running maximum, minimum, and average of the stock.

We first specify some assumptions on the additional economic factors on which the strategy may depend.

Definition 3 *A mapping* $g : [0,T] \times \mathcal{C}_{\sigma,s_0} \to \mathbb{R}$ *is a* hindsight factor, *if*

(i) $g(t,\eta) = g(t,\tilde{\eta})$ *whenever* $\eta(s) = \tilde{\eta}(s)$ *for all* $0 \le s \le t$,
(ii) $g(\cdot;\eta)$ *is of bounded variation and continuous for every* $\eta \in \mathcal{C}_{\sigma,s_0}$,
(iii) there is a constant K *such that for every continuous function* f

$$\left| \int_0^t f(s) \mathrm{d}g(s,\eta) - \int_0^t f(s) \mathrm{d}g(s,\tilde{\eta}) \right| \le K \max_{0 \le r \le t} |f(r)| \cdot \|\eta - \tilde{\eta}\|_\infty. \quad (5)$$

Property (i) is the natural assumption that the factors must not contain information about the future stock prices. Properties (ii)–(iii) are technical assumptions which we need since the forward integral is not continuous in terms of the integrands.

The running maximum, minimum, and average are denoted, respectively,

$$\begin{aligned} \eta^*(t) &:= \max_{s \in [0,t]} \eta(s), \\ \eta_*(t) &:= \min_{s \in [0,t]} \eta(s), \\ \bar{\eta}(t) &:= \int_0^t \eta(s) \, \mathrm{d}s. \end{aligned}$$

(We do not include the factor $1/t$ in the running average. This is just a matter of convenience since one can always include the factor $1/t$ in the ‘strategy function’ φ in (8).)

Proposition 1 *The running maximum, minimum, and average are hindsight factors.*

Proof Properties (i) and (ii) of Definition 3 are obviously satisfied for the running maximum, minimum, and average. Moreover, property (iii) is trivial for the running average. We now prove a somewhat stronger assertion than (iii) for the running maximum. Suppose $f,\ g,\ \tilde{g}$ are continuous functions on $[0,t]$ and define

$$\mathcal{I}(t; f, g) = \int_0^t f(s) \, \mathrm{d}g^*(s),$$

where $g^*(s) = \max_{u \in [0,s]} g(u)$. We shall show that

$$|\mathcal{I}(t; f, g) - \mathcal{I}(t; f, \tilde{g})| \le 4 \max_{r \in [0,t]} |f(r)| \max_{r \in [0,t]} |g(r) - \tilde{g}(r)|. \quad (6)$$

We first consider the case of non-negative f. Since

$$\mathrm{d}(g^* + \tilde{g}^*) \geq \mathrm{d}(g + \tilde{g})^*,$$

we obtain for non-negative f the sub-additivity of $\mathcal{I}(t; f, \cdot)$:

$$\mathcal{I}(t; f, g + \tilde{g}) \leq \mathcal{I}(t; f, g) + \mathcal{I}(t; f, \tilde{g}).$$

Hence

$$\mathcal{I}(t; f, g) - \mathcal{I}(t; f, \tilde{g}) \leq \mathcal{I}(t; f, g - \tilde{g}). \tag{7}$$

Using the Love-Young inequality (with sup-norm and total variation norm) and the fact that the total variation of the running maximum is dominated by two times the running maximum, we have for general f

$$\begin{aligned} |\mathcal{I}(t; f, g)| &\leq \max_{r \in [0,t]} |f(r)| \, \mathsf{TV}_{[0,t]}(g^*) \\ &\leq 2 \max_{r \in [0,t]} |f(r)| \max_{r \in [0,t]} |g(r)|. \end{aligned}$$

Combining this with (7) the inequality (6) follows for non-negative f with constant 2 instead of 4. From (7) and noting $\mathcal{I}(t; f, g) = -\mathcal{I}(t; -f, g)$ we get for non-positive f

$$\mathcal{I}(t; f, g) - \mathcal{I}(t; f, \tilde{g}) \leq \mathcal{I}(t; -f, \tilde{g} - g),$$

which yields (6) for non-positive f with constant 2 instead of 4. The general case with constant 4 follows now from the linearity of $\mathcal{I}(t; \cdot, g)$ and the triangle inequality.

The analogous inequality of (6) for the running minimum can be straightforwardly reduced to the case of the running maximum, since

$$\int_0^t f(s) \, \mathrm{d}g_*(s) = -\int_0^t f(s) \, \mathrm{d}(-g)^*(s).$$

Suppose hindsight factors $g_1, \ldots, g_m$ and a function $\varphi : [0,T] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is given. We consider strategies of the form

$$\Phi_t = \varphi\Big(t, S_t, g_1(t,S), \ldots, g_m(t,S)\Big). \tag{8}$$

Here Φ_t denotes the number of stocks held at time t by an investor. Hence, the wealth process corresponding to the strategy Φ is

$$V_t(\Phi, v_0; S) = v_0 + \int_0^t \Phi_u \, \mathrm{d}S_u, \tag{9}$$

where $v_0 \in \mathbb{R}$ denotes the investor's initial capital. (Recall, the stochastic integral is defined as a limit of forward sums. Thus, this definition reflects the classical and economically meaningful condition for a self-financing portfolio.)

Definition 4 *A strategy Φ is called* smooth, *if it is of form (8) with $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R} \times \mathbb{R}^m)$ and it is nds-admissible (nds stands for no-doubling-strategies) in the classical sense, i.e. there is a constant $a > 0$ such that for all $t \in [0,T]$*

$$\int_0^t \Phi_u \, \mathrm{d}S_u \geq -a \quad \mathbf{P}-\text{a.s.}.$$

(Of course, the smoothness condition can be relaxed to, say, $[0,T] \times \mathbb{R}_+ \times \mathbb{R}_+^m$ when one knows a priori, that the stock and the hindsight factors are positive).

Recall that a strategy Φ is an *arbitrage* in the market model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$, if

$$V_T(\Phi, 0; S) \geq 0 \quad \mathbf{P}-\text{a.s.} \qquad \text{and} \qquad \mathbf{P}(V_T(\Phi, 0; S) > 0) > 0.$$

Next we prove a result on absence of arbitrage with smooth Φ.

Theorem 1 *Suppose the standing assumption* **(H)** *holds. Let $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ and suppose Φ is smooth. Then Φ cannot be an arbitrage in the model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$.*

To prepare the proof we define an auxiliary wealth functional for $\tau \in [0,T]$ by

$$v : [0,\tau] \times \mathcal{C}_{\sigma,s_0} \times \mathcal{C}^1([0,\tau] \times \mathbb{R} \times \mathbb{R}^m) \to \mathbb{R}$$

as the Itô formula suggests:

$$\begin{aligned} v(t,\eta;\varphi) := {} & u(t,\eta(t), g_1(t;\eta), \ldots, g_m(t;\eta)) \\ & - \sum_{n=1}^m \int_0^t \frac{\partial}{\partial y_n} u(r,\eta(r), g_1(r;\eta), \ldots, g_m(r;\eta)) \, \mathrm{d}g_n(r;\eta) \\ & - \int_0^t \frac{\partial}{\partial t} u(r,\eta(r), g_1(r;\eta), \ldots, g_m(r;\eta)) \, \mathrm{d}r \\ & - \frac{1}{2} \int_0^t \frac{\partial}{\partial x} \varphi(r,\eta(r), g_1(r;\eta), \ldots, g_m(r;\eta)) \sigma^2(\eta(r)) \, \mathrm{d}r, \end{aligned} \tag{10}$$

where

$$u(t,x,y_1,\ldots,y_m) = \int_{s_0}^x \varphi(t,\xi,y_1,\ldots,y_m) \, \mathrm{d}\xi. \tag{11}$$

The next crucial lemma shows that v is a continuous wealth functional.

