Proof. The sketch of the proof: By a density of continuous functions in L^p, we can choose $g \in C_0(\mathbf{R}^n)$ so that $||f-g||_p$ is small. By adding and subtracting g, we can estimate

$$|(f*\phi_\varepsilon)(x) - af(x)| \leq |\phi_\varepsilon * (f-g)(x) - a(f-g)(x)| \\ + |(g*\phi_\varepsilon)(x) - ag(x)| . \tag{3.13}$$

Since $g \in C_0(\mathbf{R}^n)$, the second term tends to zero as $\varepsilon \to 0$. Thus we can focus attention on the first term on the right hand side. By Theorem 3.10, we can estimate

$$|(f*\phi_\varepsilon)(x) - af(x)| \leq |\phi_\varepsilon * (f-g)(x) - a(f-g)(x)| \\ \leq M(f-g)(x) + a\,|(f-g)(x)| .$$

Finally, we can show by using the weak type estimates that the quantities on the right hand side get small almost everywhere.

Details: **Case $1 \leq p < \infty$:**
As sketched above the weak type estimates play a key role. Theorem Hardy-Littlewood I (Theorem 2.12) implies

$$m(\{x \in \mathbf{R}^n : Mf(x) > \lambda\}) \leq \frac{C}{\lambda}\,||f||_1 \tag{3.14}$$

for $\lambda > 0$, and Hardy-Littlewood II (Theorem 2.19) imply

$$m(\{x \in \mathbf{R}^n : Mf(x) > \lambda\}) \overset{\text{Chebyshev}}{\leq} \frac{C}{\lambda^p}\,||Mf||_p^p \overset{\text{H-L II}}{\leq} C\,||f||_p^p . \tag{3.15}$$

As g is continuous at $x \in \mathbf{R}^n$ it follows that for every $\eta > 0$ there exists $\delta > 0$ such that

$$|g(x-y) - g(x)| < \eta \quad \text{whenever} \quad |y| < \delta.$$

Thus

$$|(g*\phi_\varepsilon)(x) - ag(x)| \leq \int_{\mathbf{R}^n} |g(x-y) - g(x)|\,\phi_\varepsilon(y)\,\mathrm{d}y \\ \leq \eta \underbrace{\int_{B(0,\delta)} \phi_\varepsilon(y)\,\mathrm{d}y}_{\leq ||\phi||_1} + 2\,||g||_\infty \underbrace{\int_{\mathbf{R}^n \setminus B(0,\delta)} \phi_\varepsilon(x)\,\mathrm{d}y}_{\to 0 \text{ as } \varepsilon \to 0 \text{ by Lemma 3.6}} .$$

Since η was arbitrary, it follows that

$$\lim_{\varepsilon \to 0} |(g*\phi_\varepsilon)(x) - ag(x)| = 0$$

for all $x \in \mathbf{R}^n$.

This in mind we can estimate

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} |(f*\phi_\varepsilon)(x) - af(x)| \\
&\qquad \le \limsup_{\varepsilon\to 0} |((f-g)*\phi_\varepsilon)(x) - a(f-g)(x)| \\
&\qquad\quad + \underbrace{\limsup_{\varepsilon\to 0} |(g*\phi_\varepsilon)(x) - ag(x)|}_{=0} \\
&\qquad \le \sup_{\varepsilon>0} |((f-g)*\phi_\varepsilon)(x)| + a\,|(f-g)(x)| \\
&\qquad \overset{\text{Theorem 3.10}}{\le} CM(f-g)(x) + a\,|(f-g)(x)|\,.
\end{aligned} \tag{3.16}$$

Next we define

$$A_i = \{x \in \mathbf{R}^n \,:\, \limsup_{\varepsilon\to 0} |(f*\phi_\varepsilon)(x) - af(x)| > \frac{1}{i}\}.$$

By the previous estimate,

$$A_i \subset \{x \in \mathbf{R}^n \,:\, CM(f-g)(x) > \frac{1}{2i}\} \cup \{x \in \mathbf{R}^n \,:\, a\,|f(x)-g(x)| > \frac{1}{2i}\},$$

for $i = 1, 2, \ldots$. Let $\eta > 0$, and let $g \in C_0(\mathbf{R}^n)$ be such that (density)

$$||f-g||_p \le \eta.$$

This and the previous inclusion imply

$$\begin{aligned}
m(A_i) &\le m(\{x \in \mathbf{R}^n \,:\, CM(f-g)(x) > \frac{1}{2i}\}) + m(\{x \in \mathbf{R}^n \,:\, a\,|f(x)-g(x)| > \frac{1}{2i}\}) \\
&\overset{(3.14),(3.15)}{\le} Ci^p\,||f-g||_p^p + Ci^p\,||f-g||_p^p \\
&\le Ci^p\,||f-g||_p^p \le Ci^p\eta^p
\end{aligned}$$

for every η, $i = 1, 2, \ldots$. Thus

$$m(A_i) = 0$$

and

$$m(\cup_{i=1}^\infty A_i) \le \sum_{i=1}^\infty m(A_i) = 0.$$

This gives us

$$m(\{x \in \mathbf{R}^n \,:\, \limsup_{\varepsilon\to 0} |(f*\phi_\varepsilon)(x) - af(x)| > 0\}) = 0$$

which proofs the claim

$$\lim_{\varepsilon\to 0} |(f*\phi_\varepsilon)(x) - af(x)| = 0 \quad \text{a.e. } x \in \mathbf{R}^n.$$

