HARMONIC ANALYSIS, 9. EXERCISE

1. (1p) Define the Fourier transform for $f \in S(\mathbf{R}^n)$ as
$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-i2\pi x\cdot\xi}\, d\xi.$$
Suppose that $f, g \in S(\mathbf{R}^n)$ and $\alpha, \beta \in \mathbf{C}$. Show that
   (i) $\widehat{(\alpha f + \beta g)} = \alpha\hat{f} + \beta\hat{g}$,
   (ii) $\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = 2\pi i\xi_j \hat{f}(\xi)$, $j = 1, \dots, n$,
   (iii) $\frac{\partial \hat{f}}{\partial \xi_j}(\xi) = (\widehat{-2\pi i x_j} f)(\xi)$, $j = 1, \dots, n$,
   (iv) $\hat{f}$ is continuous,
   (v) $||\hat{f}||_\infty \le ||f||_1$,
   (vi) $\widehat{f(\varepsilon x)} = \frac{1}{\varepsilon}\hat{f}(\frac{\xi}{\varepsilon}) = \hat{f}_\varepsilon(\xi), \varepsilon > 0$,
   (vii) $\widehat{fg} = \hat{f} * \hat{g}$.
2. (1p) Let
$$P_t(y) := c_n \frac{t}{(t^2 + |y|^2)^{(n+1)/2}}, \qquad Q_{k,t}(y) := c_n \frac{y_k}{(t^2 + |y|^2)^{(n+1)/2}}.$$
Show that for all $f \in S(\mathbf{R}^n)$ there is a constant c depending only on f and n such that
$$|P_t * f(x)| \le \frac{c(1+t)}{(1 + t^2 + |x|^2)^{(n+1)/2}}$$
and
$$|Q_{k,t} * f(x)| \le \frac{c}{(1 + t^2 + |x|^2)^{n/2}}$$
3. (2p) Let $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$. Suppose that at the point $x \in \mathbf{R}^n$,
$$|f(y) - f(x)| \le C|x - y|^\alpha$$
for all $y \in B(x, \delta)$, for some $C > 0$ and $\alpha, \delta > 0$. Show that the limit $(R_j f)(x)$ exists. Moreover, under the same condition on f, show that
$$\lim_{t\downarrow 0}(Q_{j,t} * f)(x) = (R_j f)(x), \qquad Q_{j,t}(y) := \frac{y_j}{(t^2 + |y|^2)^{(n+1)/2}}.$$
4. (1p) Let $f \in S(\mathbf{R}^n)$. Show that
$$\left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_2 \le \|\Delta f\|_2.$$

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