Lemma 3 *Let $0 \leq t \leq T$ and suppose Φ is a smooth strategy induced by φ in the market $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$. Then we have, for $0 \leq t \leq T$,*

$$V_t(\Phi, v_0; S) = v_0 + v(t, S; \varphi) \quad \mathbf{P}-\text{a.s.}$$

Moreover, the mapping

$$\mathcal{C}_{\sigma,s_0} \to \mathcal{C}([0,T]), \quad \eta \mapsto v(\cdot, \eta; \varphi)$$

is continuous.

Proof Applying Itô's formula to u given by (11) we obtain for $0 \leq t \leq \tau$ that

$$\int_0^t \Phi_u \mathrm{d}S_u = v(t, S; \varphi) \quad \mathbf{P}-\text{a.s.}$$

Hence,

$$V_t(\Phi, v_0; S) = v_0 + v(t, S; \varphi) \quad \mathbf{P}-\text{a.s.}$$

To prove continuity of $v(t, \cdot; \varphi)$ let $(\eta_n) \subset \mathcal{C}_{\sigma,s_0}$ be a sequence which converges to $\eta \in \mathcal{C}_{\sigma,s_0}$ in the sup-norm. Then η_n and η take values in a compact set $A_0 \subset \mathbb{R}$. Hence, due to (5) applied to $f = 1$, there is a compact set $A \subset \mathbb{R}$ such that for all $0 \leq j \leq m$, $n \in \mathbb{N}$ and $0 \leq t \leq T$ $g_j(t, \eta_n)$ and $g_j(t, \eta)$ take values in A. Moreover, applying (5) again to $f = 1$ we see that

$$|g_j(t, \eta_n) - g_j(t, \eta)| \leq K \|\eta_n - \eta\|_\infty. \tag{12}$$

Thus, by the continuity of φ, $(\partial/\partial x)\varphi$, and $(\partial/\partial t)u$, the dominated convergence theorem yields

$$\begin{aligned} &u(\Sigma(t, \eta_n)) - \int_0^t \frac{\partial}{\partial t} u(\Sigma(r, \eta_n))\,\mathrm{d}r - \frac{1}{2}\int_0^t \frac{\partial}{\partial x}\varphi(\Sigma(r, \eta_n))\sigma^2(\eta_n(r))\,\mathrm{d}r \\ \to\, &u(\Sigma(t, \eta)) - \int_0^t \frac{\partial}{\partial t} u(\Sigma(r, \eta))\,\mathrm{d}r - \frac{1}{2}\int_0^t \frac{\partial}{\partial x}\varphi(\Sigma(r, \eta))\sigma^2(\eta(r))\,\mathrm{d}r \end{aligned}$$

uniformly in t, where, for notational convenience,

$$\Sigma(t, \eta) = (t, \eta(t), g_1(t; \eta), \dots, g_m(t; \eta)).$$

To prove convergence of the integrals with respect to the hindsight factors we decompose

$$\begin{aligned} &\left| \int_0^t \frac{\partial}{\partial y_j} u(\Sigma(r, \eta_n))\,\mathrm{d}g_j(r; \eta_n) - \int_0^t \frac{\partial}{\partial y_j} u(\Sigma(r, \eta))\,\mathrm{d}g_j(r; \eta) \right| \\ \leq &\left| \int_0^t \frac{\partial}{\partial y_j} u(\Sigma(r, \eta))\,\mathrm{d}(g_j(r; \eta) - g_j(r; \eta_n)) \right| \\ &+ \left| \int_0^t \left\{ \frac{\partial}{\partial y_j} u(\Sigma(r, \eta_n)) - \frac{\partial}{\partial y_j} u(\Sigma(r, \eta)) \right\} \mathrm{d}g_j(r; \eta_n) \right| = (I) + (II). \end{aligned}$$

By (5),

$$(I) \leq K \max_{0 \leq r \leq T} \left| \frac{\partial}{\partial y_j} u(\Sigma(r, \eta)) \right| \|\eta_n - \eta\|_\infty \to 0.$$

Analogously,

$$(II) \leq K \max_{0 \leq r \leq T} \left| \frac{\partial}{\partial y_j} u(\Sigma(r, \eta)) - \frac{\partial}{\partial y_j} u(\Sigma(r, \eta_n)) \right| \|\eta_n\|_\infty.$$

By (12), given $\delta > 0$, there is an n_0 such that for all $n \geq n_0$, $0 \leq j \leq m$ and $0 \leq r \leq T$

$$|g_j(r, \eta_n) - g_j(r, \eta)| < \delta.$$

Exploiting the uniform continuity of $(\partial/\partial y_j)u$ on the compact set $[0,T]\times A_0\times A^m$ we deduce

$$\max_{0\le r\le T}\left|\frac{\partial}{\partial y_j}u(\Sigma(r,\eta))-\frac{\partial}{\partial y_j}u(\Sigma(r,\eta_n))\right|\to 0.$$

Since $\|\eta_n\|_\infty$ is bounded, $(II)\to 0$ uniformly in t.

We now proceed with the proof of Theorem 1.

Proof (Proof of Theorem 1) Let $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{S},(\tilde{\mathcal{F}}_t),\tilde{\mathbf{P}})\in\mathcal{M}_\sigma$ be the reference model and let $(\Omega,\mathcal{F},S,(\mathcal{F}_t),\mathbf{P})\in\mathcal{M}_\sigma$ be some model. Consider $\Phi_t=\varphi(t,S_t,g_1(t,S),\dots,g_m(t,S))$, with continuously differentiable φ as a strategy for the model $(\Omega,\mathcal{F},S,(\mathcal{F}_t),\mathbf{P})$. Further suppose the investor has initial capital zero and

$$V_T(\Phi,0;S)\ge 0 \quad \mathbf{P}-\text{a.s.} \tag{13}$$

By Lemma 3

$$v(T,S;\varphi)=V_T(\Phi,0;S)\quad \mathbf{P}-\text{a.s.}$$

By the small ball condition (4) and the continuity of $v(T,\cdot,\varphi)$ we see that the inequality (13) holds in the functional sense:

$$v(T,\eta;\varphi)\ge 0$$

for all $\eta\in\mathcal{C}_{\sigma,s_0}$. Indeed, otherwise $v(T,\cdot;\varphi)$ would be negative in some ball in $\mathcal{C}_{\sigma,s_0}$ by continuity. Since all balls have positive $\mathbf{P}$-measure by the small ball condition (4), the assumption (13) would be violated.

