HARMONIC ANALYSIS, 7. EXERCISE

1. (1p) Show that the Hilbert transform H is not strong $(1,1)$ or strong (∞,∞).

   Hint: take $f = \chi_{(-1,1)}$ *and calculate* Hf.

2. (3p) Show that the Hilbert transform is strong (p,p) for every $1 < p < \infty$ by verifying Steps 1-3 in the Proof number 1 of Theorem 6.11 in the lecture note.

   Every step is independent and worth of 1 point. One can proceed for example to Step 2 without showing Step 1.

3. (1p) Let $1 < p < \infty$. Suppose that $f \in L^p(\mathbf{R})$ is uniformly Dini-continuous, that is the uniform modulus of continuity
$$\omega(r) := \sup\{|f(x) - f(y)| \ : \ |x - y| < r, \ x, y \in \mathbf{R}\}$$
   satisfies that
$$\Omega(s) := \int_0^s \omega(r)\,\frac{dr}{r}$$
   is bounded for some $s = s_0 > 0$. Show that the Hilbert transform of f is continuous.

   Hints: show first that for all $0 < \varepsilon < \delta$, $|H^{(\varepsilon)}f(x)| \le \Omega(\delta) + c_\delta \|f\|_p$ *and* $|H^{(\varepsilon)}f(x+h) - H^{(\varepsilon)}f(x)| \le 2\Omega(\delta) + c_\delta \|f(\cdot + h) - f(\cdot)\|_p$. *Deduce from this that* $(H^{(\varepsilon)}f)_\varepsilon$ *is equibounded and equicontinuous, i.e. that all of* $H^{(\varepsilon)}f$ *are bounded by the same constant and continuous with the same modulus of continuity (for the latter, see the proof of Theorem 3.7). Then apply the powerful tool from Functional Analysis, Arzelà-Ascoli theorem, stating that there is a subsequence in* $(H^{(\varepsilon)}f)_\varepsilon$ *that converges to a continuous function. Finally infer that the limit* $\lim_{\varepsilon\downarrow 0} H^{(\varepsilon)}f(x)$ *exists for all* $x \in \mathbf{R}$.

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