Next we introduce dyadic cubes, which are generated using powers of 2.

Definition 4.17 (Dyadic cubes)**.** A dyadic interval on $\mathbf{R}$ is

$$[m2^{-k}, (m+1)2^{-k})$$

where $m, k \in \mathbb{Z}$. A dyadic cube in $\mathbf{R}^n$ is

$$\prod [m_j 2^{-k}, (m_j+1)2^{-k})$$

where $m_1, m_2, \ldots, m_n, k \in \mathbb{Z}$.

Observe that corners lie at $2^{-k}\mathbb{Z}^n$ and side length is 2^{-k}. Dyadic cubes have an important property that they are either disjoint or one is contained into another.

Notations

$$D_k = \text{"a collection of dyadic cubes with side length } 2^{-k}.\text{ "}$$

A collection of all the dyadic cubes is denoted by

$$D = \bigcup_{k\in\mathbb{Z}} D_k.$$

Theorem 4.18 (Local Calderón-Zygmund decomposition)**.** *Let $Q_0 \subset \mathbf{R}^n$ be a dyadic cube, and $f \in L^1(Q_0)$. Then if*

$$\lambda \geq ⨍_{Q_0} |f(x)| \, \mathrm{d}x$$

there exists a collection of dyadic cubes

$$F_\lambda = \{Q_j \, : j = 1, 2, \ldots\}$$

such that

(i)

$$Q_j \cap Q_k = \emptyset \text{ when } j \neq k,$$

(ii)

$$\lambda < ⨍_{Q_j} |f(x)| \, \mathrm{d}x \leq 2^n \lambda, \quad j = 1, 2, \ldots,$$

and

(iii)

$$|f(x)| \leq \lambda \text{ for a.e. } x \in Q_0 \setminus \cup_{j=1}^{\infty} Q_j.$$

Remark 4.19. Naturally, if $|f(x)| \leq \lambda$, then $F_\lambda = \emptyset$. Notice also the assumption that Q_0 is dyadic could be dropped, and that if the condition $\lambda \geq ⨍_{Q_0} |f(x)| \, \mathrm{d}x$ does not hold, then we can choose a larger cube to begin with so that this condition is satisfied.

Proof of Theorem 4.18. Clearly, $Q_0 \notin F_\lambda$ because of our assumption. We split Q_0 into 2^n dyadic cubes with side length $l(Q_0)/2$. Then we choose to F_λ, the cubes for which

$$\lambda < \fint_Q |f(x)| \, \mathrm{d}x.$$

Observe that (i) holds because we use dyadic cubes, and because of the estimate

$$\begin{aligned} \fint_Q |f(x)| \, \mathrm{d}x &\le \frac{m(Q_0)}{m(Q)} \fint_{Q_0} |f(x)| \, \mathrm{d}x \\ &\le 2^n \fint_{Q_0} |f(x)| \, \mathrm{d}x \le 2^n \lambda, \end{aligned} \tag{4.20}$$

also the upper bound in (ii) holds. For the cubes that were not chosen i.e. for which

$$\fint |f(x)| \, \mathrm{d}x \le \lambda,$$

we continue the process. Then the estimate (ii) holds for all the cubes that were chosen at some round. On the other hand, according to Lebesgue's density theorem

$$|f(x)| = \lim_{k\to\infty} \fint_{Q^{(k)}} |f(y)| \, \mathrm{d}y \overset{Q^{(k)} \text{ was not chosen}}{\le} \lambda$$

for a.e. $x \in \mathbf{R}^n \setminus \cup_{Q \in F_\lambda} Q$. □

Next we prove a global version of the Calderón-Zygmund decomposition. The idea in the proof is similar to the local version, but as we work in the whole of $\mathbf{R}^n$, there is no initial cube Q_0.

Theorem 4.21 (Global Calderón-Zygmund decomposition)**.** *Let $f \in L^1(\mathbf{R}^n)$ and $\lambda > 0$. Then there exists a collection of dyadic cubes*

$$F_\lambda = \{Q_j \, : j = 1, 2, \ldots\}$$

such that

(i)

$$Q_j \cap Q_k = \emptyset \text{ when } j \ne k,$$

(ii)

$$\lambda < \fint_{Q_j} |f(x)| \, \mathrm{d}x \le 2^n \lambda, \quad j = 1, 2, \ldots,$$

and

(iii)

$$|f(x)| \le \lambda \text{ for a.e. } x \in \mathbf{R}^n \setminus \cup_{j=1}^\infty Q_j.$$