We hence obtain

$$v(T,\tilde{S};\varphi)\ge 0\quad \tilde{\mathbf{P}}-\text{a.s.}$$

Analogously, the nds-admissibility of Φ implies for all $t\ge 0$

$$v(t,\tilde{S};\varphi)\ge -a\quad \tilde{\mathbf{P}}-\text{a.s.}$$

Since $\tilde{P}$ itself is an equivalent martingale measure for $\tilde{S}$, we may conclude from the classical no-arbitrage theory that

$$v(T,\tilde{S};\varphi)=0\quad \tilde{\mathbf{P}}-\text{a.s.}$$

Interchanging the roles of $\tilde{S}$ and S and applying the same argument as above yields

$$V_T(\Phi,0;S)=0\quad \mathbf{P}-\text{a.s.}$$

Hence, Φ is not an arbitrage.

Remark 4 The most important ingredient for the proof of Theorem 1 is the existence of a continuous wealth functional $v(t,\cdot;\varphi)$. This property remains unchanged when φ is only piecewise smooth. Precisely suppose $0=s_0<s_1<$

$\cdots < s_J = T$ and $\varphi_j : [s_{j-1}, s_j] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable in the first $m+2$ variables and continuous in the last one. Then the no-arbitrage results holds true for strategies of the form

$$\Phi_t = \sum_{j=1}^{J} \mathbf{1}_{(s_{j-1},s_j]}(t)\varphi_j\Big(t, S_t, g_1(t,S), \ldots, g_m(t,S), \xi_j(S)\Big),$$

where $\xi_j : \mathcal{C}_{\sigma,s_0} \to \mathbb{R}$ is continuous and $\xi_j(\eta)$ depends on the segment $\{\eta(r);\ 0 \le r \le s_{j-1}\}$ only. Note the introduction of the functionals ξ_j allows dependence of the strategy on the discretely sampled maximum, minimum, or average.

We also note that the nds-admissibility can be relaxed to

$$\int_0^t \Phi_u \,\mathrm{d}S_u \ge -a(t, S_t) \quad \mathbf{P}-\text{a.s.}$$

where $a : [0,T] \times \mathbb{R} \to \mathbb{R}_{\ge 0}$ is continuous and $\int_{\tilde\Omega} \sup_{t\in[0,T]} |a(t, \tilde S_t)| \mathrm{d}\tilde{\mathbf{P}} < \infty$.

5 Robust replication

In this section we discuss the robustness of hedges within a model class $\mathcal{M}_\sigma$. Our aim is to show that hedges for a large class of claims do not depend of the specific model in $\mathcal{M}_\sigma$ as a functional of the stock. In particular we obtain that the initial capital for such a hedge does not depend on the chosen model. In combination with the no-arbitrage result (Theorem 1) this means that the fair price of those contingent claims coincides for all models from $\mathcal{M}_\sigma$.

We first motivate a slight enhancement of the class of allowed strategies. In the standard Black-Scholes model the Black-Scholes PDE yields the hedge for a call option with strike K and maturity T as a function $\varphi(t,x)$ of time and spot. This function fails to be continuous at $t = T, x = K$. More generally, hedges which are obtained via PDEs often do not satisfy the smoothness condition at $t = T$. To overcome this difficulty we suggest to enlarge the class of allowed strategies in the following way: We relax the smoothness condition of φ to $\varphi \in \mathcal{C}^1([0,T) \times \mathbb{R} \times \mathbb{R}^m, \mathbb{R})$. The next lemma explains what happens at the terminal date $t = T$.

Lemma 4 *Suppose Φ is of the form (8) with $\varphi \in \mathcal{C}^1([0,T) \times \mathbb{R}_+ \times \mathbb{R}^m)$. Then the following assertions are equivalent:*

(i) *For all $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ and all $0 \le t \le T$ the (improper) integral*

$$\int_0^t \Phi_u \,\mathrm{d}S_u$$

exists.

(ii) *There is a dense subset $D \subset \mathcal{C}_{\sigma,s_0}$ and a limiting wealth functional $F : D \to \mathbb{R}$ such that for all $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ we have $\mathbf{P}(S \in D) = 1$ and for all $\eta \in D$ we have*

$$\lim_{t\uparrow T} v(t, \eta; \varphi) = F(\eta).$$

Proof Let $\varphi \in \mathcal{C}^1([0,T) \times \mathbb{R}_+ \times \mathbb{R}^m)$. By Lemma 3 we have for every model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ and all $0 \leq t < T$ that

$$\int_0^t \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S))\,\mathrm{d}S_r = v(t,S;\varphi). \tag{14}$$

As we always choose continuous modifications of the forward integrals the above identity holds up to $\mathbf{P}$-indistinguishably. Hence, for every model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ we may choose a set Ω^1 such that (14) holds on Ω^1 and $\mathbf{P}(\Omega^1) = 1$. Note also, that by (14) assertion (i) is equivalent to the existence of the corresponding improper forward integrals at $t = T$.

Suppose now existence of a limiting functional, i.e. (ii). For a model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ set $\Omega^2 = \Omega^1 \cap S^{-1}(D)$. Then Ω^2 has full $\mathbf{P}$-measure and for each $\omega \in \Omega^2$ we have

$$\lim_{t \uparrow T} \left(\int_0^t \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S))\,\mathrm{d}S_r \right)(\omega) = \lim_{t \uparrow T} v(t, S(\omega); \varphi) = F(\omega).$$

This means that the integral $\int_0^T \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S))\,\mathrm{d}S_r$ exists in the improper forward sense in the model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$, The claim (i) follows.

We now suppose existence of the forward integrals, i.e. (i), and construct a limiting functional. In view of (14), given a model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ we find a set Ω^3 of full $\mathbf{P}$-measure such that on Ω^3

$$\int_0^T \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S))\,\mathrm{d}S_r$$

exists and (14) holds. Define

$$D = \bigcup_{(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma} S(\Omega^3).$$

For $\eta \in D$ choose a model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ such that $\eta = S(\omega)$ with $\omega \in \Omega^3$ and define

$$F(\eta) = \left(\int_0^T \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S))\,\mathrm{d}S_r \right)(\omega).$$

Note, F is well-defined due to (14) and D is a dense set due to the small ball condition (4).

The previous lemma characterizes the minimal assumption of existence of $\int_0^{\cdot} \Phi_u\,\mathrm{d}S_u$ in terms of existence of a limiting functional of $v(t,\cdot;\varphi)$ as $t \uparrow T$. To define allowed strategies we will strengthen this minimal requirement by imposing a continuity assumption on the limiting functional.

Definition 5 *A strategy Φ is* allowed *for the model class $\mathcal{M}_\sigma$ if*

(A1) *there is a finite number of hindsight variables* $g_1, \dots, g_m$ *and a function* $\varphi \in \mathcal{C}^1([0,T) \times \mathbb{R} \times \mathbb{R}^m)$ *such that*

$$\Phi_t = \varphi(t, S_t, g_1(t,S), \dots, g_m(t,S)).$$

(A2) *There is a dense subset* $D \subset \mathcal{C}_{\sigma,s_0}$ *and a functional* $F : D \to \mathbb{R}$ *such that for all models* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ *we have* $\mathbf{P}(S \in D) = 1$ *and for all* $\eta \in D$ *we have*

$$\lim_{t \uparrow T} v(t, \eta; \varphi) = F(\eta).$$

Moreover, we assume that F *is continuous in* D.

