SELF-IMPROVING PROPERTY OF NONLINEAR HIGHER ORDER PARABOLIC SYSTEMS NEAR THE BOUNDARY

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ABSTRACT. We establish global regularity results for a wide class of non-linear higher order parabolic systems. The model problem we have in mind is the parabolic p-Laplacian system of order $2m$, $m \geq 1$,

$$\partial_t u + (-1)^m \operatorname{div}^m \left(|D^m u|^{p-2} D^m u \right) = 0$$

with prescribed boundary and initial values. We prove that if the boundary values are sufficiently regular, then $D^m u$ is globally integrable to a better power than the natural p. The method also produces a global estimate.

1. INTRODUCTION

We study the global regularity properties of solutions to a wide class of non-linear higher order parabolic systems. In particular, the parabolic p-Laplacian system of order $2m$, $m \geq 1$,

$$\partial_t u + (-1)^m \operatorname{div}^m \left(|D^m u|^{p-2} D^m u \right) = 0$$

with the initial and boundary values provides a basic example.

Under suitable conditions on the initial and boundary values, the corresponding initial boundary value problem admits a solution u such that $|D^m u|$ is integrable to the power p. Our aim is to show that $D^m u$ is actually globally integrable to a better power, that is, $|D^m u| \in L^{p+\varepsilon}$ all the way up to the boundary provided that the boundary values and the domain are sufficiently smooth. We assume that the complement of the domain satisfies a uniform capacity density condition, which is essentially sharp for higher integrability results. Moreover, the method produces an explicit estimate for the $L^{p+\varepsilon}$-norm of $D^m u$.

Higher integrability plays an important role in stability and partial regularity results for solutions and gradients in both the elliptic and parabolic cases. For elliptic regularity results with the standard and also non-standard growth conditions, see for example Acerbi-Mingione [1, 2, 3, 28]. For recent parabolic applications of higher integrability in the framework of partial regularity and Calderón-Zygmund type estimates, see for example Acerbi-Mingione [4], Acerbi-Mingione-Seregin [5], Bögelein [10], Duzaar-Mingione [16], Duzaar-Mingione-Steffen [17] and Bögelein-Duzaar-Mingione [11].

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The elliptic higher integrability techniques developed by Gehring [20], and Elcrat and Meyers [18] (see also [21]) could not directly be carried over to the parabolic case. Nevertheless, Giaquinta and Struwe proved a first parabolic analogue for systems with linear growth in [22]. Higher integrability for more general parabolic systems with non-linear growth conditions remained open for some time: The first positive result for degenerate and singular second order parabolic p-growth systems was obtained by Kinnunen and Lewis [26]. The proof employs the method of intrinsic scaling with respect to the gradient of the solution. The idea to consider parabolic cylinders whose scaling depends on the solution itself goes back to DiBenedetto and Friedman [12, 13, 14]. The local higher integrability result was recently extended to higher order parabolic systems in [9], and global higher integrability results for quasiminimizers and second order parabolic systems were obtained in [35, 36, 37].

Our basic strategy follows the guidelines of the local result in [26]. Indeed, we first derive a reverse Hölder inequality on intrinsic cylinders up to the boundary and then use a covering argument to extend the estimates to the whole space-time cylinder. The intuitive idea is to use cylinders whose space-time scaling is roughly speaking comparable to the mean value of $|D^m u|^{2-p}$ on the same cylinder. In a certain sense this space-time scaling reflects the non-homogeneity of the parabolic system, which is not present in the elliptic case. However, the boundary effects and lower order terms cause extra difficulties: The covering now consists of three kinds of intrinsic cylinders that may lie near the lateral or initial boundary, or inside the domain.

To estimate the lower order terms near the lateral boundary, we employ a boundary version of Poincaré's inequality iteratively. This step exploits the uniform capacity density condition of the complement. Near the initial boundary, we compare the solution with the mean value polynomial of the initial values. To this end, the oscillation of weighted means of the solution and lower order derivatives between the different time slices needs to be controlled. For the solution itself, we directly exploit the weak formulation of the parabolic system whereas for the derivatives, we utilize the suitable weighted means.

In the singular case, that is when $p<2$, the quadratic terms on the right-hand side of the Caccioppoli inequality usually cause technical difficulties. Therefore, we employ an iteration method in order to absorb these terms at an early stage (c.f. Lemma 4.3 and 5.5). In this way, we later avoid additional terms in the scaling which simplifies the proof considerably. Indeed, practically the same proof now runs in both the singular and degenerate cases. This observation is useful even in the local second order higher integrability proof.

2. Statement of the result

We consider initial-boundary value problems of the type

$$\begin{cases} \partial_t u + (-1)^m \displaystyle\sum_{|\alpha|=m} D^\alpha \mathscr{A}_\alpha(z, D^m u) = 0, & \text{in} \quad \Omega_T, \\ u = g, & \text{on } \partial_p \Omega_T. \end{cases} \tag{2.1}$$

Here, Ω is a bounded domain in $\mathbf{R}^n$ and $\Omega_T = \Omega \times (0,T) \subset \mathbf{R}^{n+1}$ stands for a parabolic cylinder. The initial and lateral boundary values g of the solution are prescribed on the parabolic boundary $\partial_p \Omega_T = (\Omega \times \{0\}) \cup (\partial\Omega \times (0,T))$ of Ω_T.

Moreover, $u\colon \Omega_T \to \mathbf{R}^N$ is a vector valued function and, as usual, we denote by $\partial_t u = u_t$ the derivative with respect to the time-variable t and by Du, respectively $D^k u = \{D^\alpha u_i\}_{i=1,\dots,N}^{|\alpha|=k}$, the derivatives (of order k) with respect to the space-variable x. For convenience of notation, we identify $D^m u$ as a vector in $\mathbf{R}^\ell$, $\ell = N\binom{n+m-1}{m}$, and similarly for $D^k u$. Furthermore, we adopt the shorthand notation $z = (x,t) \in \mathbf{R}^{n+1}$.

For simplicity of notation, we write $\mathscr{A} = \{\mathscr{A}_\alpha\}_{|\alpha|=m}$, where $\mathscr{A}_\alpha\colon \Omega_T \times \mathbf{R}^\ell \to \mathbf{R}^N$, and thus $\mathscr{A}\colon \Omega_T \times \mathbf{R}^\ell \to \mathbf{R}^\ell$. We assume that $\mathscr{A}$ is a Carathéodory function:

$$z \mapsto \mathscr{A}(z,w) \text{ is measurable for every } w \in \mathbf{R}^\ell,$$
$$w \mapsto \mathscr{A}(z,w) \text{ is continuous for almost every } z \in \Omega_T,$$

and satisfies the following p-growth conditions:

$$\langle \mathscr{A}(z,w), w\rangle \geq \nu\, |w|^p, \tag{2.2}$$
$$|\mathscr{A}(z,w)| \leq L\left(|w|^{p-1}+1\right), \tag{2.3}$$

for all $z \in \Omega_T$, $w \in \mathbf{R}^\ell$ and some constants $0 < \nu \leq 1$ and $1 \leq L < \infty$ and $p > \max\{1, \frac{2n}{n+2m}\}$. Above we have made several simplifications for expository reasons: we could add an inhomogeneity with controlled growth conditions into the right-hand side of (2.1) as well as additional functions to the growth bounds, cf. [9]. Nevertheless, the proofs would remain virtually the same. The restriction $p > \max\{1, \frac{2n}{n+2m}\}$ is necessary in the parabolic framework because of the Sobolev embedding $W^{m,\frac{2n}{n+2m}} \hookrightarrow L^2$ as we have to deal with the L^2-norm of u appearing in Caccioppoli's inequality.

There will naturally appear several exponents throughout the paper. Set

$$2_* = \max\{1, \tfrac{2n}{n+2m}\} \text{ and } \quad p_* = \max\{1, \tfrac{pn}{n+2m}\},$$

and observe that when $m = 1$, we simply obtain the usual Sobolev exponents. We will be able to combine the degenerate and singular cases by defining

$$\widehat{p} = \max\{2, p\}, \quad \widehat{p}_* = \max\{2_*, p_*\} \text{ and } \quad \widehat{p}' = \min\{2, \tfrac{p}{p-1}\}.$$

Next we define the space $V^p_\beta(0,T;\Omega)$ for the initial and boundary values. For $\beta \geq 0$, we denote

$$\begin{aligned} V^p_\beta(0,T;\Omega) = \Big\{\varphi \in L^{p+\beta}(0,T;W^{m,p+\beta}(\Omega;\mathbf{R}^N)) \cap W^{1,\widehat{p}'+\beta}(0,T;L^{\widehat{p}'+\beta}(\Omega;\mathbf{R}^N)) \\ \cap C([0,T);L^2(\Omega;\mathbf{R}^N)) : \varphi(\cdot,0) \in W^{m,\widehat{p}_*+\beta}(\Omega;\mathbf{R}^N)\Big\}. \end{aligned}$$

The role of the continuity assumption is to fix the right representative. Observe that even smooth boundary values lead to a nontrivial theory. Next we specify the notion of a global solution.

Definition 2.1. *Let $p > 2_*$. A function $u \in L^p(0,T;W^{m,p}(\Omega;\mathbf{R}^N)) \cap C([0,T);L^2(\Omega;\mathbf{R}^N))$ is a global (weak) solution to the initial-boundary value problem (2.1) if*

$$\int_{\Omega_T} u \cdot \varphi_t - \langle \mathscr{A}(z, D^m u), D^m \varphi\rangle \,\mathrm{d}z = 0 \tag{2.4}$$

for every test-function $\varphi \in C_0^\infty(\Omega_T;\mathbf{R}^N)$ and, moreover,

$$(u-g)(\cdot,t) \in W_0^{m,p}(\Omega;\mathbf{R}^N) \quad \textit{for almost every } t \in (0,T) \tag{2.5}$$

and

$$\frac{1}{h}\int_0^h\int_\Omega |u(x,t)-g(x,0)|^2\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \textit{as } h\downarrow 0 \tag{2.6}$$

for a given function $g\in V_0^p(0,T;\Omega)$.

Note that the space $L^p(0,T;W^{m,p}(\Omega;\mathbf{R}^N))\cap C([0,T);L^2(\Omega;\mathbf{R}^N))$ seems natural in the light of the existence theorems (see Lions [30] and Showalter [38] Chapter III, Proposition 1.2).

We work on the parabolic cylinders of the form

$$Q_{z_0}(\rho,s)=B_{x_0}(\rho)\times\Lambda_{t_0}(s)\subset\mathbf{R}^{n+1},$$

where $z_0=(x_0,t_0)\in\mathbf{R}^{n+1}$, $\rho,s>0$ and $B_{x_0}(\rho)$ denotes the open ball in $\mathbf{R}^n$ with center x_0 and radius ρ and

$$\Lambda_{t_0}(s)=(t_0-s,t_0+s)$$

the interval of length $2s$ centered at t_0. In the case $s=\rho^{2m}$, we write $Q_{z_0}(\rho)=Q_{z_0}(\rho,\rho^{2m})$. When no confusion arises, we shall omit the reference points. Furthermore, we write

$$\alpha B(\rho)=B(\alpha\rho),\ \alpha\Lambda(s)=\Lambda(\alpha^{2m}s),\text{ and } \alpha Q(\rho,s)=Q(\alpha\rho,\alpha^{2m}s),$$

for a ball, interval, and cylinder enlarged by the factor $\alpha>0$.

Next we state our main theorem. The global higher integrability is achieved under the assumption that the complement of the domain Ω satisfies a uniform capacity density condition. This regularity condition guarantees that there is "enough of complement" near every boundary point. The capacity density condition could be replaced for example by the stronger measure density condition. For the precise formulation of the condition, see Definition 3.1.

Theorem 2.2. *Suppose that u is a global solution according to Definition 2.1 with boundary and initial data* $g\in V_\beta^p(0,T;\Omega)$ *for some* $\beta>0$ *and let* $\mathbf{R}^n\setminus\Omega$ *be uniformly p-thick. Then there exists* $\varepsilon=\varepsilon(n,N,m,p,L/\nu)\in(0,\beta]$ *such that*

$$u\in L^{p+\varepsilon}(0,T;W^{m,p+\varepsilon}(\Omega;\mathbf{R}^N)).$$

Moreover, for any parabolic cylinder $Q_0=B_0\times\Lambda_0=Q_{z_0}(R,R^2)\subset\mathbf{R}^{n+1}$*, we have the following boundary estimate*

$$\begin{aligned}\frac{1}{|Q_0|}\int_{\frac14 Q_0\cap\Omega_T}|D^m u|^{p+\varepsilon}\,\mathrm{d}z \le{}& \left(\frac{c}{|Q_0|}\int_{Q_0\cap\Omega_T}\left(|D^m u|^p+G^p\right)\mathrm{d}z\right)^{(\varepsilon+d)/d}\\ &+\frac{c}{|Q_0|}\int_{Q_0\cap\Omega_T}G^{p+\varepsilon}\,\mathrm{d}z+c\\ &+\left(\frac{c\,\delta_1}{|B_0|}\int_{B_0\cap\Omega}|D^m g(\cdot,0)|^{\widehat{p}_*+\varepsilon}\,\mathrm{d}x\right)^{(\widehat{p}+\varepsilon)/(\widehat{p}_*+\varepsilon)}.\end{aligned}$$

where $c=c(n,N,m,p,L/\nu)$ *and* $\delta_1=1$ *if* $0\in\Lambda_0$ *and* $\delta_1=0$ *otherwise. Here, we have denoted*

$$G=\left(|D^m g|^p+|\partial_\tau g|^{\widehat{p}'}\right)^{1/p}\quad \textit{if } B_0\setminus\Omega\neq\emptyset \tag{2.7}$$

and $G=0$ *otherwise and*

$$d=\begin{cases}2 & \textit{if}\quad p\ge 2,\\ p-\dfrac{n(2-p)}{2m} & \textit{if}\quad 2_*<p<2.\end{cases}$$

3. PRELIMINARIES

3.1. **Variational p-capacity.** Let $1 < p < \infty$ and $\mathcal{O}$ be an open set. The *variational p-capacity* of a compact set $C \subset \mathcal{O}$ is defined to be

$$\operatorname{cap}_p(C,\mathcal{O}) = \inf_f \int_{\mathcal{O}} |\nabla f|^p \,\mathrm{d}x,$$

where the infimum is taken over all the functions $f \in C_0^\infty(\mathcal{O})$ such that $f = 1$ in C. To define the variational p-capacity of an open set $U \subset \mathcal{O}$, we take the supremum over the capacities of the compact sets belonging to U. The variational p-capacity of an arbitrary set $E \subset \mathcal{O}$ is defined by taking the infimum over the capacities of the open sets containing E.

For the capacity of a ball, we have

$$\operatorname{cap}_p(\overline{B}(\rho), B(2\rho)) = c\,\rho^{n-p}. \tag{3.1}$$

For further details, see Chapter 4 of Evans-Gariepy [19], Chapter 2 of Heinonen-Kilpeläinen-Martio [24], or Chapter 2 of Malý-Ziemer [31].

