**HARMONIC ANALYSIS, 5. EXERCISE**

1. Find a Calderón-Zygmund decomposition for $f : \mathbf{R} \to \mathbf{R}$, $f(x) = |x|^{-1/3}$ at the level $\lambda = 1$. Split $f$ into a good and a bad part $f = g + b$ according to this decomposition. Draw a picture.
2. Formulate and prove a local version of the Calderón-Zygmund decomposition with respect to measure $\mu$ (i.e. $\mu(Q) = \int_Q w \,\mathrm{d}x,\ w \in A_p,\ 1 \le p < \infty$, so that the integrals are of the form $\frac{1}{\mu(Q_j)} \int_{Q_j} fw \,\mathrm{d}x$). You can take Lebesgue's differentiation theorem for granted.
3. Let $f \in L^1(\mathbf{R}^n)$ and define a dyadic maximal function

$$M_d f(x) \sup \frac{1}{m(Q)} \int_{Q \ni x} |f(y)| \,\mathrm{d}y$$

   where the supremum is taken over all the dyadic cubes $Q \in D$ for which $x \in Q$.
   (i) Show that

$$\{x \in \mathbf{R}^n : M_d f(x) > \lambda\} = \cup_{Q \in F_\lambda} Q,$$

   where $F_\lambda$ is a Calderón-Zygmund decomposition for $f \in L^1(\mathbf{R}^n)$ at level $\lambda$.
   (ii) Show by an example that there is no $C > 0$ such that

$$C M_d f(x) \ge M f(x)$$

   for every $x \in \mathbf{R}$ and $f \in L^1(\mathbf{R})$. Draw a picture.
4. Show for the maximal function in Problem 3 that

$$m(\{x \in \mathbf{R}^n : M_d f(x) > \lambda\}) \le \frac{||f||_1}{\lambda}$$

   for every $\lambda > 0$.
5. Suppose that $g \ge 0$ is a measurable function such that

$$g \in L^\infty(\mathbf{R}^n),\ \frac{1}{g} \in L^\infty(\mathbf{R}^n).$$

   Show that if $w \in A_p,\ 1 \le p < \infty$, then $gw \in A_p$.

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