(A3) *There is a constant* $a > 0$ *such that for all* $0 \le t \le T$

$$\int_0^t \Phi_u \, \mathrm{d}S_u \ge -a \quad \mathbf{P} - \mathrm{a.s.}$$

Recall that **(A3)** is the classical concept of nds-admissibility which is typically imposed to exclude doubling strategies. Also, if **(A3)** holds for one model then it holds for all models (given **(A1)** and **(A2)**), as was shown in the proof of Theorem 1. Also note that in **(A2)** we do not assume that F can be continuously extended to the whole space $\mathcal{C}_{\sigma,s_0}$.

Obviously **(A2)** holds, if $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R} \times \mathbb{R}^m)$. Moreover, Theorem 1 carries over to allowed strategies without any additional difficulties:

Theorem 2 *Suppose condition* **(H)**. *Then every model in* $\mathcal{M}_\sigma$ *is free of arbitrage with allowed strategies.*

The next theorem states that hedges are robust as functionals of the stock within the class $\mathcal{M}_\sigma$.

Theorem 3 *Suppose condition* **(H)** *holds and* G *is a continuous functional on* $\mathcal{C}_{\sigma,s_0}$ *such that* $G(\tilde{S})$ *is replicable* $\tilde{\mathbf{P}}$*-a.s. in the reference model* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}}) \in \mathcal{M}_\sigma$ *with an allowed strategy*

$$\tilde{\Phi}^*_t = \varphi^*(t, \tilde{S}_t, g_1(t, \tilde{S}), \dots, g_m(t, \tilde{S}))$$

and initial capital v_0. *Then* $G(S)$ *is replicable* $\mathbf{P}$*-a.s. in every model* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ *with the same initial capital* v_0 *and the replicating allowed strategy is given by*

$$\Phi^*_t = \varphi^*(t, S_t, g_1(t,S), \dots, g_m(t,S)),$$

i.e. replicating allowed strategies are, as functionals of the stock prices, independent of the model.

The converse also holds, i.e. any 'functional' hedge φ^* *in some model* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ *is also a 'functional' hedge for the reference model.*

Proof Given φ^* let F be the limiting wealth functional on a dense set D as in assumption **(A2)**. Note that by Lemma 3

$$v_0 + F(\tilde{S}) = V_T(\tilde{\Phi}^*, v_0; \tilde{S}) = G(\tilde{S}), \quad \mathbf{P}\text{-a.s.}$$

Hence,

$$v_0 + F(\tilde{S}(\tilde{\omega})) = G(\tilde{S}(\tilde{\omega}))$$

for all $\tilde{\omega}$ from a set $\tilde{\Omega}^1$ of full $\tilde{\mathbf{P}}$-measure such that $\tilde{S}(\tilde{\Omega}^1) \subset D$. Hence, $F = G - v_0$ on D by the continuity of G and **(A2)**. Again by Lemma 3, $V_T(\Phi^*, v_0; S) = v_0 + F(S) = G(S)$ $\mathbf{P}$-a.s.

For the converse one simply interchanges the roles of $\tilde{S}$ and S.

Remark 5 From the previous theorem and the classical no-arbitrage theory we may derive the following result: With the notation from the previous theorem the initial capital v_0 satisfies the inequality

$$v_0 \geq \mathbf{E}^{\tilde{\mathbf{P}}}[G(\tilde{S})].$$

Moreover identity holds, if and only if $v(t, \tilde{S}; \varphi^*)$ is a martingale under $\tilde{\mathbf{P}}$. In the latter case $\mathbf{E}^{\tilde{\mathbf{P}}}[G(\tilde{S})]$ is the fair price (by no-arbitrage arguments) relative to the class of allowed strategies of the contingent claim $G(S)$ for all models $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$.

We now give some sufficient conditions for hedgeability of some relevant options via PDEs. This PDE approach to robust replication was first considered in [15] for European options.

Example 5 Suppose $G \in \mathcal{C}(\mathbb{R}^4)$. We define an option by plugging the time-T-values of the spot, the running average, maximum, and minimum into the arguments of G. To construct robust hedges for this type of option let $\Gamma_1(t, y_2) \cup \Gamma_2(t, y_3) = \{(x, y_1);\ x \leq y_2,\ y_1 \leq ty_2\} \cup \{(x, y_1);\ x \geq y_3,\ y_1 \geq ty_3\}$. Suppose for $0 \leq t < T$, $0 \leq y_3 \leq s_0 \leq y_2$, and $(x, y_1) \in \Gamma_1(t, y_2) \cup \Gamma_2(t, y_3)$ the PDE

$$\begin{aligned}
\frac{\partial}{\partial t} U(t, x, y_1, y_2, y_3) &= -\frac{\sigma(x)}{2} \frac{\partial^2}{\partial x^2} U(t, x, y_1, y_2, y_3) \\
&\quad - x \frac{\partial}{\partial y_1} U(t, x, y_1, y_2, y_3) \\
U(T, x, y_1, y_2, y_3) &= G(x, y_1, y_2, y_3) \\
\frac{\partial}{\partial y_2} U(t, \cdot, \cdot, y_2, y_3)_{|\partial \Gamma_1(t, y_2)} &= 0 \\
\frac{\partial}{\partial y_3} U(t, \cdot, \cdot, y_2, y_3)_{|\partial \Gamma_2(t, y_3)} &= 0
\end{aligned}$$

has a solution $U \in \mathcal{C}^{1,2,1}([0, T) \times \mathbb{R}^4) \cap \mathcal{C}([0, T] \times \mathbb{R}^4)$ which is bounded from below, i.e. $U \geq -a$ for some $a \geq 0$. Let $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$. By Itô's formula, for $0 \leq t_0 \leq t \leq T$, $\mathbf{P}$-almost surely,

$$\begin{aligned}
U(t, S_t, \bar{S}_t, S_t^*, S_{*,t}) &= U(t, S_{t_0}, \bar{S}_{t_0}, S_{t_0}^*, S_{*,t_0}) \\
&\quad + \int_{t_0}^t \frac{\partial}{\partial x} U(r, S_r, \bar{S}_r, S_r^*, S_{*,r}) dS_r. \qquad (15)
\end{aligned}$$

(Here, we used that

$$\int_{t_0}^{t} \frac{\partial}{\partial y_2} U(r, S_r, \bar{S}_r, S_r^*, S_{*,r}) dS_r^* = 0$$

by the boundary condition and similarly for the integral with respect to the minimum.) Define $\Phi_t = \frac{\partial}{\partial x} U(t, S_t, \bar{S}_t, S_t^*, S_{*,t})$. In particular we obtain by the usual continuity argument for $\eta \in \mathcal{C}_{\sigma,s_0}$

$$U(t, \eta(t), \bar{\eta}(t), \eta^*(t), \eta_*(t)) - U(0, s_0, 0, s_0, s_0) = v(t, \eta; \frac{\partial}{\partial x} U).$$

This shows, Φ_t is an allowed strategy. Moreover formula (15) with $t = T$ shows that Φ_t is a hedge for $G(S_t, \bar{S}_t, S_t^*, S_{*,t})$.