Next we introduce the uniform capacity density condition, which allows us to use a boundary version of a Sobolev-Poincaré type inequality. This condition is essentially sharp for our main result as shown by Kilpeläinen-Koskela [25] in the elliptic case and in [27] for the second order parabolic p-Laplace equation.

Definition 3.1. *A set $E \subset \mathbf{R}^n$ is uniformly p-thick if there exist constants $\mu, \rho_0 > 0$ such that*

$$\operatorname{cap}_p(E \cap \overline{B}_x(\rho), B_x(2\rho)) \geq \mu \operatorname{cap}_p(\overline{B}_x(\rho), B_x(2\rho)),$$

for all $x \in E$ and for all $0 < \rho < \rho_0$.

If $p > n$, the condition is superfluous since then every set is uniformly p-thick. The next lemma slightly extends the capacity estimate from the above definition (cf. [36], Lemma 3.8).

Lemma 3.2. *Let Ω be a bounded open set in $\mathbf{R}^n$ and suppose that $\mathbf{R}^n \setminus \Omega$ is uniformly p-thick. Choose $y \in \Omega$ such that $B_y(4\rho/3) \setminus \Omega \neq \emptyset$. Then there exists a constant $\tilde{\mu} = \tilde{\mu}(\mu, \rho_0, n, p) > 0$ such that*

$$\operatorname{cap}_p\left(\overline{B}_y(2\rho) \setminus \Omega, B_y(4\rho)\right) \geq \tilde{\mu} \operatorname{cap}_p\left(\overline{B}_y(2\rho), B_y(4\rho)\right).$$

A uniformly q-thick set is also uniformly ϑ-thick for all $\vartheta \geq q$. This is a consequence of Hölder's and Young's inequalities.

Lemma 3.3. *If a compact set E is uniformly q-thick, then E is uniformly ϑ-thick for all $\vartheta \geq q$.*

The next theorem states that a uniformly p-thick set has a self-improving property. This result was shown by Lewis in Theorem 1 of [29]. See also Ancona [7] and Mikkonen [33].

Theorem 3.4. *Let $1 < p \leq n$. If a set E is uniformly p-thick, then there exists $\gamma = \gamma(n, p, \mu) \in (1, p)$ for which E is uniformly γ-thick.*

Next, we formulate a well-known version of the Sobolev-type inequality. For the proof, see Chapter 10 of Maz'ja's monograph [32] or Hedberg [23] and also [36]. Later we combine this estimate with the boundary regularity condition and obtain a boundary version of Sobolev's inequality.

The lemma employs quasicontinuous representatives of Sobolev functions. We call $u \in W^{1,p}(\Omega)$ *p-quasicontinuous* if for each $\varepsilon > 0$ there exists an open set U, $U \subset \Omega \subset B_{R'}$, such that $\mathrm{cap}_p(U, B_{2R'}) \le \varepsilon$, and the restriction of u to the set $\Omega \setminus U$ is finite valued and continuous. The p-quasicontinuous functions are closely related to the Sobolev space $W^{1,p}(\Omega)$: For example, if $u \in W^{1,p}(\Omega)$, then u has a p-quasicontinuous representative.

Adopting the usual notation for the mean value integral

$$\fint_{B(\rho)} |u|^q \,\mathrm{d}x = \frac{1}{|B(\rho)|} \int_{B(\rho)} |u|^q \,\mathrm{d}x.$$

we have the following

Lemma 3.5. *Let $B = B(\rho)$ be a ball in $\mathbf{R}^n$ and suppose that $u \in W^{1,q}(B)$ is q-quasicontinuous. Denote*

$$N_{B(\rho/2)}(u) = \{x \in \overline{B}(\rho/2) : u(x) = 0\}.$$

Then there exists a constant $c = c(n,q) > 0$ such that

$$\fint_{B(\rho)} |u|^q \,\mathrm{d}x \le \frac{c}{\mathrm{cap}_q(N_{B(\rho/2)}(u), B(\rho))} \int_{B(\rho)} |Du|^q \,\mathrm{d}x.$$

3.2. **Interpolation estimates.** When dealing with higher order problems, interpolation estimates play an essential role. In several points, particularly when Poincaré's inequality cannot be applied they shall help us to treat the intermediate derivatives. First, we provide an interpolation estimate for intermediate derivatives on the annulus, cf. Adams [6], Theorem 4.14 or [8], Lemma B.1. Note that the crucial point here is the right dependence on the width of the annulus.

Lemma 3.6. *Let $B(r_1) \subset B(r_2)$ be two concentric balls in $\mathbf{R}^n$ with $0 < r_1 < r_2 \le 1$ and let $u \in W^{m,p}(B(r_2))$ with $p \ge 1$. Then for any $0 \le k \le m-1$ and $0 < \varepsilon \le 1$ there exists $c = c(n,m,p,1/\varepsilon)$, such that*

$$\int_{B(r_2)\setminus B(r_1)} \frac{|D^k u|^p}{(r_2 - r_1)^{(m-k)p}} \,\mathrm{d}x \le \varepsilon \int_{B(r_2)\setminus B(r_1)} |D^m u|^p \,\mathrm{d}x + c \int_{B(r_2)\setminus B(r_1)} \frac{|u|^p}{(r_2 - r_1)^{mp}} \,\mathrm{d}x.$$

One of the difficulties in proving the main result is the fact that both powers 2 and p play a role in Caccioppoli's inequality. We now state Gagliardo-Nirenberg-Sobolev's inequality (see Nirenberg [34]) in a form, which helps us to combine the different powers.

Theorem 3.7. *Let $B(\rho)$ be a ball in $\mathbf{R}^n$ and $u \in W^{m,q}(B(\rho))$, $m \in \mathbb{N}$ and $1 \le \sigma, q, r \le \infty$ and $\theta \in (0,1)$ and $0 \le k \le m-1$ with $k - \frac{n}{\sigma} \le \theta(m - \frac{n}{q}) - (1-\theta)\frac{n}{r}$. Then there exists $c = c(n,m,\sigma)$ such that*

$$\fint_{B(\rho)} \Big| \frac{D^k u}{\rho^{m-k}} \Big|^\sigma \,\mathrm{d}x \le c \Big(\sum_{j=0}^{m} \fint_{B(\rho)} \Big| \frac{D^j u}{\rho^{m-j}} \Big|^q \,\mathrm{d}x \Big)^{\theta\sigma/q} \Big(\fint_{B(\rho)} \Big| \frac{u}{\rho^m} \Big|^r \,\mathrm{d}x \Big)^{(1-\theta)\sigma/r}.$$

The following lemma will help us to absorb certain integrals into the left-hand side. The proof employs a standard iteration argument, see for instance Giaquinta's monograph [21], Chapter V, Lemma 3.1.

Lemma 3.8. *Let $0 < \vartheta < 1$, $A, B \ge 0$, $\alpha > 0$ and let $f \ge 0$ be a bounded function satisfying*

$$f(t) \le \vartheta f(s) + A(s-t)^{-\alpha} + B \qquad \textit{for all } \ 0 < r \le t < s \le \rho.$$

Then there exists a constant $c_{tech} = c_{tech}(\alpha, \vartheta)$, such that

$$f(r) \le c_{tech}\big(A(\rho - r)^{-\alpha} + B\big).$$

3.3. **Mean value polynomials.** In order to prove a higher integrability result for the m-th derivative of u, we shall approximate the function up to order $m-1$. For this aim, we shall employ mean value polynomials of order $m-1$. Let $B_{x_0}(r)$ be a ball in $\mathbf{R}^n$ and $f \in W^{m,1}(B_{x_0}(r); \mathbf{R}^N)$. Then its mean value polynomial $P_r^{(f)} \colon \mathbf{R}^n \to \mathbf{R}^N$ of degree $\le m-1$ is defined uniquely by the condition

$$(\delta P_r^{(f)})_{x_0;r} = (\delta f)_{x_0;r}, \tag{3.2}$$

where $\delta u = (u, Du, \ldots, D^{m-1}u)$ denotes the vector of lower order derivatives and

$$(f)_{x_0;r} = \fint_{B_{x_0}(r)} f \,\mathrm{d}z$$

denotes the mean-value of f on $B_{x_0}(r)$. Therefore, (3.2) can be rewritten as $(D^k P_r^{(f)})_{x_0;r} = (D^k f)_{x_0;r}$ for $k = 0, \ldots, m-1$. The mean value polynomial can be expressed in terms of the mean values of f as

$$P_r^{(f)}(x) = \sum_{|\alpha| \le m-1} \sum_{|\alpha+\beta| \le m-1} \frac{b_\beta}{\alpha!} (D^{\alpha+\beta} f)_{x_0;r} (x - x_0)^\alpha,$$

where

$$b_\beta = \begin{cases} 1, & \text{if } |\beta| = 0 \\ -\displaystyle\sum_{0<\gamma\le\beta} \frac{b_{\beta-\gamma}}{\gamma!} \fint_{B_{x_0}(r)} (y - x_0)^\gamma \,\mathrm{d}y, & \text{if } |\beta| \ge 1. \end{cases}$$

For more details, see for instance Duzaar-Gastel-Grotowski [15].

Due to the defining property of $P_r^{(f)}$, we can replace in the above representation $(D^{\alpha+\beta} f)_{x_0;r}$ by $(D^{\alpha+\beta} P_r^{(f)})_{x_0;r}$. Moreover, we can show that $|b_\beta| \le c(n,m)\, r^{|\beta|}$ for all multi-indices β with $0 \le |\beta| \le m-1$. This observation leads us to the estimate

$$|P(x)| \le c(n,m) \sum_{k=0}^{m-1} R^k |(D^k P)_{x_0;r}| \qquad \text{for all } x \in B_{x_0}(R), \tag{3.3}$$

valid for any polynomial $P \colon \mathbf{R}^n \to \mathbf{R}^N$ of order $\le m-1$ and balls $B_{x_0}(r)$, $B_{x_0}(R)$ in $\mathbf{R}^n$ with $0 < r \le R$. See [8], Lemma A.1, for a more detailed proof.

From this estimate, we can deduce a bound for the difference of the mean value polynomials on two different balls. The proof applies the definition of the mean value polynomials together with Poincaré's inequality.

Lemma 3.9. *Let $B_{x_0}(r)$, $B_{x_0}(R)$ be two balls in $\mathbf{R}^n$ with $R/2 \le r < R$ and suppose that $f \in W^{m,1}(B_{x_0}(R); \mathbf{R}^N)$. Denote by $P_r^{(f)}, P_R^{(f)} \colon \mathbf{R}^n \to \mathbf{R}^N$ the mean value polynomials of f of degree $\le m-1$. Then there exists $c = c(n,N,m)$ such that*

$$|P_r^{(f)}(x) - P_R^{(f)}(x)| \le c\, R^m \fint_{B_{x_0}(R)} |D^m f| \,\mathrm{d}x \qquad \textit{for all } x \in B_{x_0}(R).$$

Proof. To estimate the difference of the polynomials, we use (3.3) with $(P_r^{(f)} - P_R^{(f)})$ instead of P and exploit the defining property of the polynomial $P_r^{(f)}$ to find

$$|(P_r^{(f)} - P_R^{(f)})(x)| \le c \sum_{k=0}^{m-1} R^k \Big| \fint_{B_{x_0}(r)} D^k (P_r^{(f)} - P_R^{(f)}) \,\mathrm{d}y \Big|$$

$$= c\sum_{k=0}^{m-1} R^k \Big| \fint_{B_{x_0}(r)} D^k(f - P_R^{(f)})\,\mathrm{d}y\Big|.$$

Next we enlarge the domain of integration in the integrals on the right-hand side and recall that $|B_{x_0}(R)|/|B_{x_0}(r)| \le 2^n$ since $R/2 \le r$. Finally, applying Poincaré's inequality $m-k$ times to $D^k(f - P_R^{(f)})$ which is allowed since $(D^k(f-P_R^{(f)}))_{x_0,r} = 0$ leads us to

$$|(P_r^{(f)} - P_R^{(f)})(x)| \le c\sum_{k=0}^{m-1} R^k \fint_{B_{x_0}(R)} |D^k(f-P_R^{(f)})|\,\mathrm{d}x \le c\,R^m \fint_{B_{x_0}(R)} |D^m f|\,\mathrm{d}x.$$

This is the desired estimate. □

3.4. **Steklov-means.** Since weak solutions do not a priori possess any differentiability properties with respect to the time variable t, it is standard to use a mollification in time. Therefore, given a function $f \in L^1(\Omega_T)$, we define its Steklov-mean by

$$f_h(x,t) = [f]_h(x,t) = \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} f(x,s)\,\mathrm{d}s, & t \in (0, T-h), \\ 0, & t \in (T-h, T), \end{cases}$$

for $0 < h < T$ and $(x,t) \in \Omega_T$. Using Steklov-means, we get for a.e. $t \in (0,T)$ an equivalent system:

$$\int_\Omega \partial_t u_h(\cdot,t)\cdot\varphi + \big\langle [\mathscr{A}(\cdot, D^m u)]_h(\cdot,t), D^m\varphi\big\rangle\,\mathrm{d}x = 0, \tag{3.4}$$

for all $\varphi \in L^2(\Omega;\mathbf{R}^N) \cap W_0^{m,p}(\Omega;\mathbf{R}^N)$.

4. Estimates near the lateral boundary

In this chapter, we derive estimates on parabolic cylinders lying near the lateral boundary $\partial\Omega \times (0,T)$. For notational convenience, in this chapter we will combine the boundary terms and the constant coming from the growth bounds as follows

$$\widetilde{G} = \big(|D^m g|^p + |\partial_\tau g|^{\widehat{p}'}\big)^{1/p} + 1.$$

Since we now are in the lateral boundary situation we have $\widetilde{G} = G + 1$, where G is from (2.7). As usual, the first step when proving higher integrability is to derive suitable Caccioppoli's inequality. Although we state it for arbitrary cylinders in $\mathbf{R}^{n+1}$, it will be needed later only for cylinders intersecting the lateral boundary.