Case $p = \infty$: Now $f \in L^\infty(\mathbf{R}^n)$. We show that

$$\lim_{\varepsilon\to 0} (f*\phi_\varepsilon)(x) = af(x)$$

for almost every $x \in B(0,r), r > 0$. Let

$$f_1(x) = f\chi_{B(0,r+1)}(x) = \begin{cases} f(x), & x \in B(0,r+1) \\ 0, & \text{otherwise,} \end{cases}$$

and $f_2 = f - f_1$. Now $f_1 \in L^1(\mathbf{R}^n)$ and by the previous case

$$\lim_{\varepsilon \to 0}(f_1 * \phi_\varepsilon)(x) = af_1(x)$$

for almost every $x \in \mathbf{R}^n$. By utilizing this, we obtain for almost every $x \in B(0,r)$ that

$$\begin{aligned} \lim_{\varepsilon \to 0}(f * \phi_\varepsilon)(x) &= \lim_{\varepsilon \to 0}(f_1 * \phi_\varepsilon)(x) + \lim_{\varepsilon \to 0}(f_2 * \phi_\varepsilon)(x) \\ &= af(x) + \lim_{\varepsilon \to 0}(f_2 * \phi_\varepsilon)(x), \end{aligned}$$

and it remains to show that $\lim_{\varepsilon \to 0}(f_2 * \phi_\varepsilon)(x) = 0$ for almost all $x \in B(0,r)$. To this end, let $x \in B(0,r)$ so that $f_2(x-y) = 0$ for $y \in B(0,1)$ and calculate

$$\begin{aligned} |(f_2 * \phi_\varepsilon)(x)| &= \left| \int_{\mathbf{R}^n} f_2(x-y)\phi_\varepsilon(y)\,\mathrm{d}y \right| \\ &= \left| \int_{\mathbf{R}^n \setminus B(0,1)} f_2(x-y)\phi_\varepsilon(y)\,\mathrm{d}y \right| \\ &= ||f_2||_\infty \int_{\mathbf{R}^n \setminus B(0,1)} \phi_\varepsilon(y)\,\mathrm{d}y \to 0 \end{aligned}$$

as $\varepsilon \to 0$. □

By choosing

$$\phi(x) = \chi_{B(0,1)}(x)/m(B(0,1)),$$

so that

$$\phi_\varepsilon(x) = \chi_{B(0,\varepsilon)}/(\varepsilon^n m(B(0,1))) = \chi_{B(0,\varepsilon)}/m(B(0,\varepsilon)),$$

we immediately obtain

Theorem 3.17 (Lebesgue density theorem)**.** *If $f \in L^1_{loc}(\mathbf{R}^n)$, then*

$$\lim_{r \to 0} \fint_{B(x,R)} f(y)\,\mathrm{d}y = f(x)$$

for almost every $x \in \mathbf{R}^n$.

Example 3.18. *Let*

$$\phi(x) = P(x) = \frac{C(n)}{(1+|x|^2)^{(n+1)/2}}$$

where the constant is chosen so that

$$\int_{\mathbf{R}^n} P(x)\,\mathrm{d}x = 1.$$

Next we define

$$P_t(x) = \frac{1}{t^n} P(\frac{x}{t}) = C(n) \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}, \ t > 0$$

and

$$u(x,t) = (f * P_t)(x) = \int_{\mathbf{R}^n} P_t(x-y) f(y) \, \mathrm{d}y.$$

This is called the Poisson integral for f. It has the following properties

(i) $\Delta u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1^2} + \ldots + \frac{\partial^2 u}{\partial x_2^2} = 0$ *and*

(ii) $\lim_{t \to 0} u(x,t) = f(x)$ *for almost every $x \in \mathbf{R}^n$ by Theorem 3.12.*

Let

$$\mathbf{R}^{n+1}_+ = \{(x_1, x_2, \ldots, t) \in \mathbf{R}^{n+1} \ : \ t > 0\}$$

denote the upper half space. As stated above u is harmonic in $\mathbf{R}^{n+1}_+$ so that $u(x,t) = \int_{\mathbf{R}^n} P_t(x-y) f(y) \, \mathrm{d}y$ solves

$$\begin{cases} \Delta u(x,t) = 0, & (x,t) \in \mathbf{R}^{n+1}_+ \\ u(x,0) = f(x), & , x \in \partial \mathbf{R}^{n+1}_+ = \mathbf{R}^n, \end{cases}$$

where the boundary condition is obtained in the sense

$$\lim_{t \to 0} u(x,t) = f(x)$$

almost everywhere on $\mathbf{R}^n$. As $(x,t) \to (x,0)$ along a perpendicular axis, we call this radial convergence.

Question Does the Poisson integral converge better than radially?

Definition 3.19. Let $x \in \mathbf{R}^n$ and $\alpha > 0$. Then

(i) We define a cone

$$\Gamma_\alpha(x) = \{(y,t) \in \mathbf{R}^{n+1}_+ \ : \ |x-y| < \alpha t\}.$$

(ii) Function $u(x,t)$ converges nontangentially, if $u(y,t) \to f(x)$ and $(y,t) \to (x,0)$ so that (y,t) remains inside the cone $\Gamma_\alpha(x)$.

Theorem 3.20. *Let $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, and $u(x,t) = (f * P_t)(x)$. Then for every $\alpha > 0$, there exists $C = C(n, \alpha)$ such that*

$$u^*_\alpha(x) := \sup_{(y,t) \in \Gamma_\alpha(x)} |u(y,t)| \le C M f(x)$$

for every $x \in \mathbf{R}^n$.

u^* is called a nontangential maximal function. 23.9.2010

Proof. First we show that

$$P_t(y-z) \le C(\alpha, n) P_t(x-z) \quad \text{for every } (y,t) \in \Gamma_\alpha(x), \ z \in \mathbf{R}^n.$$