We note, the above PDE has two difficulties: (i) it is degenerate parabolic, since the second derivative in y_1-direction does not appear; (ii) the boundary conditions in terms of the derivatives in direction of the parameters y_2 and y_3 are a rather unusual. Nonetheless there are some well known existence results for practically important exotic options such as lookback options (which depend on the maximum or minimum only) and Asian options (which depend on the average only). Of course European options with continuous payoff functions are also covered by this PDE approach. For details we refer to [19].

Remark 6 Theorem 3 requires that the option is continuous as a function of the paths. The most prominent option which fails to satisfy this assumption is the digital option $G(\eta) = \mathbf{1}_{[K,\infty)}(\eta(T))$ with strike K. A straightforward modification of the argument in Theorem 3 shows that hedges for the digital option are robust in any subclass of $\mathcal{M}_\sigma$ which contains only models that satisfy $\mathbf{P}(S_T = K) = 0$.

A drawback of the results obtained so far is the following: The class of allowed strategies (with respect to which we price), does not contain the realistic simple predictable strategies of the form

$$\sum_{i=1}^{n} \Phi_i \mathbf{1}_{(\tau_i, \tau_{i+1}]}$$

where the τ_i's are stopping times bounded by T and Φ_i are $\mathcal{F}_{\tau_i}$-measurable. Hence, it would be desirable to extend the no arbitrage result to a wider class of strategies, which

1. contains allowed strategies and typical simple predictable strategies;
2. is a linear space (up to the nds-admissibility).

This program is carried out in the next section, partly under a stronger small ball condition. We also discuss *approximate arbitrage*.

6 Arbitrage opportunities

6.1 Results on no-arbitrage with stopping times

Note that by Remark 4 the arbitrage is excluded if one changes the trading strategy at *deterministic* stopping times. It turns out that this extends to a certain class of stopping times, having some weak continuity properties. To avoid technicalities and to be able to use results from [9] we shall assume that the function σ, which determines the quadratic variation structure of the model class, is chosen such that

$$\mathcal{C}_{\sigma,s_0} = \{\eta \in \mathcal{C}([0,T] \times \mathbb{R}_+); \eta(0) = s_0\}.$$

Here we recall that the space $\mathcal{C}_{\sigma,s_0}$ was introduced in (3).

We shall also strengthen the small ball condition (4) as follows:

We assume that the following *conditional small-ball property* is satisfied:

$$\mathbf{P}\left(\sup_{t\in[\tau,T]} |S_t - \eta(t)| < \varepsilon \,\middle|\, \mathcal{F}_\tau\right) > 0 \tag{16}$$

$\mathbf{P}$-a.s. for all stopping times $\tau < T$, all paths $\eta > 0$ such that $\eta(\tau) = S_\tau$, and every positive ε.

So, we here work with a ‘conditional’ model class which is restricted only by the conditional small ball property and an assumption on the quadratic variation.

Remark 7 Here we do not discuss how to check the conditional small-ball property. In a recent work by Guasoni et.al. [9] the authors show that (16) follows from the weaker condition

$$\mathbf{P}\left(\sup_{t\in[u,T]} |S_t - \eta(t)| < \varepsilon \,\middle|\, \mathcal{F}_u\right) > 0, \tag{17}$$

where $u \in (0,T)$ is a deterministic time.

For example, if S is the mixed fractional Black-Scholes model, i.e. $S_t = f_\sigma(X_t)$, where $X = W + Z$, W is a standard Brownian motion, Z is a fractional Brownian motion with Hurst index $H > \frac{1}{2}$, independent of W, then we can prove (17) along the same lines as in [9]. Another argument to verify (17) is to check the condition with respect to filtration $(\mathcal{F}_t^W \vee \mathcal{F}_T^Z)$, where one can use the independence of W and Z, and then it will be true also with respect to $(\mathcal{F}_t^X)$. As mentioned, we do not want to go in details here.

Now we turn to continuity properties of stopping times. We work with $[0,T] \times \mathcal{C}_{\sigma,s_0}$, and consider this as a topological space, where the basis of the topology are open sets of the form $B_\varepsilon(t_0) \times B_\gamma(\eta_0)$ with

$$B_\varepsilon(t_0) = \{t \in [0,T] : |t - t_0| < \varepsilon\} \;\text{ and }\; B_\gamma(\eta_0) = \{\eta \in \mathcal{C}_{\sigma,s_0} : ||\eta - \eta_0|| < \gamma\}.$$

Let τ be a stopping time with $\tau(\eta) \le T$. Put $A = [[\tau, T]]$, where $[[\tau, T]]$ is the stochastic interval

$$[[\tau, T]] = \{(t, \eta) \in [0,T] \times \mathcal{C}_{\sigma,s_0} : \tau(\eta) \le t \le T\}.$$

We have that $\tau(\eta) = \inf\{t : (t,\eta) \in A\}$, so the set A is *stopping set* of the stopping time τ.

Definition 6 *Let $\mathcal{X}$ and $\mathcal{Y}$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is* locally continuous *if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $f(x_n) \to f(x)$ whenever $x_n \to x$ in U_x.*

Local continuity at x is continuity from the 'direction' U_x. If $x \in U_x$ then local continuity is continuity at x.

In the remainder of this section we will mainly work with locally continuous stopping times. In order to give examples of such stopping times, the following lemma is useful. Its elementary proof is omitted.

Lemma 5 *If the stopping set $A = [[\tau, T]]$ is closed in the topology of $[0,T] \times \mathcal{C}_{\sigma,s_0}$, then τ is lower semicontinuous, i.e.*

$$\liminf_n \tau(\eta_n) \geq \tau(\eta_0)$$

if $\eta_n \to \eta_0$ in $\mathcal{C}_{\sigma,s_0}$.

We now give some examples of locally continuous stopping times which are widely used.

Example 6 (i) For continuous functions a, b, $0 \leq a < s_0 < b$, we consider

$$\tau(\eta) := \inf\{t \geq 0;\ \eta(t) \leq a(t) \text{ or } \eta(t) \geq b(t)\} \wedge T.$$

Note that the stopping set

$$[[\tau, T]] = (\{T\} \times \{\eta : \tau(\eta) = T\}) \cup \{(t,\eta);\ \eta(t) \leq a(t)\} \cup \{(t,\eta);\ \eta(t) \geq b(t)\}$$

of τ is closed and therefore τ is lower semicontinuous. We fix some path η_0 and set $t_0 = \tau(\eta_0)$. We treat the case that $t_0 > 0$ and $\eta_0(t_0) = b(t_0)$. Then, we can define $U_0 := \{\eta;\ \exists_{t<t_0}\ \eta(t) > b(t)\}$. U_0 clearly is an open set in C_{σ,s_0} with η_0 on its boundary. Suppose $(\eta_n) \subset U_0$ with $\eta_n \to \eta_0$. Then $\tau(\eta_n) < t_0$, i.e. $\limsup_{n\to\infty} \tau(\eta_n) \leq \tau(\eta_0)$. From the lower semicontinuity of τ we derive that U_0 is a local continuity set for τ at η_0. The other cases are treated similarly.

Of course, one of the boundaries can be set to $\pm\infty$.

(ii) In an analogous way one can prove that, for continuous functions a, b, $0 < a < s_0 < b$, the stopping time

$$\inf\{t \geq 0;\ \min_{0\leq s\leq t} \eta(s) \leq a(t) \text{ and } \max_{0\leq s\leq t} \eta(s) \geq b(t)\} \wedge T$$

is locally continuous.