Lemma 4.1. *Let u be a global solution according to Definition 2.1. Then there exists $c_{Cac} = c_{Cac}(n,m,p,L/\nu)$ such that for all parabolic cylinders $Q_{z_0}(r,s)$, $Q_{z_0}(R,S) \subset \mathbf{R}^{n+1}$ with $0 < R/2 \le r < R \le 1$, $s = \lambda^{2-p} r^{2m}$, $S = \lambda^{2-p}R^{2m} \le 1$, $\lambda > 0$ there holds*

$$\sup_{t \in \Lambda_{t_0}(s)\cap(0,T)} \int_{B_{x_0}(r)\cap\Omega} |(u-g)(\cdot,t)|^2\,\mathrm{d}x + \int_{Q_{z_0}(r,s)\cap\Omega_T} |D^m u|^p\,\mathrm{d}z$$
$$\le c_{Cac}\int_{Q_{z_0}(R,S)\cap\Omega_T} \lambda^{p-2}\Big|\frac{u-g}{(R-r)^m}\Big|^2 + \Big|\frac{u-g}{(R-r)^m}\Big|^p + \widetilde{G}^p\,\mathrm{d}z.$$

Proof. We choose $r \le r_1 < r_2 \le R$ and $\eta \in C_0^\infty(B_{x_0}(r_2))$, $\zeta \in C^1(\mathbf{R})$ to be two cut-off functions with

$$\text{(4.1)} \quad \begin{cases} \eta \equiv 1 \text{ in } B_{x_0}(r_1),\ 0 \le \eta \le 1,\ |D^k\eta| \le \frac{c_\eta}{(r_2-r_1)^k} \text{ for all } 0 \le k \le m; \\ \zeta \equiv 0 \text{ on} \left(-\infty, t_0-S\right),\ \zeta \equiv 1 \text{ on } \left(t_0-s,\infty\right),\ 0 \le \zeta \le 1,\ 0 \le \zeta' \le \frac{2}{S-s}. \end{cases}$$

Choosing the test-function $\varphi_h = \eta\zeta^2(u_h - g_h)$ in the Steklov-formulation (3.4) and integrating with respect to τ over $(0,t)$, we get for $t \in (0,T)$

$$\text{(4.2)} \qquad \int_{\Omega_t} \partial_\tau u_h \cdot \varphi_h + \left\langle [\mathscr{A}(\cdot, D^m u)]_h, D^m \varphi_h \right\rangle \mathrm{dz} = 0,$$

where we abbreviated $\Omega_t = \Omega \times (0,t)$. For the first term on the left-hand side, we find that

$$\begin{aligned} \int_{\Omega_t} \partial_\tau u_h \cdot \varphi_h \,\mathrm{dz} &= \int_{\Omega_t} \partial_\tau (u_h - g_h) \cdot \varphi_h + \partial_\tau g_h \cdot \varphi_h \,\mathrm{dz} \\ &\to \tfrac{1}{2} \int_\Omega |(u-g)(\cdot,t)|^2 \eta \zeta(t)^2 \,\mathrm{d}x - \int_{\Omega_t} |u-g|^2 \eta\zeta\zeta' \,\mathrm{dz} + \int_{\Omega_t} \partial_\tau g \cdot (u-g)\eta\zeta^2 \,\mathrm{dz}, \end{aligned}$$

as $h \downarrow 0$. Here we have also taken into account that the initial boundary term vanishes at $\tau = 0$ because of the initial condition. The last integral on the right-hand side is now further estimated with the help of Young's inequality with exponents $(2,2)$ if $p < 2$, respectively $(p, p/(p-1))$ when $p \ge 2$. Note also that $r_2 - r_1 \le R \le 1$, respectively $\lambda^{2-p}(r_2-r_1)^{2m} \le S \le 1$. We get

$$\left| \int_{\Omega_t} \partial_\tau g \cdot (u-g)\eta\zeta^2 \,\mathrm{dz} \right| \le \int_{Q_{z_0}(R,S)\cap\Omega_T} |\partial_\tau g|^{\widehat{p}'} + \lambda^{p-2} \frac{|u-g|^2}{(r_2-r_1)^{2m}} + \frac{|u-g|^p}{(r_2-r_1)^{mp}} \,\mathrm{dz}.$$

Passing to the limit $h \downarrow 0$ also in the second term on the right side of (4.2), we find

$$\begin{aligned} &\int_{\Omega_t} \left\langle [\mathscr{A}(\cdot, D^m u)]_h, D^m \varphi_h \right\rangle \mathrm{dz} \\ &\to \int_{\Omega_t} \langle \mathscr{A}(\cdot, D^m u), D^m u \rangle \eta\zeta^2 - \langle \mathscr{A}(\cdot, D^m u), D^m g \rangle \eta\zeta^2 + \langle \mathscr{A}(\cdot, D^m u), \mathrm{LOT} \rangle \zeta^2 \,\mathrm{dz}, \end{aligned}$$

where we abbreviated the lower order terms by

$$\mathrm{LOT} = \sum_{k=0}^{m-1} \binom{m}{k} D^{m-k}\eta \odot D^k(u-g).$$

From the ellipticity (2.2) of $\mathscr{A}$, we infer for the first term that

$$\int_{\Omega_t} \langle \mathscr{A}(\cdot, D^m u), D^m u \rangle \eta\zeta^2 \,\mathrm{dz} \ge \nu \int_{\Omega_t} |D^m u|^p \eta\zeta^2 \,\mathrm{dz},$$

while for the second one, we obtain by the growth bound (2.3) of $\mathscr{A}$ and Young's inequality for $\varepsilon > 0$ that

$$\left| \int_{\Omega_t} \langle \mathscr{A}(\cdot, D^m u), D^m g \rangle \eta\zeta^2 \,\mathrm{dz} \right| \le \varepsilon \int_{\Omega_t} \left(|D^m u|^p + 1 \right) \eta\zeta^2 \,\mathrm{dz} + c_\varepsilon \int_{\Omega_t} |D^m g|^p \,\mathrm{dz},$$

where $c_\varepsilon = c_\varepsilon(p, L, 1/\varepsilon)$. Similarly, for the third term, we get

$$\left| \int_{\Omega_t} \langle \mathscr{A}(\cdot, D^m u), \mathrm{LOT} \rangle \zeta^2 \,\mathrm{dz} \right| \le \varepsilon \int_{\Omega_t \cap \operatorname{spt}\eta} \left(|D^m u|^p + 1 \right) \zeta^2 \,\mathrm{dz} + c_\varepsilon \int_{\Omega_t} |\mathrm{LOT}|^p \zeta^2 \,\mathrm{dz},$$

where $c_\varepsilon = c_\varepsilon(p, L, 1/\varepsilon)$. To estimate the integral involving the terms of lower order, we first note that $D^k\eta = 0$ on $B_{x_0}(r_1)$ for $k \ge 1$. Due to our boundary condition (2.5) we can extend $u - g$ by zero outside Ω_T to an $L^p - W^{m,p}$ function, i.e. for the extended function we know $u-g\in L^p(0,T;W^{m,p}(B_{x_0}(r_2);\mathbf{R}^N))$. This allows us to replace the domain of integration by $B_{x_0}(r_2)\backslash B_{x_0}(r_1)\times(0,t)$ and then apply the Interpolation-Lemma 3.6 slicewise on the annulus $B_{x_0}(r_2)\backslash B_{x_0}(r_1)$. This yields for $0<\tilde\varepsilon\le 1$ that

$$\int_{\Omega_t}|\mathrm{LOT}|^p\,\zeta^2\,\mathrm{d}z\le c\sum_{k=0}^{m-1}\int_0^t\int_{B_{x_0}(r_2)\backslash B_{x_0}(r_1)}\frac{|D^k(u-g)|^p}{(r_2-r_1)^{p(m-k)}}\,\zeta^2\,\mathrm{d}z$$
$$\le\tilde\varepsilon\int_0^t\int_{B_{x_0}(r_2)\backslash B_{x_0}(r_1)}|D^m(u-g)|^p\zeta^2\,\mathrm{d}z+c_{\tilde\varepsilon}\int_0^t\int_{B_{x_0}(r_2)}\frac{|u-g|^p}{(r_2-r_1)^{mp}}\zeta^2\,\mathrm{d}z,$$

where $c_{\tilde\varepsilon}=c_{\tilde\varepsilon}(n,m,p,1/\tilde\varepsilon)$.

Combining the previous observations with (4.2), recalling that $\eta=1$ on $B_{x_0}(r_1)$ and choosing $\tilde\varepsilon\ll 1$ with respect to p,L and ε we infer for a.e. $t\in(0,T)$ that

$$\tfrac12\int_{B_{x_0}(r_1)\cap\Omega}|(u-g)(\cdot,t)|^2\,\zeta^2(t)\,\mathrm{d}x+\nu\int_0^t\int_{B_{x_0}(r_1)\cap\Omega}|D^mu|^p\,\zeta^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le 3\varepsilon\int_0^t\int_{B_{x_0}(r_2)\cap\Omega}|D^mu|^p\,\zeta^2\,\mathrm{d}z$$
$$+c\int_{Q_{z_0}(R,S)\cap\Omega_T}\lambda^{p-2}\frac{|u-g|^2}{(r_2-r_1)^{2m}}+\frac{|u-g|^p}{(r_2-r_1)^{mp}}+\widetilde G^p\,\mathrm{d}z,$$

where $c=c(n,m,p,L,1/\varepsilon)$. Now, we choose $\varepsilon=\nu/6$ and multiply with $1/\nu$. Applying Lemma 3.8, we get rid of the term involving $|D^mu|$ on the right-hand side. Then, we take the supremum over $t\in\Lambda_{t_0}(s)\cap(0,T)$ in the first term on the left-hand side of the resulting inequality and choose $t=\min\{t_0+S,T\}$ in the second term. Finally we recall that $\zeta\equiv 1$ on $\Lambda_{t_0}(s)$ to conclude desired Caccioppoli's inequality. □

Next, we derive a Poincaré type inequality for solutions on parabolic cylinders intersecting the lateral boundary $\partial\Omega\times(0,T)$. The capacity density condition and the boundary condition allow us to apply Poincaré's inequality slicewise to $u-g$. Therefore, in contrast to the local situation, we do not need to compare mean value polynomials between different time slices.

Lemma 4.2. *Let u be a global solution according to Definition 2.1 and suppose that $\mathbf{R}^n\setminus\Omega$ is uniformly p-thick. Furthermore, let $Q_{z_0}(\rho,s)\subset\mathbf{R}^{n+1}$ be a parabolic cylinder such that $B_{x_0}(\rho/3)\setminus\Omega\neq\emptyset$. Then there exist $\gamma=\gamma(n,p,\mu)\in(1,p)$ such that for all $0\le k\le m-1$ and $\gamma\le\vartheta\le p$, we have*

$$\int_{Q_{z_0}(\rho,s)\cap\Omega_T}|D^k(u-g)|^\vartheta\,\mathrm{d}z\le c\,\rho^{\vartheta(m-k)}\int_{Q_{z_0}(\rho,s)\cap\Omega_T}|D^mu|^\vartheta+G^\vartheta\,\mathrm{d}z,$$

where $c=c(n,m,N,\mu,\rho_0,\vartheta)$ and G was defined in (2.7).

Proof. Let $\gamma=\gamma(n,p,\mu)\in(1,p)$ be the constant from Theorem 3.4. Then we know that $\mathbf{R}^n\setminus\Omega$ is uniformly γ-thick, and therefore also uniformly ϑ-thick by Lemma 3.3. Then we extend $u-g$ by zero outside of Ω_T, use the same notation for the extension. We fix $k\le j\le m-1$ and $t\in\Lambda(s)\cap(0,T)$ and denote

$$N^j_{B(\rho/2)}=\{x\in\overline B(\rho/2):D^j(u-g)(x,t)=0\}.$$

From Lemma 3.5, we get (here, we consider for the moment the ϑ-quasicontinuous representative of u)

$$\int_{B(\rho)\cap\Omega} |D^j(u-g)(\cdot,t)|^\vartheta \,dx = \int_{B(\rho)} |D^j(u-g)(\cdot,t)|^\vartheta \,dx \leq \frac{c\,\rho^n}{\operatorname{cap}_\vartheta(N^j_{B(\rho/2)}, B(\rho))} \int_{B(\rho)} |D^{j+1}(u-g)(\cdot,t)|^\vartheta \,dx,$$

with $c = c(n,N,\vartheta)$. Since $\mathbf{R}^n \setminus \Omega$ is uniformly ϑ-thick, Lemma 3.2 and (3.1) imply

$$\operatorname{cap}_\vartheta\left(N^j_{B(\rho/2)}, B(\rho)\right) \geq \tilde{\mu}\operatorname{cap}_\vartheta\left(\overline{B}(\rho/2), B(\rho)\right) = c\,\rho^{n-\vartheta}.$$

Note that $\tilde{\mu} = \tilde{\mu}(n,\mu,\rho_0,\vartheta)$. Combining this capacity estimate with the previous one, we conclude

$$\int_{B(\rho)\cap\Omega} |D^j(u-g)(\cdot,t)|^\vartheta \,dx \leq c\,\rho^\vartheta \int_{B(\rho)\cap\Omega} |D^{j+1}(u-g)(\cdot,t)|^\vartheta \,dx,$$

where $c = c(n,N,\mu,\rho_0,\vartheta)$. Integrating with respect to t over $\Lambda(s)\cap(0,T)$ and iterating the resulting estimate for $j = k,\dots,m-1$, we deduce the following Poincaré's inequality

$$\int_{Q(\rho,s)\cap\Omega_T} |D^k(u-g)|^\vartheta \,dz \leq c\,\rho^{\vartheta(m-k)} \int_{Q(\rho,s)\cap\Omega_T} |D^m(u-g)|^\vartheta \,dz.$$

The assertion now follows by Young's inequality and the definition of G. □

Also in the singular case, i.e. when $p<2$, we will have to estimate the L^2-norm of u, since it appears on the right-hand side of Caccioppoli's inequality in Lemma 4.1. Therefore, we prove a suitable L^2-estimate in the following lemma. This lemma simplifies the proof in the singular case considerably since we absorb the additional terms into the left-hand side. Indeed, due to this lemma, we can apply the same scaling as in the degenerate case.