Proof. We study a subcollection

$$F_\lambda \subset D$$

of dyadic cubes, which are the largest possible cubes such that

$$\fint_Q |f(x)| \, \mathrm{d}x > \lambda \tag{4.22}$$

holds. In other words, $Q \in F_\lambda$ if $Q \in D_k$ for some k, if (4.22) holds and for all the larger dyadic cubes $\tilde{Q}$, $Q \subset \tilde{Q}$, it holds that

$$\fint_{\tilde{Q}} |f(y)| \, \mathrm{d}y \leq \lambda.$$

The largest cube exists, if (4.22) holds for Q, because

$$\fint_{\tilde{Q}} |f(x)| \, \mathrm{d}x \leq \frac{||f||_1}{m(\tilde{Q})} \to 0$$

as $m(\tilde{Q}) \to \infty$ because $f \in L^1(\mathbf{R}^n)$. As the cubes in F_λ are maximal, they are disjoint, because if this were not the case the smaller cube would be contained to larger one as they are dyadic and thus we could replace it by the larger one. A similar calculation as in (4.20) shows that also the upper bound in (*ii*) holds. The proof is completed similarly as in the local version: (*iii*) is a consequence of Lebesgue's density theorem Theorem 3.17. □

Example 4.23. *Calderón-Zygmund decomposition for*

$$f : \mathbf{R} \to [0, \infty], \;\; f(x) = |x|^{-1/2}$$

with $\lambda = 1$.

Example 4.24. *By using the Calderón-Zygmund decomposition, we can split any* $f \in L^1(\mathbf{R}^n)$ *into a good and a bad part as (further details during the lecture)*

$$f = g + b$$

as

$$g = \begin{cases} f(x), & x \in \mathbf{R}^n \setminus \cup_{j=1}^\infty Q_j, \\ \fint_{Q_j} f(y) \, \mathrm{d}y, & x \in Q_j \in F_\lambda \end{cases}$$

and

$$b(x) = \sum_{j=1}^\infty b_j(x),$$

$$b_j(x) = (f(x) - \fint_{Q_j} f(y) \, \mathrm{d}y) \chi_{Q_j}(x).$$

Observe that $g \leq 2^n \lambda$ *and* $\int_{Q_j} b(y) \, \mathrm{d}y = 0$. *Split* $f : \mathbf{R} \to [0, \infty], \; f(x) = |x|^{-1/2}$ *in this way with* $\lambda = 1$.

Lemma 4.25. *Let* $f \in L^1(\mathbf{R}^n)$ *and*

$$F_\lambda = \{Q_j \, : j = 1, 2, \ldots\}$$

Calderón-Zygmund decomposition with $\lambda > 0$ *from Theorem 4.21. Then*

$$\{x \in \mathbf{R}^n \, : \, Mf(x) > 4^n\lambda\} \subset \cup_{j=1}^{\infty} 3Q_j.$$

Proof. The Calderón-Zygmund decomposition gives bounds for the averages, so our task is passing from the averages to the maximal function. To this end, let

$$x \in \mathbf{R}^n \setminus \cup_{j=1}^{\infty} 3Q_j$$

and $Q \subset \mathbf{R}^n$ is a cube (not necessarily dyadic) s.t. $x \in Q$. If we choose, k so that

$$2^{-k-1} \le l(Q) < 2^{-k},$$

then there exists at the most 2^n dyadic cubes $R_1, \ldots R_l \in D_k$ such that

$$R_m \cap Q \neq \emptyset, \; m = 1, \ldots, l.$$

Because R_m and Q intersect, $Q \subset 3R_m$. On the other hand R_m is not contained to any $Q_j \in F_\lambda$, because otherwise we would have $x \in Q \subset 3Q_j$ which contradicts our assumption $x \in \mathbf{R}^n \setminus \cup_{j=1}^{\infty} 3Q_j$. As R_m is not in F_λ, it follows by definition that

$$\fint_{R_m} |f(y)| \, \mathrm{d}y \le \lambda$$

for $m = 1, \ldots, l$. Thus

$$\begin{aligned}
\fint_Q |f(y)| \, \mathrm{d}y &= \frac{1}{m(Q)} \sum_{m=1}^{l} \int_{R_m \cap Q} |f(y)| \, \mathrm{d}y \\
&\le \sum_{m=1}^{l} \frac{m(R_m)}{m(Q)} \frac{1}{m(R_m)} \int_{R_m} |f(y)| \, \mathrm{d}y \\
&\le l 2^n \lambda \le 4^n \lambda.
\end{aligned}$$

Moreover,

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)| \, \mathrm{d}y \le 4^n \lambda$$

for every $x \in \mathbf{R}^n \setminus \cup_{j=1}^{\infty} 3Q_j$. Thus

$$\mathbf{R}^n \setminus \cup_{j=1}^{\infty} 3Q_j \subset \{x \in \mathbf{R}^n \, : \, Mf(x) \le 4^n\lambda\}. \qquad \square$$

5.10.2010

Corollary 4.26. *Let* $f \in L^1(\mathbf{R}^n)$ *and*

$$F_\lambda = \{Q_j \, : j = 1, 2, \ldots\}$$

Calderón-Zygmund decomposition with $\lambda > 0$ *from Theorem 4.21. Then*