(iii) With a little extra effort one can verify that a stopping time τ is locally continuous, if its stopping set $[[\tau, T]]$ is of the form

$$[[\tau, T]] = (\{T\} \times \{\eta : \tau(\eta) = T\}) \cup \bar{U}$$

where $\bar{U}$ is the closure of an open set U in the topology of $[0,T] \times \mathcal{C}_{\sigma,s_0}$. Many more concrete examples of locally continuous stopping times can be derived from this generic example.

Definition 7 *A trading strategy Φ is* stopping-smooth *if it is of the form*

$$\Phi_t = \sum_{k=1}^{n} \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t),$$

where the $\Phi^{(k)}$'s are smooth, and τ_k's are locally continuous.

The definition 7 is understood in the conditional sense, i.e. $\Phi^{(k)}$ may depend on on $\mathcal{F}_{\tau_k}$ and $\tau_{k+1} \geq \tau_k$ is locally continuous in the conditioned, or quotient, space $\mathcal{C}_{S_{\tau_k},\sigma}[\tau_k, T]$.

The next lemma is a stopping-smooth analogue to Lemma 3 with one locally continuous stopping time. We emphasize that it holds true with the simple small ball property and does not require the conditional version.

Lemma 6 *Let $0 \leq t \leq T$ and suppose $\Phi \mathbf{1}_{(0,\tau]}$ is a stopping-smooth strategy induced by φ and τ in the market $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$. Then we have, for $0 \leq t \leq T$,*

$$V_t(\Phi \mathbf{1}_{(0,\tau]}, v_0; S) = v_0 + v(\tau(S) \wedge t, S; \varphi) \quad \mathbf{P}-\text{a.s.}$$

Moreover, for every $0 \leq t \leq T$, the mapping

$$\mathcal{C}_{\sigma,s_0} \to \mathbb{R}, \quad \eta \mapsto v(\tau(\eta) \wedge t, \eta; \varphi)$$

is locally continuous, and the local continuity sets are independent of t.

Proof The representation of the wealth process is a direct consequence of Lemma 3, since by the pathwise nature of the forward integral

$$V_t(\Phi \mathbf{1}_{(0,\tau]}, v_0; S) = V_{\tau(S)}(\Phi, v_0; S).$$

To prove the continuity assertion, fix some $\eta \in \mathcal{C}_{\sigma,s_0}$ and let U_η be a local continuity set at η. For any sequence $(\eta_n)_{n\in\mathbb{N}} \subset U_\eta$ with $\eta_n \to \eta$ we can write

$$\begin{aligned} &|v(\tau(\eta_n) \wedge t, \eta_n; \varphi) - v(\tau(\eta) \wedge t, \eta; \varphi)| \\ \leq &\sup_{0\leq s\leq T} |v(s, \eta_n; \varphi) - v(s, \eta; \varphi)| + |v(\tau(\eta_n) \wedge t, \eta; \varphi) - v(\tau(\eta) \wedge t, \eta; \varphi)| \end{aligned}$$

The first term does not depend on the stopping time and converges to zero by the continuity property of v proved in Lemma 3. The second term converges to zero, since v is continuous in the time variable and $\tau(\eta_m) \to \tau(\eta)$. (Recall that the sequence belongs to the local continuity set). Therefore U_η is a local continuity set for the mapping

$$\mathcal{C}_{\sigma,s_0} \to \mathbb{R}, \quad \eta \mapsto v(\tau(\eta) \wedge t, \eta; \varphi)$$

at the point η as well.

The previous lemma implies absence of arbitrage with stopping-smooth strategies with one locally continuous stopping time for models of class $\mathcal{M}_\sigma$.

Theorem 4 *Suppose condition **(H)** is in force, and a model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$ from class $\mathcal{M}_\sigma$ is given. Then there is no-arbitrage in this model with stopping-smooth strategies of the form $\Phi \mathbf{1}_{(0,\tau]}$.*

Proof The same proof as for Theorem 1 applies with a minor modification: From

$$v(\tau(S), S; \varphi) = V_T(\Phi \mathbf{1}_{(0,T]}, 0; S) \geq 0 \quad \mathbf{P} - a.s.$$

one can obtain the functional inequality

$$v(\tau(\eta), \eta; \varphi) \geq 0$$

for all $\eta \in \mathcal{C}_{\sigma, s_0}$ as follows: Thanks to Lemma 6, we can consider a local continuity set U_η at η. Since U_η is open with η in its closure, and due to the small ball condition (4), we can retrieve a sequence $(\omega_n) \subset \Omega$ such that $S(\omega_n) \in U_\eta$, $S(\omega_n) \to \eta$ and

$$v(\tau(S(\omega_n)), S(\omega_n); \varphi) \geq 0$$

for all $n \in \mathbb{N}$. This implies

$$v(\tau(\eta), \eta; \varphi) = \lim_{n \to \infty} v(\tau(S(\omega_n)), S(\omega_n); \varphi) \geq 0.$$

To obtain more general results we have to assume that the conditional small-ball property (16) holds.

Theorem 5 *Assume the quadratic variation property (1) and the* conditional small-ball *property (16), and let Φ be a stopping-smooth strategy. Then Φ is not an arbitrage opportunity.*

Proof With the conditional small ball property we can show, as in Theorem 4 that $\Phi^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}$ is not an arbitrage opportunity. Recall that here $\Phi^{(k)}$ can additionally depend on F_{τ_k} and $\tau_{k+1} > \tau_k$ is locally continuous in the conditioned, or quotient, space $\mathcal{C}_{S_{\tau_k}, \sigma}[\tau_k, T]$.

But this means that

$$\Phi = \sum_{k=1}^{N} \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(\cdot),$$

does not allow arbitrage on the interval $(\tau_k, \tau_{k+1}]$, and hence does not allow arbitrage on the interval $[0, T]$

Remark 8 One can easily verify that Theorem 5 remains true if the class of strategies is enriched to consist of sums of stopping-smooth and allowed strategies. This class of strategies is linear up to nds-admissibility, contains many typical simple predictable strategies and hedges for many relevant options.

Note that in Theorem 5 one trades continuously between the locally continuous stopping times. If we allow only simple strategies, we can relax even more on the continuity properties of stopping times.

Definition 8 *Let $\mathcal{X}$ be a metric space and let $\mathcal{Y}$ be an ordered complete metric space. A function $f : \mathcal{X} \to \mathcal{Y}$ is* locally lower semi-continuous *if for all $x \in \mathcal{X}$ there exists an open $U_x \subset \mathcal{X}$ such that $x \in \bar{U}_x$ and $\liminf f(x_n) \geq f(x)$ whenever $x_n \to x$ in U_x.*

Definition 9 *A trading Φ strategy is* simple *if it is of the form*

$$\Phi_t = \sum_{k=1}^{n} \xi_k \mathbf{1}_{(\tau_k, \tau_{k+1}]},$$

where τ_k is locally lower semi-continuous stopping times (relative to τ_{k-1}), and ξ_k's are $\mathcal{F}_{\tau_k}$ measurable.