Lemma 4.3. *Let $\kappa \geq 1$, $2_* < p < 2$, and u be a global solution according to Definition 2.1. Furthermore, let $Q = Q_{z_0}(\rho,s) \subset \mathbf{R}^{n+1}$ with $0<\rho\leq 1$, $s = \lambda^{2-p}\rho^{2m} \leq 1$, and $\lambda > 0$ be a parabolic cylinder such that $B_{x_0}(2\rho/3)\setminus\Omega \neq \emptyset$. If*

$$\frac{1}{|2Q|}\int_{2Q\cap\Omega_T} \left(|D^m u|^p + \widetilde{G}^p\right) dz \leq \kappa\,\lambda^p, \tag{4.3}$$

then there exists $c = c(n,N,m,p,L/\nu,\mu,\rho_0,\kappa)$ such that

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T} |u-g|^2 \,dz \leq c\,\rho^{2m}\lambda^p.$$

Proof. We first extend $u-g$ by zero outside of Ω_T. Next, we choose $1 \leq \alpha_1 < \alpha_2 \leq 2$ and denote $\alpha_i Q = Q_{z_0}(\alpha_i\rho, \alpha_i^{2m}s)$ for $i = 1,2$. Applying Gagliardo-Nirenberg-Sobolev's inequality, i.e. Theorem 3.7 with (σ,q,θ,r,k) replaced by $(2,p,p/2,2,0)$ slicewise to $(u-g)(\cdot,t)$, we obtain

$$\begin{aligned}\int_{\alpha_1 Q\cap\Omega_T} |u-g|^2 \,dz &= \int_{\alpha_1 Q} |u-g|^2 \,dz \\ &\leq c\,\rho^{mp}\int_{\alpha_1\Lambda}\sum_{k=0}^m \int_{\alpha_1 B}\left|\frac{D^k(u-g)}{\rho^{m-k}}\right|^p dx \left(\fint_{\alpha_1 B}|u-g|^2\,dx\right)^{(2-p)/2} dt \\ &\leq c\,\rho^{mp}\sum_{k=0}^m \int_{\alpha_1 Q}\left|\frac{D^k(u-g)}{\rho^{m-k}}\right|^p dz \left(\sup_{t\in\alpha_1\Lambda}\fint_{\alpha_1 B}|(u-g)(\cdot,t)|^2\,dx\right)^{(2-p)/2}. \end{aligned} \tag{4.4}$$

To estimate the integrals in the sum on the right-hand side, we recall that $\alpha_1 Q \subset 2Q$ and $u-g=0$ on $2Q\setminus\Omega_T$. Therefore, we can replace the domain of integration by $2Q\cap\Omega_T$ which allows us to apply Poincaré's inequality from Lemma 4.2 on $2Q\cap\Omega_T$. Finally, using hypothesis (4.3) and the fact that $|2Q|=2^{n+2m}|Q|$ we infer for $0\le k\le m$ that

$$\int_{\alpha_1 Q}\Big|\frac{D^k(u-g)}{\rho^{m-k}}\Big|^p\,\mathrm{d}z \le c\int_{2Q\cap\Omega_T}|D^m u|^p+G^p\,\mathrm{d}z\le c\,\lambda^p\,|Q|,$$

where $c=c(n,m,N,\mu,\rho_0,\kappa)$. We now come to the estimate for the sup-term in (4.4). Here, we first apply Caccioppoli's inequality, i.e. Lemma 4.1. Then we use Young's inequality, note that $p<2$, to estimate the second term on the right-hand side and the assumption (4.3) to estimate the term involving $\widetilde{G}$. Finally, we recall that $|Q|=2\rho^{2m}\lambda^{2-p}|B|$. Proceeding this way, we obtain for a.e. $t\in\alpha_1\Lambda$ that

$$\begin{aligned}
⨍_{\alpha_1 B}|(u-g)(\cdot,t)|^2\,\mathrm{d}x &= \frac{1}{\alpha_1^n|B|}\int_{\alpha_1 B\cap\Omega}|(u-g)(\cdot,t)|^2\,\mathrm{d}x\\
&\le \frac{c_{Cac}}{|B|}\int_{\alpha_2 Q\cap\Omega_T}\lambda^{p-2}\frac{|u-g|^2}{(\alpha_2\rho-\alpha_1\rho)^{2m}}+\frac{|u-g|^p}{(\alpha_2\rho-\alpha_1\rho)^{mp}}+\widetilde{G}^p\,\mathrm{d}z\\
&\le \frac{c}{|B|}\int_{\alpha_2 Q\cap\Omega_T}\lambda^{p-2}\frac{|u-g|^2}{(\alpha_2\rho-\alpha_1\rho)^{2m}}+\lambda^p\,\mathrm{d}z\\
&= \frac{c}{|Q|}\int_{\alpha_2 Q\cap\Omega_T}\frac{|u-g|^2}{(\alpha_2-\alpha_1)^{2m}}\,\mathrm{d}z+c\,\rho^{2m}\lambda^2,
\end{aligned}$$

where $c=c(n,m,p,L/\nu,\kappa)$. Joining the previous estimates with (4.4), applying Young's inequality and recalling that $s=\lambda^{2-p}\rho^{2m}$, we arrive at

$$\int_{\alpha_1 Q\cap\Omega_T}|u-g|^2\,\mathrm{d}z\le\tfrac12\int_{\alpha_2 Q\cap\Omega_T}|u-g|^2\,\mathrm{d}z+\frac{c}{(\alpha_2-\alpha_1)^{2m(2-p)/p}}\,\rho^{2m}\,|Q|\,\lambda^p,$$

where $c=c(n,N,m,p,L/\nu,\mu,\rho_0,\kappa)$. Applying Lemma 3.8, we deduce the desired estimate. □

Now, we have the prerequisites to prove a reverse Hölder type inequality for parabolic cylinders lying near the lateral boundary.

Lemma 4.4. *Let $\kappa\ge1$, and u be a global solution according to Definition 2.1 and suppose that $\mathbf{R}^n\setminus\Omega$ is uniformly p-thick. Furthermore, let $Q=Q_{z_0}(\rho,s)\subset\mathbf{R}^{n+1}$ with $0<\rho\le1$ and $s=\lambda^{2-p}\rho^{2m}\le1$, $\lambda\ge1$ be a parabolic cylinder such that $B_{x_0}(4\rho/3)\setminus\Omega\neq\emptyset$. Suppose that*

$$\lambda^p\le\frac{\kappa}{|Q|}\int_{Q\cap\Omega_T}(|D^m u|^p+\widetilde{G}^p)\,\mathrm{d}z,\qquad \frac{1}{|8Q|}\int_{8Q\cap\Omega_T}(|D^m u|^p+\widetilde{G}^p)\,\mathrm{d}z\le\kappa\lambda^p. \tag{4.5}$$

and let $\gamma=\gamma(n,p,\mu)$ be the constant from Lemma 4.2. Then, for any q with $\max\{\gamma,\widehat{p}_*\}\le q<p$ there exists $c=c(n,N,m,p,L/\nu,\mu,\rho_0,\kappa)$ such that

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}|D^m u|^p\,\mathrm{d}z\le\left(\frac{c}{|4Q|}\int_{4Q\cap\Omega_T}|D^m u|^q\,\mathrm{d}z\right)^{p/q}+\frac{c}{|4Q|}\int_{4Q\cap\Omega_T}\widetilde{G}^p\,\mathrm{d}z.$$

Proof. First, we extend $u-g$ by zero outside of Ω_T and use the same notation for the extension. From Caccioppoli's inequality, i.e. Lemma 4.1, we get

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}|D^m u|^p\,\mathrm{d}z\le\frac{c_{Cac}}{|Q|}\int_{2Q\cap\Omega_T}\lambda^{p-2}\Big|\frac{u-g}{\rho^m}\Big|^2+\Big|\frac{u-g}{\rho^m}\Big|^p+\widetilde{G}^p\,\mathrm{d}z. \tag{4.6}$$

In the following, we will infer bounds for the first two terms on the right-hand side. Therefore, we abbreviate (note that $u-g=0$ on $2Q\setminus\Omega_T$)

$$I_\sigma = \lambda^{p-\sigma} \fint_{2Q} \Big|\frac{u-g}{\rho^m}\Big|^\sigma \mathrm{d}z$$

for $\sigma=2$ or $\sigma=p$. First, observe that $(u-g)(\cdot,t)\in W^{m,p}(2B;\mathbf{R}^N)$ when extended by zero on $2B\setminus\Omega$. Now we fix $q\in[max\{\gamma,\widehat{p}_*\},p)$, apply Gagliardo-Nirenberg-Sobolev's inequality, i.e. Theorem 3.7 in the case $r=2$ and $\theta=q/\sigma$ slicewise to $(u-g)(\cdot,t)$ and take the supremum over $t\in 2\Lambda$ in the second integral to infer

$$I_\sigma \le c(n,m,p)\,\lambda^{p-\sigma}\sum_{k=0}^{m}\fint_{2Q}\Big|\frac{D^k(u-g)}{\rho^{m-k}}\Big|^q \mathrm{d}z\,\cdot J^{(\sigma-q)/2}, \tag{4.7}$$

where

$$J=\sup_{t\in 2\Lambda}\fint_{2B}\Big|\frac{(u-g)(\cdot,t)}{\rho^m}\Big|^2 \mathrm{d}x.$$

Let us first observe that we can replace the domain of integration in the above integrals by $2Q\cap\Omega$, respectively $2B\cap\Omega$, since $u-g=0$ on the set $2Q\setminus\Omega_T$. We first consider the sum of integrals on the right-hand side of (4.7). Here, we apply Poincaré's inequality from Lemma 4.2 to find for $0\le k\le m$ that

$$\fint_{2Q}\Big|\frac{D^k(u-g)}{\rho^{m-k}}\Big|^q \mathrm{d}z \le \frac{c}{|4Q|}\int_{4Q\cap\Omega_T}\Big|\frac{D^k(u-g)}{\rho^{m-k}}\Big|^q \mathrm{d}z \le \frac{c}{|4Q|}\int_{4Q\cap\Omega_T}|D^m u|^q+G^q\,\mathrm{d}z,$$

where $c=c(\mu,\rho_0,n,m,N,\kappa)$. Next, we derive an estimate for J. Using $|Q|=2\lambda^{2-p}\rho^{2m}|B|$ and applying Caccioppoli's inequality, Lemma 4.1, we get

$$J\le\frac{c_{Cac}}{|Q|}\int_{4Q\cap\Omega_T}\Big|\frac{u-g}{\rho^m}\Big|^2+\lambda^{2-p}\Big|\frac{u-g}{\rho^m}\Big|^p+\lambda^{2-p}\widetilde{G}^p\,\mathrm{d}z.$$

Our aim is to bound the right-hand side in terms of λ^2. For the first term we either apply Lemma 4.3 when $p<2$, which is applicable due to the second inequality in (4.5), or when $p\ge 2$, we in turn apply Poincaré's inequality from Lemma 4.2, Hölder's inequality and the second inequality in (4.5). To estimate the second term on the right-hand side, we use Poincaré's inequality from Lemma 4.2 and the second inequality in (4.5) in any case. Finally, the term involving $\widetilde{G}^p$ is estimated in terms of λ^p also due to our assumption (4.5). Observe that here we utilize the fact that the scaling also takes the boundary terms into account. Proceeding this way we find that

$$J\le c\,\lambda^2.$$

Combining this and the second last estimate with (4.7) and applying Young's inequality, we obtain for $\varepsilon>0$ that

$$\begin{aligned}I_\sigma &\le c\,\lambda^{p-\sigma}\frac{1}{|4Q|}\int_{4Q\cap\Omega_T}(|D^m u|^q+G^q)\,\mathrm{d}z\cdot\lambda^{\sigma-q}\\ &\le \varepsilon\,\lambda^p+\bigg(\frac{c_\varepsilon}{|4Q|}\int_{4Q\cap\Omega_T}|D^m u|^q\,\mathrm{d}z\bigg)^{p/q}+\frac{c_\varepsilon}{|4Q|}\int_{4Q\cap\Omega_T}G^p\,\mathrm{d}z,\end{aligned}$$

where $c_\varepsilon=c_\varepsilon(n,N,m,p,L/\nu,\mu,\rho_0,\kappa,1/\varepsilon)$. Inserting this estimate for $\sigma=2$ and $\sigma=p$ in (4.6), we arrive at

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}|D^m u|^p\,\mathrm{d}z$$

$$\leq 2c_{Cac}\,\varepsilon\,\lambda^p + \bigg(\frac{c_\varepsilon}{|4Q|}\int_{4Q\cap\Omega_T}|D^m u|^q\,\mathrm{d}z\bigg)^{p/q} + \frac{c_\varepsilon}{|4Q|}\int_{4Q\cap\Omega_T}\widetilde{G}^p\,\mathrm{d}z,$$

where $c_\varepsilon = c_\varepsilon(n,N,m,p,L/\nu,\mu,\rho_0,\kappa,1/\varepsilon)$. Now we use the first inequality in (4.5) to bound λ^p in terms of the integral on the left-hand side of the preceding inequality. Choosing ε small enough, we can absorb this term on the left. Proceeding this way, we deduce the desired reverse Hölder inequality. □

5. ESTIMATES NEAR THE INITIAL BOUNDARY

In this chapter, we are concerned with parabolic cylinders lying near the initial boundary $\Omega\times\{0\}$. We shall use the following abbreviation for the initial values

$$g_0(x) = g(x,0) \qquad \text{for } x\in\Omega.$$

Due to our assumptions the initial values are well defined. Moreover, given a ball $B_{x_0}(r)$ in $\mathbf{R}^n$, we denote by $P_r^{(g_0)}\colon \mathbf{R}^n\to\mathbf{R}^N$ the mean value polynomial of g_0 of degree $\leq m-1$ on $B_{x_0}(r)$ defined by $(\delta P_r^{(g_0)})_{x_0;r} = (\delta g_0)_{x_0;r}$, as introduced in Section 3.3. As usual, we first prove suitable Caccioppoli's inequality for parabolic cylinders lying near the initial boundary.

Lemma 5.1. *Suppose that u is a global solution according to Definition 2.1. Then there exists $c_{Cac} = c_{Cac}(n,m,p,L/\nu)$ such that for all parabolic cylinders $Q_{z_0}(r,s), Q_{z_0}(R,S)\subset\mathbf{R}^{n+1}$ with $0<R/2\leq r<R\leq 1$, $s=\lambda^{2-p}r^{2m}$, $S=\lambda^{2-p}R^{2m}$, $\lambda\geq 1$ satisfying $B_{x_0}(R)\subset\Omega$ and $0\in\Lambda_{t_0}(S)$ there holds*

$$\begin{aligned}\sup_{t\in\Lambda_{t_0}(s)\cap(0,T)}\int_{B_{x_0}(r)}|(u-P_R^{(g_0)})(\cdot,t)|^2\,\mathrm{d}x + \int_{Q_{z_0}(r,s)\cap\Omega_T}|D^m u|^p\,\mathrm{d}z \\ \leq c_{Cac}\int_{Q_{z_0}(R,S)\cap\Omega_T}\bigg(\lambda^{p-2}\Big|\frac{u-P_R^{(g_0)}}{(R-r)^m}\Big|^2 + \Big|\frac{u-P_R^{(g_0)}}{(R-r)^m}\Big|^p + 1\bigg)\,\mathrm{d}z \\ + c_{Cac}\,R^{n+2m}\bigg(⨍_{B_{x_0}(R)}|D^m g_0|^{2_*}\,\mathrm{d}x\bigg)^{2/2_*}.\end{aligned}$$

Proof. Since the proof is very much similar to the one of Lemma 4.1, we only point out the differences. We again start with the Steklov-formulation (4.2) of the parabolic system. But now we take $\varphi_h = \eta\zeta^2(u_h - P_R^{(g_0)})$ as test-function with cut-off functions η,ζ as in (4.1). The main difference compared to the proof of Lemma 4.1 is related to the first term on the left-hand side of (4.2). Therefore, we shall only accomplish how to treat this term. Taking into account that $\partial_t P_R^{(g_0)} = 0$ and the initial condition (2.6), we find in the limit $h\downarrow 0$

$$\begin{aligned}&\int_{\Omega_t}\partial_\tau u_h\cdot\varphi_h\,\mathrm{d}z = \int_{\Omega_t}\partial_\tau(u_h-P_R^{(g_0)})\cdot\varphi_h\,\mathrm{d}z\\ &\to \tfrac12\int_\Omega |(u-P_R^{(g_0)})(\cdot,t)|^2\eta\zeta(t)^2 - |g_0-P_R^{(g_0)}|^2\eta\zeta(0)^2\,\mathrm{d}x - \int_{\Omega_t}|u-P_R^{(g_0)}|^2\eta\zeta\zeta'\,\mathrm{d}z.\end{aligned}$$

To estimate the second integral on the right-hand side we iterate Sobolev-Poincaré's inequality (recall that $\operatorname{spt}\eta\subset B(R)$ and $\eta,\zeta\leq 1$, that $(D^k(g_0 - P_R^{(g_0)}))_R = 0$ for $0\leq k\leq m-1$ by the definition of $P_R^{(g_0)}$ and that $D^m P_R^{(g_0)} = 0$)

$$\int_\Omega |g_0-P_R^{(g_0)}|^2\eta\zeta(0)^2\,\mathrm{d}x \leq c\,|B(R)|\,R^2\bigg(⨍_{B(R)}|D(g_0-P_R^{(g_0)})|^{2n/(n+2)}\,\mathrm{d}x\bigg)^{(n+2)/n}$$

$$\vdots$$

$$\leq c(n,m)\,|B(R)|\,R^{2m}\Big(\fint_{B(R)}|D^m g_0|^{2_*}\,\mathrm{d}x\Big)^{2/2_*}.$$

The remaining terms in (4.2) are estimated similarly as in the proof of Lemma 4.1 with $P_R^{(g_0)}$ instead of g (note again that $D^m P_R^{(g_0)}=0$). This leads us to the desired Caccioppoli's inequality. □

In our notion of a global weak solution we did not impose any differentiability assumption with respect to time. Therefore, we cannot apply usual Poincaré's inequality. Nevertheless, we can exploit the parabolic system to gain the needed regularity with respect to time. Indeed, in the next lemma, we will show that the weighted means $(D^k u)_\eta(t)$ of $D^k u(\cdot,t)$ - defined below - possess an absolutely continuous representative. This is first deduced for the weighted means of u by using the system. The result then extends to the weighted means of derivatives $D^k u$ with integration by parts.

