HARMONIC ANALYSIS, 10. EXERCISE

In these exercises, T stands for the singular integral operator of convolution type satisfying properties $(i - iv)$ in the lecture note unless otherwise stated.

1. Let n=1. Show that the singular integral operator T is a multiple of the Hilbert transform.
2. Suppose that condition (iv) is violated in the lecture note, that is
$$\int_{S^{n-1}} \Omega(x)\, d\mathcal{H}^{n-1}(x) \neq 0,$$
but Ω satisfies $(i-iii)$. Show that the principal value does not exist in general.
3. Let $f \in S(\mathbf{R}^n)$. Show that
$$\widehat{Tf}(\xi) = \int_{S^{n-1}} \Omega(u) \left(\frac{1}{\log |u \cdot \xi/|\xi||} - i\frac{\pi}{2} \operatorname{sgn}(u \cdot \xi) \right) d\mathcal{H}^{n-1}(u) \widehat{f}(\xi)$$
for all $\xi \in \mathbf{R}^n \setminus \{0\}$.
4. Using Exercise 3, calculate the Fourier transform of the Riesz transform R_j.
5. Define
$$\Omega_o(u) = \frac{1}{2} \left(\Omega(u) - \Omega(-u)\right), \qquad \Omega_e(u) = \frac{1}{2} \left(\Omega(u) + \Omega(-u)\right).$$
Let $f \in S(\mathbf{R}^n)$. Instead of the boundedness assumption (ii) for Ω, suppose that $\Omega_o \in L^1(S^{n-1})$ and $\Omega_e \in (L \log(L))(S^{n-1})$, i.e.
$$|\Omega_e| \max\left(\log(|\Omega_e|), 0\right) \in L^1(S^{n-1}).$$
Show that T is strong $(2,2)$ in $S(\mathbf{R}^n)$.

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