Theorem 6 *Assume* conditional small-ball property *(16). Let Φ be a simple strategy. Then Φ is not an arbitrage opportunity.*

Remark 9 Note that Theorem 6 is true without the *quadratic variation* condition (1).

Because of the "time-linearity" of the arbitrage and conditional small-ball property it is enough to show the following (see [18, Theorem V.2.A*, p. 425]):

Lemma 7 *Let τ be locally lower semi-continuous stopping time. Then*

$$\mathbf{P}(S_\tau > s_0) > 0 \quad \textit{and} \quad \mathbf{P}(S_\tau < s_0) > 0.$$

Proof We show that $\mathbf{P}(S_\tau > s_0) > 0$; the case $\mathbf{P}(S_\tau < s_0) > 0$ is symmetric. We show that the set $\{S_\tau > s_0\} = \{\eta; \eta(\tau(\eta)) > s_0\}$ contains a ball. Then the claim will follow from the small-ball property.

Fix an increasing and concave path η_0 with $\eta_0(0) = s_0$ and a local lower semi-continuity set U_{η_0} of τ at η_0. Since τ is lower semi-continuous on U_{η_0} we can find such an $\varepsilon < 1/2\ (\eta_0(\tau(\eta_0)) - s_0)$ that $\tau(\eta) \geq 1/2\,\tau(\eta_0)$ whenever $\eta \in B$, where B is some ball contained in $B_{\eta_0}(\varepsilon) \cap U_{\eta_0}$. Since η_0 is increasing and concave

$$\begin{aligned}
\eta(\tau(\eta)) &> \eta_0(\tau(\eta)) - 1/2\ (\eta_0(\tau(\eta_0)) - s_0) \\
&\geq \eta_0\,(1/2\,\tau(\eta_0)) - 1/2\,\eta_0(\tau(\eta_0)) + 1/2\,s_0 \\
&\geq 1/2\,\eta_0(0) + 1/2\,s_0 = s_0.
\end{aligned}$$

So, the ball B is contained in the set $\{S_\tau > s_0\}$, which implies that $\mathbf{P}[S_\tau > s_0] > 0$.

Remark 10 Recall that is F is a closed set, then $\tau := \inf\{t : \eta(t) \in F\}$ is lower-semicontinuous stopping time.

Remark 11 The Lemma 7 is true with local lower semi-continuity replaced by a weaker assumption of ε-*delay*:

For all η_0 there are positive $\varepsilon = \varepsilon(\eta_0)$ and $\delta = \delta(\eta_0)$ such that

$$\tau(\eta) \geq \varepsilon \quad \text{when } \eta \in B_{\eta_0}(\delta).$$

Remark 12 Cheridito has studied the no-arbitrage with models driven by fractional Brownian motion. Concerning arbitrage possibilities he shows that for any $c > 0$ there exists an almost simple self-financing strategy Φ such that

$$\mathbf{P}(V_T(\Phi, 0; S) \geq c) \geq 1 - \frac{1}{c},$$

$$\inf_{t \in (0,T]} V_t(\Phi, 0; S) \geq -\frac{1}{c};$$

here almost simple is an extension of Definition 9 with n replaced by a finite random integer and $S = e^{B^H}$, where B^H is a fractional Brownian motion with $H \in (0,1) \setminus \{\frac{1}{2}\}$. Such a strategy Φ is called $\frac{1}{c}$- *admissible c-arbitrage* strategy (see [3, pp. 539- 540]). Note that model allowing $\frac{1}{c}$- admissible c-arbitrage strategies also have NFLVR asymptotic arbitrage, but do not necessarily admit arbitrage possibilities in the classical sense. Cheridito's contruction is based on the fact that fractional Brownian motion with $H \in (0,1) \setminus \{\frac{1}{2}\}$ has either infinite or zero quadratic variation. Since the construction of the stopping times in [3] is not explicit, it is not clear weather they are locally continuous in our sense. Since in our case $S \in \mathcal{M}_\sigma$, and thus has a non-trivial quadratic variation, the above contruction to obtain approximate arbitrage with $\frac{1}{c}$- admissible c- arbitrage strategies is not applicable in the situation of Theorem 5.

Next, we consider no-arbitrage result in Theorem 6 and compare it to the corresponding result in [3, Theorem 4.3, p.549]. We ask that the stock price process has conditional small ball property, and geometric fractional Brownian motion has this property. In this respect Theorem 6 is more general than Cheridito's result. Moreover he proves the no-arbitrage property under the assumption that the stopping times in the construction satisfy $\tau_{k+1} - \tau_k \geq h$, where $h > 0$ is a fixed and deterministic constant. We, on the other hand, assume that the stopping times satisfy the ε- delay (see Remark 11), and this is weaker than Cheridito's condition with deterministic h- delay.

6.2 Approximate arbitrage exemplified

For fractional Brownian motion several explicit arbitrage opportunities are known (cf. [5,17]). These examples do not generate arbitrage in the mixed fractional Black-Scholes model. Below we give a sequence of (allowed) strategies that generate arbitrage in the mixed model "in the limit", although there is no limiting arbitrage strategy.

The basic idea of the no-arbitrage result in Theorem 1 was to extend absence of arbitrage in the reference model by means of the continuous wealth functional. This reasoning does not carry over to notions of approximate arbitrage or similar limiting procedures in general. Indeed, we will now show that there can exist an approximate arbitrage in some models in $\mathcal{M}_\sigma$, which fail to be an approximate arbitrage in the reference model. Although we think that the construction below is interesting in its own right, it also gives a clear hint at the limitations of our no-arbitrage pricing approach for non-semimartingales.

Notice that our notion of approximate arbitrage is different from the notion of a free lunch with vanishing risk. However it admits to construct a very intuitive example in the context of mixed fractional Black-Scholes model.

Example 7 Suppose that $\sigma(x) = \sigma x$. Hence, the reference model $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}})$ is the risk neutral Black-Scholes model. Now, the mixed fractional Black-Scholes model

$$\mathrm{d}S_t = \sigma S_t \,\mathrm{d}X_t$$

where $X_t = W_t + Z_t$ is a sum of an independent Brownian motion and a fractional Brownian motion belongs to same class $\mathcal{M}_\sigma$ if the Hurst parameter H of the fractional Brownian motion satisfies $H \in (1/2, 3/4)$.

We define a sequence of strategies via the functionals

$$\varphi^n(t,\eta) = n^{2H-1} \sum_{k=2}^{n} \mathbf{1}_{\left(T\frac{k-1}{n}, T\frac{k}{n}\right]}(t) \frac{\theta(T\frac{k-1}{n}) - \theta(T\frac{k-2}{n})}{\eta(t)},$$

with $\theta(t) = \log(\eta(t))/\sigma + 1/2\sigma^2 t$ which fits into the context of Remark 4.