We say $\eta\in C_0^\infty(B_{x_0}(\rho))$ is a nonnegative weight-function on $B_{x_0}(\rho)\subset\mathbf{R}^n$, if

$$\eta\geq 0,\quad \fint_{B_{x_0}(\rho)}\eta\,\mathrm{d}x=1\ \text{and}\quad \|D^\ell\eta\|_\infty\leq c_\eta/\rho^\ell\ \text{for}\ \ 0\leq\ell\leq 2m. \tag{5.1}$$

Note that the smallest possible value of c_η depends on n and m. Let $Q_{z_0}(\rho,s)\subset\mathbf{R}^{n+1}$ be a parabolic cylinder and $v\in L^1(Q_{z_0}(\rho,s);\mathbf{R}^k)$, $k\in\mathbb{N}$. Then we define the weighted mean of $v(\cdot,t)$ on $B_{x_0}(\rho)$ for a.e. $t\in\Lambda_{t_0}(s)$ by

$$(v)_\eta(t)=\fint_{B_{x_0}(\rho)}v(\cdot,t)\,\eta\,\mathrm{d}x. \tag{5.2}$$

Lemma 5.2. *Suppose that u is a global solution according to Definition 2.1 and let $Q_{z_0}(\rho,s)\subset\mathbf{R}^{n+1}$ be a parabolic cylinder with $0<\rho\leq 1$, $s>0$ and $B_{x_0}(\rho)\subset\Omega$. Let $\eta\in C_0^\infty(B_{x_0}(\rho))$ be a nonnegative weight-function satisfying (5.1). Then there exists $c=c(N,L,c_\eta)$ such that for the weighted means of $D^k u$, $0\leq k\leq m-1$, and a.e. $t_1,t_2\in\Lambda_{t_0}(s)\cap(0,T)$, there holds*

$$|(D^k u)_\eta(t_2)-(D^k u)_\eta(t_1)|\leq\frac{c}{\rho^{m+k}}\int_{t_1}^{t_2}\fint_{B_{x_0}(\rho)}\big(|D^m u|^{p-1}+1\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Proof. Let $i\in\{1,\dots N\}$ and η be as above and choose $\varphi\colon\mathbf{R}^{n+1}\to\mathbf{R}^N$ with $\varphi_i=\eta$, $\varphi_j\equiv 0$ for $j\neq i$ as a test-function in the Steklov-formulation (3.4). For a.e. $t_1,t_2\in\Lambda_{t_0}(s)\cap(0,T)$, we get

$$\begin{aligned}\big([u_i]_h\big)_\eta(t_2)-\big([u_i]_h\big)_\eta(t_1)&=\int_{t_1}^{t_2}\partial_t\big([u_i]_h\big)_\eta\,\mathrm{d}t\\&=-\int_{t_1}^{t_2}\fint_{B_{x_0}(\rho)}\big\langle[\mathscr{A}_i(\cdot,D^m u)]_h,D^m\eta\big\rangle\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\mathscr{A}_i\colon\Omega_T\times\mathbf{R}^\ell\to\mathbf{R}^{\ell/N}$ denotes a component of $\mathscr{A}$. To infer the assertion for the case $k=0$, we use the growth conditions (2.3) for $\mathscr{A}$ and the fact that $\|D^m\eta\|_\infty\leq c/\rho^m$. After passing to the limit $h\downarrow 0$ and summing over $i=1,\dots,N$ we obtain

$$|(u)_\eta(t_2)-(u)_\eta(t_1)|\leq\frac{cL}{\rho^m}\int_{t_1}^{t_2}\fint_{B_{x_0}(\rho)}\big(|D^m u|^{p-1}+1\big)\,\mathrm{d}x\,\mathrm{d}t.$$

For the general case, we consider a multi-index α of order k and obtain with integration by parts

$$(D^\alpha u)_\eta(t) = \fint_{B_{x_0}(\rho)} D^\alpha u(\cdot,t)\eta\,\mathrm{d}x = (-1)^k \fint_{B_{x_0}(\rho)} u(\cdot,t)D^\alpha\eta\,\mathrm{d}x = (-1)^k (u)_{D^\alpha\eta}(t).$$

Therefore the asserted estimate follows from the case $k=0$ by exchanging η with $D^\alpha\eta$ and summing over $|\alpha| = k$. □

Since we have achieved some regularity with respect to time of our solution u, we are now in a position to prove a Poincaré type inequality.

Lemma 5.3. *Suppose that u is a global solution according to Definition 2.1 and let $Q = Q_{z_0}(\rho,s) \subset \mathbf{R}^{n+1}$ be a parabolic cylinder with $0<\rho\le 1$ and $s = \lambda^{2-p}\rho^{2m}$, $\lambda > 1$ and $B = B_{x_0}(\rho)\subset\Omega$ and $0\in\Lambda_{t_0}(s)$. Then for all $0\le k\le m-1$ and $1\le\vartheta\le p$ there exists $c = c(n,N,m,L,\vartheta)$ such that*

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}\Big|\frac{D^k(u-P_\rho^{(g_0)})}{\rho^{m-k}}\Big|^\vartheta \mathrm{d}z \le c\bigg[\frac{1}{|Q|}\int_{Q\cap\Omega_T}|D^m u|^\vartheta\,\mathrm{d}z + \Big(\frac{\lambda^{2-p}}{|Q|}\int_{Q\cap\Omega_T}(|D^m u|^{p-1}+1)\,\mathrm{d}z + \fint_B |D^m g_0|\,\mathrm{d}x\Big)^\vartheta\bigg].$$

Proof. Let $\eta\in C_0^\infty(B)$ be a nonnegative weight-function satisfying (5.1). For $k\le j\le m-1$ and a.e. $t\in\Lambda\cap(0,T)$ and $0<h<t_0+s$, we decompose

$$\begin{aligned}\fint_B |D^j(u(\cdot,t)-P_\rho^{(g_0)})|^\vartheta\,\mathrm{d}x &\le 4^\vartheta\bigg[\fint_B \big|D^j(u(\cdot,t)-P_\rho^{(g_0)}) - \big(D^j(u(\cdot,t)-P_\rho^{(g_0)})\big)_\eta\big|^\vartheta\,\mathrm{d}x \\ &\quad + \Big|\fint_0^h (D^j u)_\eta(t) - (D^j u)_\eta(\tau)\,\mathrm{d}\tau\Big|^\vartheta \\ &\quad + \Big|\fint_0^h (D^j u)_\eta(\tau) - (D^j g_0)_\eta\,\mathrm{d}\tau\Big|^\vartheta \\ &\quad + |(D^j g_0)_\eta - (D^j P_\rho^{(g_0)})_\eta|^\vartheta\bigg] \\ &= 4^\vartheta\big(I(t)+II(t)+III+IV\big). \end{aligned} \tag{5.3}$$

To estimate $I(t)$, we apply Poincaré's inequality slicewise to $D^j(u-P_\rho^{(g_0)})(\cdot,t)$ and find for a.e. $t\in\Lambda\cap(0,T)$

$$I(t)\le c(n,\vartheta)\,\rho^\vartheta \fint_B |D^{j+1}(u(\cdot,t)-P_\rho^{(g_0)})|^\vartheta\,\mathrm{d}x.$$

To estimate $II(t)$, we use Lemma 5.2 (note that $|Q| = 2\rho^{2m}\lambda^{2-p}|B|$). It implies for a.e. $t\in\Lambda\cap(0,T)$ that

$$\begin{aligned} II(t) &\le \operatorname*{ess\,sup}_{t_1,t_2\in\Lambda\cap(0,T)} |(D^j u)_\eta(t_1) - (D^j u)_\eta(t_2)|^\vartheta \\ &\le c\,\rho^{\vartheta(m-j)}\Big(\frac{\lambda^{2-p}}{|Q|}\int_{Q\cap\Omega_T}(|D^m u|^{p-1}+1)\,\mathrm{d}z\Big)^\vartheta. \end{aligned}$$

Note that the previous bound is independent of h. To estimate III, we first consider a multi-index α of order j. With integration by parts, we obtain

$$\Big|\fint_0^h (D_\alpha u)_\eta(\tau) - (D_\alpha g_0)_\eta\,\mathrm{d}\tau\Big| = \Big|\fint_0^h \fint_B D_\alpha(u(\cdot,\tau)-g_0)\eta\,\mathrm{d}x\,\mathrm{d}\tau\Big|$$

$$= \Big| \fint_0^h \fint_B (u(\cdot,\tau)-g_0) D_\alpha \eta \,\mathrm{d}x\,\mathrm{d}\tau \Big|$$
$$\le \frac{c}{\rho^j} \fint_0^h \fint_B |u(\cdot,\tau)-g_0| \,\mathrm{d}x\,\mathrm{d}\tau \to 0$$

as $h \downarrow 0$ by our initial condition (2.6). Observe that the weight function helped us to complete the previous step. Summing over all indices α of order j, we therefore conclude that $III \to 0$ as $h \downarrow 0$. Finally, to estimate IV, we recall that $\eta \le c(n,m)$ and apply Poincaré's inequality $m-j$ times to $D^j(g_0 - P_\rho^{(g_0)})$ (note that $(D^\ell(g_0 - P_\rho^{(g_0)}))_B = 0$ for $\ell = j, \dots, m-1$ and $D^m P_\rho^{(g_0)} = 0$) to infer

$$IV \le \Big(\fint_B |D^j(g_0 - P_\rho^{(g_0)})| \,\mathrm{d}x \Big)^\vartheta \le \dots \le c\,\rho^{\vartheta(m-j)} \Big(\fint_B |D^m g_0| \,\mathrm{d}x \Big)^\vartheta .$$

Combining the previous estimates for $I(t) - IV$ with (5.3), passing to the limit $h \downarrow 0$ and integrating with respect to t over $\Lambda \cap (0,T)$, we get for $k \le j \le m-1$

$$\frac{1}{|Q|} \int_{Q\cap\Omega_T} |D^j(u - P_\rho^{(g_0)})|^\vartheta \,\mathrm{d}z \le \frac{c\,\rho^\vartheta}{|Q|} \int_{Q\cap\Omega_T} |D^{j+1}(u - P_\rho^{(g_0)})|^\vartheta \,\mathrm{d}z$$
$$+ c\,\rho^{\vartheta(m-j)} \Big(\frac{\lambda^{2-p}}{|Q|} \int_{Q\cap\Omega_T} (|D^m u|^{p-1} + 1) dz + \fint_B |D^m g_0| \,\mathrm{d}x \Big)^\vartheta ,$$

where $c = c(n,N,m,L,\vartheta)$. Iterating this estimate for $j = k, \dots, m-1$, we find

$$\frac{1}{|Q|} \int_{Q\cap\Omega_T} |D^k(u - P_\rho^{(g_0)})|^\vartheta \,\mathrm{d}z$$
$$\le \frac{c\,\rho^\vartheta}{|Q|} \int_{Q\cap\Omega_T} |D^{k+1}(u - P_\rho^{(g_0)})|^\vartheta \,\mathrm{d}z + c\,\rho^{\vartheta(m-k)} \Big(\dots \Big)^\vartheta$$
$$\le \frac{c\,\rho^{2\vartheta}}{|Q|} \int_{Q\cap\Omega_T} |D^{k+2}(u - P_\rho^{(g_0)})|^\vartheta \,\mathrm{d}z + c\,\rho^{\vartheta(m-k)} \Big(\dots \Big)^\vartheta$$
$$\vdots$$
$$\le \frac{c\,\rho^{\vartheta(m-k)}}{|Q|} \int_{Q\cap\Omega_T} |D^m u|^\vartheta \,\mathrm{d}z + c\,\rho^{\vartheta(m-k)} \Big(\dots \Big)^\vartheta ,$$

with the obvious meaning of $(\dots)^\vartheta$. This proves the asserted Poincaré type inequality. □

The integral $(\int |D^m u|^{p-1} \,\mathrm{d}z)^\vartheta$ on the right-hand side of the previous Poincaré type inequality has the "wrong exponent". Therefore, we shall exploit the intrinsic scaling of the cylinders, which depends via hypothesis (5.4) on the solution itself. This will help us to "compensate" the degeneracy.