Write

$$\begin{aligned} S_t \varphi^n(t,S) \;&=\; n^{2H-1} \sum_{k=2}^{n} \mathbf{1}_{\left(T\frac{k-1}{n}, T\frac{k}{n}\right]}(t) \left(W_{T\frac{k-1}{n}} - W_{T\frac{k-2}{n}} \right) \\ &\quad + n^{2H-1} \sum_{k=2}^{n} \mathbf{1}_{\left(T\frac{k-1}{n}, T\frac{k}{n}\right]}(t) \left(Z_{T\frac{k-1}{n}} - Z_{T\frac{k-2}{n}} \right) \\ &=: K_t^n + L_t^n. \end{aligned}$$

Since $H \in (\frac{1}{2}, \frac{3}{4})$, $S_t \varphi^n(t,S)$ converges uniformly to zero in probability with respect to $\mathbf{P}$. (Indeed, this convergence holds for K_t^n and L_t^n). Hence the risk of the strategies $\Phi_t^n = \varphi^n(t,S)$ becomes smaller and smaller in the sense that the number of risky assets held by the investor tends to zero.

We now decompose

$$\int_0^T \Phi_t^n \,\mathrm{d}S_t = \int_0^T K_t^n \,\mathrm{d}W_t + \int_0^T L_t^n \,\mathrm{d}W_t + \int_0^T K_t^n \,\mathrm{d}Z_t + \int_0^T L_t^n \,\mathrm{d}Z_t.$$

The first and the second term go to zero in probability by Theorem II.11 in [13]. The third terms converges to zero in probability, since, by the independence of W and Z, and the stationarity of the increments of W,

$$\begin{aligned} \int_0^T K_t^n \,\mathrm{d}Z_t &\overset{\text{Law}}{=} \int_0^T L_t^n \,\mathrm{d}W_t - n^{2H-1} Z_{\frac{T}{n}} \left(W_{T\frac{2}{n}} - W_{T\frac{1}{n}} \right) \\ &\quad + n^{2H-1} \left(Z_T - Z_{T-\frac{T}{n}} \right) \left(W_{T+\frac{T}{n}} - W_T \right). \end{aligned}$$

However, for the fourth term, we obtain,

$$\begin{aligned} \int_0^T L_s^n \,\mathrm{d}Z_s \;&=\; n^{2H-1} \sum_{k=1}^{n-1} \left(Z_{T\frac{k+1}{n}} - Z_{T\frac{k}{n}} \right) \left(Z_{T\frac{k}{n}} - Z_{T\frac{k-1}{n}} \right) \\ &\overset{\text{Law}}{=} T^H \frac{1}{n} \sum_{k=1}^{n-1} \left((Z_{k+1} - Z_k)(Z_k - Z_{k-1}) \right) \to T^{2H}(2^{2H-1} - 1) \end{aligned}$$

in $L^1(\mathbf{P})$ by [10, Theorem 9.5.2., p. 479]. Note that this limit is the same for $H \leq \frac{1}{2}$, too. Hence, the limiting wealth of the strategies Φ_t^n is strictly positive, namely,

$$\lim_{n\to\infty} \int_0^T \Phi_t^n \, \mathrm{d}S_t = T^{2H}(2^{2H-1} - 1)$$

in probability. We consider such sequence an *approximate arbitrage*.

Of course, the corresponding sequence of strategies does not constitute an approximate arbitrage in the reference model. As above, we obtain that

$$\tilde{S}_t \varphi^n(t, \tilde{S}) = n^{2H-1} \sum_{k=2}^{n} \mathbf{1}_{\left(T\frac{k-1}{n}, T\frac{k}{n}\right]}(t) \left(\tilde{W}_{T\frac{k-1}{n}} - \tilde{W}_{T\frac{k-2}{n}} \right)$$

converges to zero uniformly in probability with respect to $\tilde{\mathbf{P}}$. Hence, with $\tilde{\Phi}_t^n = \varphi^n(t, \tilde{S})$,

$$\lim_{n\to\infty} \int_0^t \tilde{\Phi}_t^n \, \mathrm{d}\tilde{S}_t = \lim_{n\to\infty} \int_0^t \sigma \tilde{S}_t \varphi^n(t, \tilde{S}) \, \mathrm{d}\tilde{W}_t = 0$$

as the Itô integral is continuous in terms of the integrand.

Remark 13 The construction of the approximate arbitrage in the mixed fractional Black-Scholes model above follows an easy intuition. Due to the memory of the fractional Brownian motion the stock tends to increase, if it already increased in the previous time period. How to exploit this intuition is made precise above. The example also shows that integrals with respect to the mixed fractional Brownian motion with $H \in (1/2, 3/4)$ are not continuous in terms of the integrands. Hence, it may be considered a simple proof that mixed fractional Brownian motion is not a semimartingale for this range of the Hurst parameter. The reader is invited to compare our argument with the proof by [2].

7 Conclusion

In this paper we discussed no-arbitrage pricing beyond the semimartingale setting for models with continuous trajectories. To this end we imposed two conditions on the models. The conditional small ball condition guarantees absence of arbitrage for a large class of simple predictable strategies. This condition is satisfied e.g. in many log-Gaussian models such as the fractional Black-Scholes model and the mixed fractional Black-Scholes model. As mentioned in Remark 12 our no-arbitrage result (Theorem 6) admits trading with a somewhat larger class of simple strategies than the class considered by Cheridito [3, Theorem 4.3] in the context of the fractional Black-Scholes model.

Although simple predictable strategies are economically meaningful in the sense that they cover all strategies that can be implemented in reality, they do not include hedges for relevant options. Note also that approximate hedging arguments with respect to the class of simple predictable strategies may induce arbitrages (and are therefore meaningless from the perspective of no-arbitrage pricing) in non-semimartingale models, because the forward integral with respect to the stock lacks continuity in the integrand. Therefore, we added a condition on the quadratic variation of the stock paths. It rules out all models with zero quadratic variation such as the fractional Black-Scholes model, but still is satisfied by many non-semimartingale models, for example by the mixed fractional Black-Scholes model. The condition on the quadratic variation ensures that the model becomes free of arbitrage with trading strategies that depend smoothly on the spot and other relevant quantities (hindsight factors) such as the running maximum, minimum, and average of the stock (Theorem 1). These strategies are crucial from the perspective of hedging, as they cover hedges for many European, lookback, and Asian options. In combination with the conditional small ball condition, we can even show that absence of arbitrage still holds, if one changes the strategy abruptly at stopping times (which satisfy a mild local continuity condition), and changes it smoothly (as functional of the spot and hindsight factors) in between the stopping times (Theorem 5). This is the main no-arbitrage result of the present paper, as it holds for a class of trading strategies which is linear (up to nds-admissibility), contains many realistic simple predictable strategies and hedges for relevant options.

Although one could think of larger classes of relevant strategies (e.g. containing simple strategies where the portfolio is readjusted at a random, possibly unbounded, number of stopping times), our no-arbitrage result allows to extend no-arbitrage pricing for many options in a sensible way. We consider this a major novelty, since no-arbitrage results on simple strategies only cannot be used for pricing purposes beyond semimartingales. We find that the corresponding no-arbitrage replication prices of options depend essentially only on the quadratic variation. As the quadratic variation is a path property, it does not tell much about the probabilistic structure. Because of this we can change the distributional properties of the Black-Scholes model (and, more general, of related local volatility models) without changing the prices of the options. For example, the fractional Black-Scholes model incorporates memory effects and modifies the variance without any changes in the quadratic variation. This example also shows that the covariance structure of the stock returns is not relevant for option pricing, but the quadratic variation is. So, one should not be surprised if the historical and implied volatility do not agree: The former is an estimate of the variance and the latter is an estimate of the quadratic variation.

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