Corollary 5.4. *Let $\kappa \ge 1$ and u be a global solution according to Definition 2.1. Furthermore, let $Q = Q_{z_0}(\rho,s) \subset \mathbf{R}^{n+1}$ be a parabolic cylinder with $0 < \rho \le 1$, $\lambda > 0$ and $s = \lambda^{2-p}\rho^{2m}$ and $B = B_{x_0}(\rho) \subset \Omega$ and $0 \in \Lambda_{t_0}(s)$. Suppose that*

$$\frac{1}{|Q|} \int_{Q\cap\Omega_T} (|D^m u|^p + 1) \,\mathrm{d}z \le \kappa \lambda^p . \tag{5.4}$$

Then there exists $c = c(n,N,m,L,\vartheta,\kappa)$ such that for all $0 \le k \le m-1$ and $1 \le \vartheta \le p$, we have

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}\Big|\frac{D^k(u-P_\rho^{(g_0)})}{\rho^{m-k}}\Big|^\vartheta \,\mathrm{d}z \le c\,\left(\lambda + \fint_B |D^m g_0|\,\mathrm{d}x\right)^\vartheta.$$

Proof. We first apply Poincaré's inequality from Lemma 5.3 (note that $s/\rho^{2m} = \lambda^{2-p}$). Then we use Hölder's inequality and hypothesis (5.4) to infer

$$\begin{aligned}
&\frac{1}{|Q|}\int_{Q\cap\Omega_T}\Big|\frac{D^k(u-P_\rho^{(g_0)})}{\rho^{m-k}}\Big|^\vartheta \,\mathrm{d}z \\
&\quad\le c\left[\frac{1}{|Q|}\int_{Q\cap\Omega_T}|D^m u|^\vartheta\,\mathrm{d}z + \Big(\frac{\lambda^{2-p}}{|Q|}\int_{Q\cap\Omega_T}(|D^m u|^{p-1}+1)\,\mathrm{d}z + \fint_B |D^m g_0|\,\mathrm{d}x\Big)^\vartheta\right] \\
&\quad\le c\left[\lambda^\vartheta + \Big(\lambda^{2-p}\lambda^{p-1} + \fint_B |D^m g_0|\,\mathrm{d}x\Big)^\vartheta\right] \\
&\quad= c\Big(\lambda + \fint_B |D^m g_0|\,\mathrm{d}x\Big)^\vartheta,
\end{aligned}$$

where $c = c(n,N,m,L,\vartheta,\kappa)$. This proves the desired estimate. □

The next lemma is an analogue of Lemma 4.3 for parabolic cylinders near the initial boundary. Later, in the proof of the reverse Hölder inequality it will help us to bound the L^2-norm of u in the case $p<2$.

Lemma 5.5. *Let $\kappa \ge 1$, $2_* < p < 2$ and let u be a global solution according to Definition 2.1. Furthermore, let $Q = Q_{z_0}(\rho,s) \subset \mathbf{R}^{n+1}$ be a parabolic cylinder with $0<\rho\le 1$, $\lambda > 0$ and $s = \lambda^{2-p}\rho^{2m}$ and $2B = B_{x_0}(2\rho) \subset \Omega$ and $0 \in \Lambda_{t_0}(s)$. If*

$$\frac{1}{|2Q|}\int_{2Q\cap\Omega_T}(|D^m u|^p + 1)\,\mathrm{d}z \le \kappa\lambda^p, \tag{5.5}$$

then there exists $c = c(n,N,m,p,L/\nu,\kappa)$ such that

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}|u-P_\rho^{(g_0)}|^2\,\mathrm{d}z \le c\,\rho^{2m}\left(\lambda^{2_*} + \fint_{2B}|D^m g_0|^{2_*}\,\mathrm{d}x\right)^{2/2_*}.$$

Proof. Let $1 \le \alpha_1 < \alpha_2 \le 2$. Applying Gagliardo-Nirenberg-Sobolev's inequality, i.e. Theorem 3.7 with (σ,q,θ,r,k) replaced by $(2,p,p/2,2,0)$ slicewise to $(u - P_{\alpha_1\rho}^{(g_0)})(\cdot,t)$, we obtain

$$\int_{\alpha_1 Q\cap\Omega_T}|u-P_{\alpha_1\rho}^{(g_0)}|^2\,\mathrm{d}z \le c\,\rho^{mp}\sum_{k=0}^m\int_{\alpha_1 Q\cap\Omega_T}\Big|\frac{D^k(u-P_{\alpha_1\rho}^{(g_0)})}{\rho^{m-k}}\Big|^p\,\mathrm{d}z\cdot J^{(2-p)/2}, \tag{5.6}$$

where

$$J = \sup_{t\in\alpha_1\Lambda\cap(0,T)}\fint_{\alpha_1 B}|u(\cdot,t) - P_{\alpha_1\rho}^{(g_0)}|^2\,\mathrm{d}x.$$

We first estimate the integrals in the sum on the right-hand side of (5.6). For this aim we apply Corollary 5.4 on $\alpha_1 Q$ (note that the hypothesis (5.4) follows from (5.5) since $\alpha_1 Q \subset 2Q$ and $|2Q|/|\alpha_1 Q| \le 2^{n+2m}$) yielding that

$$\frac{1}{|Q|}\int_{\alpha_1 Q\cap\Omega_T}\Big|\frac{D^k(u-P_{\alpha_1\rho}^{(g_0)})}{\rho^{m-k}}\Big|^p\,\mathrm{d}z \le c\,\left(\lambda + \fint_{\alpha_1 B}|D^m g_0|\,\mathrm{d}x\right)^p. \tag{5.7}$$

We now come to the estimate of J. Lemma 3.9 provides the following estimate for difference of the mean value polynomials on $\alpha_1 Q$ and $\alpha_2 Q$:

$$|P^{(g_0)}_{\alpha_1\rho}(x) - P^{(g_0)}_{\alpha_2\rho}(x)| \le c\,\rho^m \fint_{\alpha_1 B} |D^m g_0|\,\mathrm{d}x \le c\,\rho^m \Big(\fint_{2B} |D^m g_0|^{2_*}\,\mathrm{d}x\Big)^{1/2_*},$$

for all $x \in \alpha_2 B$ and thus we can use the Caccioppoli inequality, Lemma 5.1 to obtain the following estimate for J:

$$\begin{aligned} J &\le \frac{c}{\alpha_1^n |B|} \int_{\alpha_2 Q\cap\Omega_T} \bigg(\lambda^{p-2} \frac{|u-P^{(g_0)}_{\alpha_2\rho}|^2}{(\alpha_2\rho-\alpha_1\rho)^{2m}} + \frac{|u-P^{(g_0)}_{\alpha_2\rho}|^p}{(\alpha_2\rho-\alpha_1\rho)^{mp}} + 1\bigg)\,\mathrm{d}z \\ &\qquad\qquad + c\,\rho^{2m}\Big(\fint_{2B} |D^m g_0|^{2_*}\,\mathrm{d}x\Big)^{2/2_*} \\ &\le \frac{c}{|Q|}\int_{\alpha_2 Q\cap\Omega_T} \frac{|u-P^{(g_0)}_{\alpha_2\rho}|^2}{(\alpha_2-\alpha_1)^{2m}}\,\mathrm{d}z + c\,\rho^{2m}\Big(\lambda^{2_*} + \fint_{2B}|D^m g_0|^{2_*}\,\mathrm{d}x\Big)^{2/2_*}. \end{aligned}$$

Here, in the last line, we have used Young's inequality and the fact that $s = \lambda^{2-p}\rho^{2m}$ and $\lambda \ge 1$ (similarly as in the proof of Lemma 4.3). Joining the previous estimate and (5.7) with (5.6) and applying Young's inequality, we find

$$\begin{aligned} \int_{\alpha_1 Q\cap\Omega_T} |u-P^{(g_0)}_{\alpha_1\rho}|^2\,\mathrm{d}z &\le \tfrac{1}{2}\int_{\alpha_2 Q\cap\Omega_T} |u-P^{(g_0)}_{\alpha_2\rho}|^2\,\mathrm{d}z \\ &\qquad + \frac{c}{(\alpha_2-\alpha_1)^{2m(2-p)/p}}\;\rho^{2m}\Big(\lambda^{2_*}|Q| + \fint_{2B}|D^m g_0|^{2_*}\,\mathrm{d}x\Big)^{2/2_*}. \end{aligned}$$

Applying Lemma 3.8 we obtain the desired estimate. □

Now we are in a position to prove a reverse Hölder inequality on cylinders intersecting the initial boundary. We take $\tilde G$ from Section 4 into account in the scaling because then we can later use the same scaling in all the cases.

Lemma 5.6. *Let $\kappa \ge 1$ and let u be a global solution according to Definition 2.1. Furthermore, let $Q = Q_{z_0}(\rho,s) \subset \mathbf{R}^{n+1}$ be a parabolic cylinder with $0<\rho\le 1$, $\lambda > 0$ and $s = \lambda^{2-p}\rho^{2m}$ and $8B \subset \Omega$, where $8B = B_{x_0}(8\rho)$ and $0 \in 2\Lambda_{t_0}(s)$. Suppose that*

$$\text{(5.8)}\quad \lambda^p \le \frac{\kappa}{|Q|}\int_{Q\cap\Omega_T}(|D^m u|^p + \widetilde G^p)\,\mathrm{d}z, \qquad \frac{1}{|8Q|}\int_{8Q\cap\Omega_T}(|D^m u|^p + \widetilde G^p)\,\mathrm{d}z \le \kappa\lambda^p,$$

for some $\widetilde G \in L^p(8Q)$ with $\widetilde G \ge 1$. Then there exists a constant $c = c(n,N,m,p,L/\nu,\kappa)$, such that for any q with $\max\{p-1,\widehat p_*\} \le q < p$ there holds

$$\begin{aligned} \frac{1}{|Q|}\int_{Q\cap\Omega_T} |D^m u|^p\,\mathrm{d}z &\le \Big(\frac{c}{|4Q|}\int_{4Q\cap\Omega_T}|D^m u|^q\,\mathrm{d}z\Big)^{p/q} \\ &\qquad + \frac{c}{|4Q|}\int_{4Q\cap\Omega_T}\widetilde G^p\,\mathrm{d}z + c\,\lambda^{p-\widehat p}\Big(\fint_{4B}|D^m g_0|^{\widehat p_*}\,\mathrm{d}x\Big)^{\widehat p/\widehat p_*}. \end{aligned}$$

Proof. From Caccioppoli's inequality, i.e. Lemma 5.1, we get

$$\begin{aligned} \frac{1}{|Q|}\int_{Q\cap\Omega_T} |D^m u|^p\,\mathrm{d}z &\le \frac{c_{Cac}}{|Q|}\int_{2Q\cap\Omega_T}\bigg(\lambda^{p-2}\Big|\frac{u-P^{(g_0)}_{2\rho}}{\rho^m}\Big|^2 + \Big|\frac{u-P^{(g_0)}_{2\rho}}{\rho^m}\Big|^p + 1\bigg)\,\mathrm{d}z \\ &\qquad + c_{Cac}\,\lambda^{p-2}\Big(\fint_{2B}|D^m g_0|^{2_*}\,\mathrm{d}x\Big)^{2/2_*}. \end{aligned} \tag{5.9}$$

We still have to bound the first two terms on the right-hand side. Therefore, we abbreviate

$$I_\sigma = \frac{\lambda^{p-\sigma}}{|Q|} \int_{2Q\cap\Omega_T} \Big|\frac{u-P_{2\rho}^{(g_0)}}{\rho^m}\Big|^\sigma \,\mathrm{d}z$$

for $\sigma = 2$ or $\sigma = p$. Now, we fix $q \in [\max\{p-1, \widehat{p}_*\}, p)$. In order to estimate I_σ we apply Gagliardo-Nirenberg-Sobolev's inequality, i.e. Theorem 3.7 with $r = 2$ and $\theta = q/\sigma$ slicewise to $(u-P_{2\rho}^{(g_0)})(\cdot,t)$ yielding that

$$I_\sigma \le c(n,m,p)\, \frac{\lambda^{p-\sigma}}{|2Q|} \sum_{k=0}^{m} \int_{2Q\cap\Omega_T} \Big|\frac{D^k(u-P_{2\rho}^{(g_0)})}{\rho^{m-k}}\Big|^q \,\mathrm{d}z \cdot J^{(\sigma-q)/2}, \tag{5.10}$$

where

$$J \equiv \sup_{t\in 2\Lambda\cap(0,T)} \fint_{2B} \Big|\frac{u(\cdot,t)-P_{2\rho}^{(g_0)}}{\rho^m}\Big|^2 \,\mathrm{d}x.$$

To estimate the integrals in the sum on the right-hand side, we enlarge the domain of integration from $2Q$ to $4Q$ and apply Poincaré's inequality from Lemma 5.3 to infer for $0 \le k \le m$ that (recall that $\widetilde{G} \ge 1$)

$$\begin{aligned}\frac{1}{|2Q|} \int_{2Q\cap\Omega_T} \Big|\frac{D^k(u-P_{2\rho}^{(g_0)})}{\rho^{m-k}}\Big|^q \,\mathrm{d}z \le c \Bigg[\frac{1}{|4Q|} \int_{4Q\cap\Omega_T} |D^m u|^q \,\mathrm{d}z \\ + \Big(\frac{\lambda^{2-p}}{|4Q|} \int_{4Q\cap\Omega_T} (|D^m u| + \widetilde{G})^{p-1} \,\mathrm{d}z + \fint_{4B} |D^m g_0| \,\mathrm{d}x \Big)^q \Bigg].\end{aligned}$$

To further estimate the second term on the right-hand side, we use Hölder's inequality and hypothesis (5.8) (i.e. the first inequality in (5.8) when $p < 2$, respectively the second inequality in (5.8) when $p > 2$) to find

$$\begin{aligned}\frac{\lambda^{2-p}}{|4Q|} \int_{4Q\cap\Omega_T} (|D^m u| + \widetilde{G})^{p-1} \,\mathrm{d}z &= \lambda^{2-p} \Big(\dots \Big)^{1-1/(p-1)} \Big(\dots \Big)^{1/(p-1)} \\ &\le \lambda^{2-p} \Big(\frac{1}{|4Q|} \int_{4Q\cap\Omega_T} (|D^m u| + \widetilde{G})^p \,\mathrm{d}z \Big)^{(p-2)/p} \Big(\frac{1}{|4Q|} \int_{4Q\cap\Omega_T} (|D^m u| + \widetilde{G})^q \,\mathrm{d}z \Big)^{1/q} \\ &\le c\, \lambda^{2-p} \lambda^{\frac{p(p-2)}{p}} \Big(\frac{1}{|4Q|} \int_{4Q\cap\Omega_T} (|D^m u| + \widetilde{G})^q \,\mathrm{d}z \Big)^{1/q} \\ &= c \Big(\frac{1}{|4Q|} \int_{4Q\cap\Omega_T} (|D^m u| + \widetilde{G})^q \,\mathrm{d}z \Big)^{1/q}.\end{aligned}$$

Inserting this above to bound the second term on the right-hand side, we deduce

$$\begin{aligned}&\frac{1}{|2Q|} \int_{2Q\cap\Omega_T} \Big|\frac{D^k(u-P_{2\rho}^{(g_0)})}{\rho^{m-k}}\Big|^q \,\mathrm{d}z \\ &\quad \le c \Bigg[\Big(\frac{1}{|4Q|} \int_{4Q\cap\Omega_T} (|D^m u|^q + \widetilde{G}^q) \,\mathrm{d}z \Big)^{1/q} + \fint_{4B} |D^m g_0| \,\mathrm{d}x \Bigg]^q.\end{aligned} \tag{5.11}$$

The estimate of J now is similar to the one from Lemma 5.5. We first apply Caccioppoli's inequality in Lemma 5.1 with $Q_{z_0}(r,s)$ and $Q_{z_0}(R,S)$ replaced by $2Q$ and $4Q$, note that $|Q| = 2\lambda^{2-p}\rho^{2m}|B|$ and then estimate the terms appearing on the right-hand side as follows: In the case $p \ge 2$, we now use Corollary 5.4 to bound the first integral on the right-hand side, while in the case $p < 2$ we use Lemma 5.5.

Moreover, from Corollary 5.4, we also infer a bound for the second integral on the right-hand side. Note that the second inequality in (5.8) ensures that the hypothesis of Corollary 5.4 and Lemma 5.5 are satisfied. Proceeding this way we arrive at

$$\sup_{t\in 2\Lambda\cap(0,T)} \fint_{2B}\Big|\frac{u(\cdot,t)-P_{4\rho}^{(g_0)}}{\rho^m}\Big|^2 \mathrm{d}x \le c\left(\lambda^{2_*}+\fint_{4B}|D^m g_0|^{2_*}\,\mathrm{d}x\right)^{2/2_*}.$$

Since by Lemma 3.9, we can bound the difference of the mean value polynomials on $2B$ and $4B$ as

$$|P_{2\rho}^{(g_0)}(x)-P_{4\rho}^{(g_0)}(x)| \le c\,\rho^m \fint_{4B}|D^m g_0|\,\mathrm{d}x \qquad \text{for all } x\in 4B,$$

this leads us to

$$J \le c\left(\lambda^{2_*}+\fint_{4B}|D^m g_0|^{2_*}\,\mathrm{d}x\right)^{2/2_*}.$$

Inserting the previous estimate and (5.11) in (5.10) for $\sigma=2$ and $\sigma=p$ and applying Young's inequality, we obtain for $\varepsilon>0$ that

$$I_\sigma \le \varepsilon\,\lambda^p + c_\varepsilon\left(\frac{1}{|4Q|}\int_{4Q\cap\Omega_T}(|D^m u|^q+\widetilde{G}^q)\,\mathrm{d}z\right)^{p/q} + \left(\fint_{4B}|D^m g_0|^{2_*}\,\mathrm{d}x\right)^{p/2_*},$$

where $c_\varepsilon = c_\varepsilon(n,N,m,p,L/\nu,\kappa,1/\varepsilon)$. Inserting this estimate for $\sigma=2$ and $\sigma=p$ in (5.9) and using Young's inequality for the term involving g_0 (note that $(\dots)^{p/2_*}+\lambda^{p-2}(\dots)^{2/2_*}\le \varepsilon\lambda^p + c_\varepsilon\lambda^{p-\max\{2,p\}}(\dots)^{\max\{2,p\}/2_*}$), we arrive at

$$\fint_Q |D^m u|^p\,\mathrm{d}z \le 3c_{Cac}\,\varepsilon\lambda^p + \left(\frac{c_\varepsilon}{|4Q|}\int_{4Q\cap\Omega_T}(|D^m u|^q+\widetilde{G}^q)\,\mathrm{d}z\right)^{p/q} + c_\varepsilon\,\lambda^{p-\widehat{p}}\left(\fint_{4B}|D^m g_0|^{2_*}\,\mathrm{d}x\right)^{\widehat{p}/2_*}.$$

Now we use the first inequality in (5.8) to bound λ^p in terms of the integral on the left-hand side of the preceding inequality. Choosing ε small enough we can absorb this term on the left. Finally, applying Hölder's inequality to the term involving $\widetilde{G}$ and to the initial term if necessary (i.e. if $p>2$) we deduce the desired reverse Hölder inequality. □

6. Proof of the higher integrability

In this chapter, we prove the global higher integrability result from Theorem 2.2. To deduce the desired estimate on the set Ω_T, we will cover Ω_T by intrinsic cylinders. Therefore, we have to take into account three different configurations, that is, cylinders lying in the interior of Ω_T and those lying near the lateral or initial boundaries. For the latter two, we have proved reverse Hölder type inequalities in Lemma 4.4 and 5.6. For the interior cylinders there holds an analogue of Lemma 5.6 without the initial boundary term, of course (cf. [9], Lemma 13).

Proof of Theorem 2.2. We fix a cylinder $Q_0 = Q_{z_0}(R,R^2)\subset \mathbf{R}^{n+1}$ which might intersect the complement of Ω_T. As usual, we denote $\frac14 Q_0 = Q_{z_0}(R/4,(R/4)^2)$ and also $B_0 = B_{z_0}(R)$ and $\Lambda_0=\Lambda_{t_0}(R^2)$. At the end, a choice $\Omega_T\subset \frac14 Q_0$ leads to the global higher integrability result.

To begin with, we cover Q_0 by Whitney-type cylinders

$$Q_i = Q_{z_i}(r_i, r_i^{2m}),\;\; i=1,2,\dots,$$

where r_i is comparable to the parabolic distance of Q_i to the boundary ∂Q_0 of Q_0 (see, for example, page 15 of [39]). The *parabolic distance* of two sets $E, F \subset \mathbf{R}^{n+1}$ is defined to be

$$\operatorname{dist}_p(E,F) = \inf\left\{ |x-\bar{x}| + |t-\bar{t}|^{1/(2m)} \,:\, (x,t) \in E, (\bar{x},\bar{t}) \in F \right\}.$$

In addition, the cylinders Q_i are of bounded overlap, meaning that every z belongs at most to a fixed finite number of cylinders depending only on n and m, and

$$5Q_i \subset Q_0.$$

Later we shall divide Q_0 into a good and a bad set, i.e. into certain level sets according to a level $\lambda > 0$. In order to apply the reverse Hölder inequality from Lemma 4.4, respectively Lemma 5.6, we aim to find cylinders having the scaling factor λ^{2-p} and satisfying (4.5), respectively (5.8) around each point lying in the bad set. For this we first set

$$\lambda_0' = \left(\frac{1}{|Q_0|} \int_{Q_0 \cap \Omega_T} \left(|D^m u|^p + \widetilde{G}^p \right) \mathrm{d}z \right)^{1/d},$$

where d was defined in the statement of Theorem 2.2 and $\widetilde{G}$ is defined by

$$\widetilde{G}^p = \begin{cases} 1 & \text{if } B_0 \subset \Omega \\ G^p + 1 & \text{if } B_0 \setminus \Omega \neq \emptyset. \end{cases}$$

We now choose λ such that

$$\lambda > \max\{\lambda_0', 1\} = \lambda_0.$$

For $z \in Q_0 \cap \Omega_T$, we define

$$h(z) = \frac{1}{c_1 |Q_0|^{1/d}} \min\{ |Q_i|^{1/d} : z \in Q_i \} \, |D^m u(z)|,$$

where $c_1 \geq 1$ will be fixed later. Further, we consider $\tilde{z} \in Q_0 \cap \Omega_T$ such that

$$h(\tilde{z}) > \lambda$$

and fix a Whitney-cylinder $Q_i = Q_{z_i}(r_i, r_i^{2m})$ such that $\tilde{z} \in Q_i \cap \Omega_T$. We define

$$\alpha = \alpha(\tilde{z}) = \left(\frac{|Q_0|}{|Q_i|} \right)^{1/d},$$

and

$$\theta = \theta(\tilde{z}) = \left(\lambda \alpha(\tilde{z}) \right)^{2-p}.$$

Now we will use the stopping time argument to find an intrinsic cylinder around $\tilde{z}$ of the type $Q_{\tilde{z}}(r, \theta r^2)$ on which the assumptions (4.5), respectively (5.8) of Lemma 4.4, respectively 5.6 are satisfied. To begin with, we show that the first inequality in (4.5), respectively the first inequality in (5.8), is valid for suitably small cylinders due to Lebesgue's differentiation theorem. Indeed, for almost every $\tilde{z} \in Q_i \cap \Omega_T$ such that $h(\tilde{z}) > \lambda$, we have

$$\lim_{r' \to 0} \frac{1}{|Q(r', \theta r'^{2m})|} \int_{Q_{\tilde{z}}(r', \theta r'^{2m}) \cap \Omega_T} \left(|D^m u|^p + \widetilde{G}^p \right) \mathrm{d}z > {c_1}^p \alpha^p \lambda^p. \tag{6.1}$$

Note that $Q_{\tilde{z}}(r', \theta r'^{2m}) \cap \Omega_T = Q_{\tilde{z}}(r', \theta r'^{2m})$ for $r' > 0$ small enough.

Our next aim is to show that the second inequality in (4.5), respectively the second inequality in (5.8), is valid due to the definition of λ_0. For this we have to distinguish the cases $p \geq 2$ and $p < 2$ since the scaling factor θ is smaller that one in the former case, respectively larger than one in the latter case.

We start considering **the case** $p \geq 2$, where we have $d = 2$ and $\theta \leq 1$. For an intrinsic cylinder $Q_{\tilde z}(r, \theta r^{2m})$, with radius r such that $r_i/64 \leq r \leq r_i$, (note that $Q_{\tilde z}(r, \theta r^{2m}) \subset 2Q_i \subset Q_0$), we obtain due to the definition of λ_0, α and d

$$\begin{aligned} &\frac{1}{|Q(r,\theta r^{2m})|}\int_{Q_{\tilde z}(r,\theta r^{2m})\cap\Omega_T}\big(|D^m u|^p + \widetilde{G}^p\big)\,\mathrm{d}z \\ &\qquad \leq \frac{c\,|Q_0|}{|Q_i|\theta}\frac{1}{|Q_0|}\int_{Q_0\cap\Omega_T}\big(|D^m u|^p + \widetilde{G}^p\big)\,\mathrm{d}z \leq {c_1}^p\alpha^p\lambda^p, \end{aligned} \tag{6.2}$$

where c_1 is chosen to be large enough. This fixes the constant $c_1 \geq 1$ in the definition of h in dependence of n, m and p.

Now we want to use a similar reasoning for **the case** $2_* < p < 2$. Here, the basic difference compared to the case $p \geq 2$ is that the scaling factor θ for the parabolic cylinders is now larger than 1. Nevertheless, we still have to ensure that the considered intrinsic cylinders are contained in Q_0. To accomplish this, we consider radii r with $\theta^{-1/2m} r_i/64 \leq r \leq \theta^{-1/2m} r_i$. Thus, $Q_{\tilde z}(r, \theta r^{2m}) \subset Q_0$, and due to the definition of λ_0, α and d (note that $d = p - n(2-p)/(2m)$ in the present case), we obtain

$$\frac{1}{|Q(r,\theta r^{2m})|}\int_{Q_{\tilde z}(r,\theta r^{2m})\cap\Omega_T}\big(|D^m u|^p + \widetilde{G}^p\big)\,\mathrm{d}z \leq c\,\frac{|Q_0|}{|Q_i|}\,\theta^{n/(2m)}\lambda^d \leq {c_1}^p\alpha^p\lambda^p. \tag{6.3}$$

According to (6.1) and (6.2) when $p \geq 2$, respectively (6.3) when $p < 2$ and due to the fact that the integrals above depend continuously on the radius of the cylinder, there exists one largest radius $\rho = \rho(\tilde z)$ with $\rho \in (0, r_i/64]$ when $p \geq 2$, respectively $\rho \in (0, \theta^{-1/2m} r_i/64]$ when $p < 2$ such that equality holds. In the following, we denote briefly $Q = Q_{\tilde z}(\rho, \theta\rho^{2m})$, $B = B_{\tilde x}(\rho)$, $4Q = Q_{\tilde z}(4\rho, \theta(4\rho)^{2m})$, $4B = B_{\tilde x}(4\rho)$, $4\Lambda = \Lambda(\theta(4\rho)^{2m})$ etc. Note that $64Q \subset Q_0$ by the choice of ρ. Hence, from the previous reasoning we have

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}\big(|D^m u|^p + \widetilde{G}^p\big)\,\mathrm{d}z = c_1^p\big(\alpha\lambda\big)^p \geq c_1^{-p}\big(\alpha\lambda\big)^p \tag{6.4}$$

and

$$\frac{1}{|64Q|}\int_{64Q\cap\Omega_T}\big(|D^m u|^p + \widetilde{G}^p\big)\,\mathrm{d}z \leq c_1^p\big(\alpha\lambda\big)^p. \tag{6.5}$$

Now we set $q = \max\{p-1, \gamma, \widehat{p}_*\}$, where $\gamma = \gamma(n,p,\mu)$ is the constant from Lemma 4.4. From (6.4) and (6.5), we see that (after enlarging the domain of integration if necessary) conditions (4.5) and (5.8) are satisfied with λ replaced by $\alpha\lambda$ and, further, $s = \theta\rho^{2m} = (\alpha\lambda)^{2-p}\rho^{2m}$. We now distinguish several cases. If $8B \setminus \Omega \neq \emptyset$, we can apply Lemma 4.4 with $8Q$ instead of Q (then the condition $B_{x_0}(4\rho/3) \setminus \Omega \neq \emptyset$ has to be replaced by $B_{x_0}(32\rho/3) \setminus \Omega \neq \emptyset$ which is fulfilled). In this case, the first inequality in (4.5) is satisfied on $8Q$ with $8^{n+2m}c_1$ instead of κ after enlarging the domain of integration in (6.4) from Q to $8Q$. We get

$$\begin{aligned} &\frac{1}{|8Q|}\int_{8Q\cap\Omega_T}|D^m u|^p\,\mathrm{d}z \\ &\qquad \leq \left(\frac{c}{|32Q|}\int_{32Q\cap\Omega_T}|D^m u|^q\,\mathrm{d}z\right)^{p/q} + \frac{c}{|32Q|}\int_{32Q\cap\Omega_T}\widetilde{G}^p\,\mathrm{d}z. \end{aligned} \tag{6.6}$$

On the other hand, if $8B \subset \Omega$, we still have to distinguish the cases when the cylinder lies near, respectively far from the initial boundary. If $8B \subset \Omega$ and $2Q$ intersects the initial boundary, i.e. $0 \in 2\Lambda$, then we are in a position to apply Lemma 5.6 (note that the second inequality in (5.8) is satisfied on $8Q$ with $8^{n+2m}c_1$ instead of κ due to (6.5) after enlarging the domain of integration from $8Q$ to $64Q$). Therefore the application of the lemma implies

$$\begin{aligned} \frac{1}{|Q|}\int_{Q\cap\Omega_T}|D^m u|^p\,\mathrm{d}z &\le \left(\frac{c}{|4Q|}\int_{4Q\cap\Omega_T}|D^m u|^q\,\mathrm{d}z\right)^{p/q} \\ &\quad + \frac{c}{|4Q|}\int_{4Q\cap\Omega_T}\widetilde{G}^p\,\mathrm{d}z + c\,(\alpha\lambda)^{p-\widehat{p}}\left(⨍_{4B}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\right)^{\widehat{p}/\widehat{p}_*}. \end{aligned} \tag{6.7}$$

In the remaining case $8B \subset \Omega$ and $2\Lambda \subset (0,T)$, we have $2Q \subset \Omega_T$. In that case we can use the interior analogue of Lemma 5.6 to obtain

$$\frac{1}{|Q|}\int_Q |D^m u|^p\,\mathrm{d}z \le \left(\frac{c}{|2Q|}\int_{2Q}|D^m u|^q\,\mathrm{d}z\right)^{p/q} + \int_{2Q}\widetilde{G}^p\,\mathrm{d}z. \tag{6.8}$$

The proof can be deduced from [9], Lemma 13 by modifying the involved radii and it is akin to the proof of Lemma 5.6. Observe that (6.4) and (6.5) are valid on any cylinder lying between Q and $2Q$, with possibly larger constants.

We now note that $h(z) \le c_1^{-1}\alpha^{-1}|D^m u(z)| \le c\,h(z)$ for $z \in 64Q$ since all the Whitney cylinders Q_i intersecting $64Q$ are comparable. Moreover, we have $\alpha^{-1} = (|Q_i|\,/\,|Q_0|)^{1/d} \le 1$. Therefore, multiplying (6.6) - (6.8) by α^{-p}, we deduce that in any case, we have

$$\begin{aligned} \frac{1}{|Q|}\int_{Q\cap\Omega_T}(h^p + \alpha^{-p}\widetilde{G}^p)\,\mathrm{d}z &\le \left(\frac{c}{|32Q|}\int_{32Q\cap\Omega_T}h^q\,\mathrm{d}z\right)^{p/q} \\ &\quad + \frac{c}{|32Q|}\int_{32Q\cap\Omega_T}\widetilde{G}^p\,\mathrm{d}z + c\,\delta_1'(\alpha\lambda)^{p-\widehat{p}}\left(⨍_{4B}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\right)^{\widehat{p}/\widehat{p}_*}, \end{aligned} \tag{6.9}$$

where we also added $\frac{1}{|Q|}\int_{Q\cap\Omega_T}\alpha^{-p}\widetilde{G}^p\,\mathrm{d}z$ on both sides and then estimated

$$\frac{1}{|Q|}\int_{Q\cap\Omega_T}\alpha^{-p}\widetilde{G}^p\,\mathrm{d}z \le \frac{c}{|32Q|}\int_{32Q\cap\Omega_T}\widetilde{G}^p\,\mathrm{d}z$$

on the right hand side. Here we have set $\delta_1' = 1$ if $8B \subset \Omega$ and $0 \in 2\Lambda$ and $\delta_1' = 0$ otherwise.

Next, we decompose Q_0 into level sets and define

$$\mathscr{G}(\lambda) = \{z \in Q_0 \cap \Omega_T \,:\, h(z) > \lambda\},$$

and

$$\mathscr{G}_{\mathrm{lat}}(\lambda) = \{z \in Q_0 \cap \Omega_T \,:\, \widetilde{G}(z) > \lambda\}.$$

Finally, we define

$$\mathscr{G}_{\mathrm{ini}}(\lambda) = \{x \in B_0 \cap \Omega \,:\, |D^m g_0(x)| > \lambda\} \qquad \text{if } 0 \in \Lambda_0$$

and $\mathscr{G}_{\mathrm{ini}}(\lambda) = \emptyset$ otherwise. Since $h(z) > \lambda$ in $\mathscr{G}(\lambda)$, we can later cover $\mathscr{G}(\lambda)$ by cylinders of the type considered above. Observe that for $\eta \ge 0$ we have $h(z) \le \eta\lambda$

whenever $z \in (32Q \cap \Omega_T) \setminus \mathscr{G}(\eta\lambda)$, and similarly for $\widetilde{G}$ and $|D^m g_0|$. Therefore, by (6.5), (6.4) and (6.9), we obtain

$$\frac{1}{|64Q|}\int_{64Q\cap\Omega_T}(h^p + \alpha^{-p}\widetilde{G}^p)\,\mathrm{d}z \le c\,\eta^p\lambda^p + \left(\frac{c}{|32Q|}\int_{32Q\cap\mathscr{G}(\eta\lambda)} h^q\,\mathrm{d}z\right)^{p/q} + \frac{c}{|32Q|}\int_{32Q\cap\mathscr{G}_{\mathrm{lat}}(\eta\lambda)}\widetilde{G}^p\,\mathrm{d}z + c\,\delta_1'(\alpha\lambda)^{p-\widehat{p}}\left(\frac{1}{|4B|}\int_{4B\cap\mathscr{G}_{\mathrm{ini}}(\eta\lambda)}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\right)^{\widehat{p}/\widehat{p}_*}.$$

Choosing $\eta > 0$ small enough and employing a suitable version of (6.4) as in [36] for h, we can absorb the first term on the right-hand side into the left, because the left-hand side matches with (6.4). Moreover, with the help of Hölder's inequality and (6.5), multiplied by α^{-p}, we deduce

$$\left(\frac{1}{|32Q|}\int_{32Q\cap\Omega_T} h^q\,\mathrm{d}z\right)^{(p-q)/q} \le c\,\lambda^{p-q}$$

which helps us to estimate the second term on the right-hand side of the preceding inequality. Finally, estimating the left-hand side from below, and multiplying both sides by $|64Q|$ (note that $(\alpha\lambda)^{p-\widehat{p}}|64Q|/|4B|^{p/\widehat{p}_*} \le 16^{n+2m}\rho^{n+2m-np/\widehat{p}_*}$ since $(\alpha\lambda)^{p-\widehat{p}}\theta = (\alpha\lambda)^{2-\widehat{p}} \le 1$), we arrive at

$$\int_{64Q\cap\Omega_T} h^p\,\mathrm{d}z \le c\,\lambda^{p-q}\int_{32Q\cap\mathscr{G}(\eta\lambda)} h^q\,\mathrm{d}z + c\int_{32Q\cap\mathscr{G}_{\mathrm{lat}}(\eta\lambda)}\widetilde{G}^p\,\mathrm{d}z + c\,\delta_1'\rho^{n+2m-n\widehat{p}/\widehat{p}_*}\left(\int_{4B\cap\mathscr{G}_{\mathrm{ini}}(\eta\lambda)}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\right)^{\widehat{p}/\widehat{p}_*}. \tag{6.10}$$

Let us mention that the exponent $n+2m-n\widehat{p}/\widehat{p}_*$ of ρ is non-negative due to the definition of $\widehat{p}_*$.

As a next step, we use a covering argument to extend the estimates to the whole of $\mathscr{G}(\lambda)$. Recall that up to now, we have found, for any $\tilde{z} \in \mathscr{G}(\lambda)$, a parabolic cylinder $Q = Q_{\tilde{z}}(\rho(\tilde{z}), \theta(\tilde{z})\rho(\tilde{z})^{2m})$ satisfying (6.10). By Vitali's covering theorem, we can extract a family of cylinders $\{Q_{\tilde{z}_j}(\rho_j, \theta\rho_j^2)\}_{j=1}^{\infty}$ with $\tilde{z}_j \in \mathscr{G}(\lambda)$ such that $Q_{\tilde{z}_j}(64\rho_j, \theta(64\rho_j)^{2m}) \subset Q_0$ and

$$\mathscr{G}(\lambda) \subset \bigcup_{j=1}^{\infty} Q_{\tilde{z}_j}(64\rho_j, \theta(64\rho_j)^{2m})$$

when possibly neglecting a set of measure zero. Moreover, at most $c(n,m)$ of the cylinders $Q_{\tilde{z}_j}(32\rho_j, \theta(32\rho_j)^{2m})$ intersect in each point z of Q_0 and (6.10) holds in each of the cylinders. Then we sum over j, note that $\rho^{n+2m-n\widehat{p}/\widehat{p}_*} \le R^{n+2m-n\widehat{p}/\widehat{p}_*} \le c|Q_0|/|B_0|^{\widehat{p}/\widehat{p}_*}$, and obtain

$$\int_{\mathscr{G}(\lambda)} h^p\,\mathrm{d}z \le c\,\lambda^{p-q}\int_{\mathscr{G}(\eta\lambda)} h^q\,\mathrm{d}z + c\int_{\mathscr{G}_{\mathrm{lat}}(\eta\lambda)}\widetilde{G}^p\,\mathrm{d}z + |Q_0|\left(\frac{c}{|B_0|}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda)}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\right)^{\widehat{p}/\widehat{p}_*}. \tag{6.11}$$

When summing over the initial boundary terms, we used the fact that $\widehat{p}/\widehat{p}_* > 1$.

Now we multiply the previous estimate by $\lambda^{\varepsilon-1}$, where $\varepsilon \in (0,1]$ is to be fixed later, integrate with respect to λ over (λ_0, ∞) and then apply Fubini's theorem to each of the resulting terms. We will only work out the details for the last term in (6.11) in order to show how to deal with the exponent $\widehat{p}/\widehat{p}_*$. The computations for the remaining terms are similar but easier and can be deduced from the local or second order proofs. Hence, with Hölder's inequality and Fubini's theorem, we get

$$\begin{aligned}
\int_{\lambda_0}^{\infty}\lambda^{\varepsilon-1}&\left(\frac{1}{|B_0|}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda)}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\right)^{\widehat{p}/\widehat{p}_*}\mathrm{d}\lambda\\
&\le\left(\frac{1}{|B_0|}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda_0)}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\right)^{\widehat{p}/\widehat{p}_*-1}\frac{1}{|B_0|}\int_{\lambda_0}^{\infty}\lambda^{\varepsilon-1}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda)}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}x\,\mathrm{d}\lambda\\
&\le\left(\frac{1}{|B_0|}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda_0)}|D^m g_0|^{\widehat{p}_*+\varepsilon}\,\mathrm{d}x\right)^{(\widehat{p}-\widehat{p}_*)/(\widehat{p}_*+\varepsilon)}\\
&\qquad\qquad\cdot\frac{1}{|B_0|}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda_0)}\int_{\lambda_0}^{|D^m g_0|/\eta}\lambda^{\varepsilon-1}|D^m g_0|^{\widehat{p}_*}\,\mathrm{d}\lambda\,\mathrm{d}x\\
&\le\frac{1}{\varepsilon}\left(\frac{1}{|B_0|}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda_0)}|D^m g_0|^{\widehat{p}_*+\varepsilon}\,\mathrm{d}x\right)^{(\widehat{p}+\varepsilon)/(\widehat{p}_*+\varepsilon)}.
\end{aligned}$$

Therefore, treating the remaining terms in (6.11) in a similar way (see also [36], proof of Theorem 4.7) we end up with

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\mathscr{G}(\lambda_0)}h^{p+\varepsilon}\,\mathrm{d}z&\le\frac{c}{p+\varepsilon-q}\int_{\mathscr{G}(\lambda_0)}h^{p+\varepsilon}\,\mathrm{d}z+\frac{c\lambda_0^{\varepsilon}}{\varepsilon}\int_{\mathscr{G}(\eta\lambda_0)}h^{p}\,\mathrm{d}z\\
&\quad+\frac{c}{\varepsilon}\int_{\mathscr{G}_{\mathrm{lat}}(\eta\lambda_0)}\widetilde{G}^{p+\varepsilon}\,\mathrm{d}z+\frac{c}{\varepsilon}|Q_0|\left(\frac{1}{|B_0|}\int_{\mathscr{G}_{\mathrm{ini}}(\eta\lambda_0)}|D^m g_0|^{\widehat{p}_*+\varepsilon}\,\mathrm{d}x\right)^{(\widehat{p}+\varepsilon)/(\widehat{p}_*+\varepsilon)}.
\end{aligned}$$

Now we multiply both sides by ε and absorb the integral involving $h^{p+\varepsilon}$ on the left-hand side by choosing ε small enough. As usual, in order to exclude the possibility that the term we would like to absorb is infinite, we can replace h by $h_k=\min\{h,k\}$, $k>\lambda_0$, repeat the previous calculations and then pass to the limit $k\to\infty$.

Next we note that since $h\le\lambda_0$ in $(Q_0\cap\Omega_T)\setminus\mathscr{G}(\lambda_0)$, our estimate extends to the whole of $\frac{1}{4}Q_0\cap\Omega_T$ as follows

$$\begin{aligned}
\int_{\frac{1}{4}Q_0\cap\Omega_T}h^{p+\varepsilon}\,\mathrm{d}z&\le(\lambda_0)^{\varepsilon}\int_{(Q_0\cap\Omega_T)\setminus\mathscr{G}(\lambda_0)}h^{p}\,\mathrm{d}z+\int_{\mathscr{G}(\lambda_0)}h^{p+\varepsilon}\,\mathrm{d}z\\
&\le c(\lambda_0)^{\varepsilon}\int_{Q_0\cap\Omega_T}h^{p}\,\mathrm{d}z+c\int_{Q_0\cap\Omega_T}\widetilde{G}^{p+\varepsilon}\,\mathrm{d}z\\
&\quad+c\,\delta_1|Q_0|\left(\frac{1}{|B_0|}\int_{B_0\cap\Omega}|D^m g_0|^{\widehat{p}_*+\varepsilon}\,\mathrm{d}x\right)^{(\widehat{p}+\varepsilon)/(\widehat{p}_*+\varepsilon)},
\end{aligned}$$

where $\delta_1=1$ if $0\in\Lambda_0$ and $\delta_1=0$ otherwise. This is a consequence of the definitions for the sets $\mathscr{G}_{\mathrm{lat}}(\eta\lambda_0)$ and $\mathscr{G}_{\mathrm{ini}}(\eta\lambda_0)$. We divide the estimate by $|Q_0|$ and apply the definition of $h(z)$. Since $\frac{1}{4}Q_0$ lies far away from the boundary of Q_0, there exists $c=c(n,m)>0$, independent of R, such that for every $z\in\frac{1}{4}Q_0\cap\Omega_T$ we have

$$c<\min\{|Q_i|^{1/d}:z\in Q_i\}/|Q_0|^{1/d}\le 1.$$

The upper bound is due to the fact that the Whitney-cylinders Q_i are contained in Q_0. By the definition of h and recalling that $\lambda_0=\max\{\lambda_0',1\}$ we therefore deduce from the previous estimate

$$\frac{1}{|Q_0|}\int_{\frac{1}{4}Q_0\cap\Omega_T}|D^m u|^{p+\varepsilon}\,\mathrm{d}z \le \left(\frac{c}{|Q_0|}\int_{Q_0\cap\Omega_T}\left(|D^m u|^p+\widetilde{G}^p\right)\mathrm{d}z\right)^{(\varepsilon+d)/d}+c$$
$$+\frac{c}{|Q_0|}\int_{Q_0\cap\Omega_T}\widetilde{G}^{p+\varepsilon}\,\mathrm{d}z+\left(\frac{c\delta_1}{|B_0|}\int_{B_0\cap\Omega}|D^m g_0|^{\widehat{p}_*+\varepsilon}\,\mathrm{d}x\right)^{(\widehat{p}+\varepsilon)/(\widehat{p}_*+\varepsilon)}.$$

Recalling that $\widetilde{G}^p = 1$ if $B_0 \subset \Omega$ and $\widetilde{G}^p = G^p + 1$ if $B_0 \setminus \Omega \neq \emptyset$, this proves the estimate in Theorem 2.2, and since $Q_0 \subset \mathbf{R}^{n+1}$ is an arbitrary parabolic cylinder and Ω_T is bounded, we conclude that $|D^m u| \in L^{p+\varepsilon}(\Omega_T)$. This finishes the proof of the theorem. □

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