3. Approximation by convolution

Definition 3.1 (Convolution)**.** Suppose that $f, g : \mathbf{R}^n \to [-\infty, \infty]$ are Lebesgue-measurable functions. The convolution

$$(f * g)(x) = \int_{\mathbf{R}^n} f(y)g(x-y)\,\mathrm{d}y$$

is defined if $y \mapsto f(y)g(x-y)$ is integrable for almost every $x \in \mathbf{R}^n$.

Observe that: $f, g \in L^1(\mathbf{R}^n)$ does not imply $fg \in L^1(\mathbf{R}^n)$ which can be seen by considering for example $f = g = \frac{\chi_{(0,1)}(x)}{\sqrt{x}}$.

Theorem 3.2 (Minkowski's/Young's inequality)**.** *If $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$ and $g \in L^1(\mathbf{R}^n)$, then $(f * g)(x)$ exists for almost all $x \in \mathbf{R}^n$ and*

$$||f * g||_p \leq ||f||_p \, ||g||_1 \, .$$

Proof. **Case $p = 1$:** Because

$$|(f * g)(x)| \leq \int_{\mathbf{R}^n} |f(y)|\,|g(x-y)|\,\mathrm{d}y$$

we have

$$\begin{aligned}
\int_{\mathbf{R}^n} |(f * g)(x)|\,\mathrm{d}x &\leq \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |f(y)|\,|g(x-y)|\,\mathrm{d}y\,\mathrm{d}x \\
&\overset{\text{Fubini}}{=} \int_{\mathbf{R}^n} |f(y)| \Big(\int_{\mathbf{R}^n} |g(x-y)|\,\mathrm{d}x \Big)\,\mathrm{d}y \\
&= \int_{\mathbf{R}^n} |f(y)|\,\mathrm{d}y \int_{\mathbf{R}^n} |g(x)|\,\mathrm{d}x \\
&= ||f||_1 \, ||g||_1 \, .
\end{aligned}$$

Case $p = \infty$:

$$\begin{aligned}
|(f * g)(x)| &\leq \int_{\mathbf{R}^n} |f(y)|\,|g(x-y)|\,\mathrm{d}y \\
&\leq \operatorname*{ess\,sup}_{y \in \mathbf{R}^n} |f(x)| \int_{\mathbf{R}^n} |g(x-y)|\,\mathrm{d}y \\
&= ||f||_\infty \, ||g||_1 \, .
\end{aligned}$$

Case $1 < p < \infty$: Set

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Then

$$\begin{aligned}
|(f*g)(x)| &\le \int_{\mathbf{R}^n} |f(y)|\,|g(x-y)|\;\mathrm{d}y \\
&= \int_{\mathbf{R}^n} |f(y)|\,|g(x-y)|^{1/p}\,|g(x-y)|^{1/p'}\;\mathrm{d}y \\
&\overset{\text{Hölder}}{\le} \Big(\int_{\mathbf{R}^n} |f(y)|^p\,|g(x-y)|\;\mathrm{d}y\Big)^{1/p}\Big(\int_{\mathbf{R}^n} |g(x-y)|\;\mathrm{d}y\Big)^{1/p'} \\
&= \Big(\int_{\mathbf{R}^n} |f(y)|^p\,|g(x-y)|\;\mathrm{d}y\Big)^{1/p}\,||g||_1^{1/p'}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathbf{R}^n} |(f*g)(x)|^p\;\mathrm{d}x &\le ||g||_1^{p/p'}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |f(y)|^p\,|g(x-y)|\;\mathrm{d}y\,\mathrm{d}x \\
&\overset{\text{Fubini}}{=} ||g||_1^{p/p'}\int_{\mathbf{R}^n} |f(y)|^p\int_{\mathbf{R}^n} |g(x-y)|\;\mathrm{d}x\,\mathrm{d}y \\
&= ||g||_1^{p/p'}\,||g||_1\,||f||_p^p = ||g||_1^p\,||f||_p^p,
\end{aligned}$$

because

$$\frac{p}{p'}+1 = p\Big(\frac{1}{p'}+\frac{1}{p}\Big) = p. \qquad \square$$

We state the following simple properties of convolution without a proof.

Lemma 3.3 (Basic properties of convolution)**.** *Let $f, g, h \in L^1(\mathbf{R}^n)$. Then*

(i) $f*g = g*f$.
(ii) $f*(g*h) = (f*g)*h$.
(iii) $(\alpha f+\beta g)*h = \alpha(f*h)+\beta(g*h),\ \alpha,\beta\in\mathbf{R}^n$.

For $\phi\in L^1(\mathbf{R}^n)$, $\varepsilon>0$, we denote

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi\Big(\frac{x}{\varepsilon}\Big),\ x\in\mathbf{R}^n. \tag{3.4}$$

Example 3.5. *(i) Let $\phi(x)=\frac{\chi_{B(0,1)}(x)}{m(B(0,1))}$. Then*

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\frac{\chi_{B(0,1)}(\frac{x}{\varepsilon})}{m(B(0,1))} = \frac{\chi_{B(0,\varepsilon)(x)}}{m(B(0,\varepsilon))}.$$

Then for $f\in L^1(\mathbf{R}^n)$, a mollification

$$\begin{aligned}
(f*\phi_\varepsilon)(x) &= \int_{\mathbf{R}^n} f(y)\phi_\varepsilon(x-y)\,\mathrm{d}y \\
&= \fint_{B(x,\varepsilon)} f(y)\,\mathrm{d}y.
\end{aligned}$$

turns out to be useful. Observe also that $||\phi_\varepsilon||_1 = 1$ *for any* $\varepsilon > 0$ *so that*

$$||f * \phi_\varepsilon||_1 \le ||f||_1 \, ||\phi_\varepsilon||_1 = ||f||_1 .$$

(ii)

$$\varphi = \begin{cases} \exp\left(\frac{1}{|x|^2-1}\right), & x \in B(0,1) \\ 0, & \textit{else.} \end{cases}$$

It holds that $\varphi \in C_0^\infty(\mathbf{R}^n)$ *and thus also* $\varphi \in L^1(\mathbf{R}^n)$*. Let*

$$\phi = \frac{\varphi}{||\varphi||_1}.$$

Then $\phi_\varepsilon \in C_0^\infty(\mathbf{R}^n)$, $\operatorname{spt}(\phi_\varepsilon) \subset \overline{B}(0,\varepsilon)$, *and*

$$\begin{aligned} \int_{\mathbf{R}^n} \phi_\varepsilon(x)\,\mathrm{d}x &= \frac{1}{\varepsilon^n}\int_{\mathbf{R}^n} \phi(x/\varepsilon)\,\mathrm{d}x \\ &\overset{y=\frac{x}{\varepsilon},\ \mathrm{d}x=\varepsilon^n\,\mathrm{d}y}{=} \frac{1}{\varepsilon^n}\int_{\mathbf{R}^n} \phi(y)\varepsilon^n\,\mathrm{d}y \\ &= \int_{\mathbf{R}^n} \phi(y)\,\mathrm{d}y \\ &= \int_{\mathbf{R}^n} \frac{\varphi(y)}{||\varphi||_1}\,\mathrm{d}y = \frac{||\varphi||_1}{||\varphi||_1} = 1, \end{aligned}$$

for all $\varepsilon > 0$*. The function* ϕ_ε *is called a standard mollifier in this case. As before, if* $f \in L^1(\mathbf{R}^n)$*, then*

$$||f * \phi_\varepsilon||_1 \le ||f||_1 .$$

Lemma 3.6. *Let* $\phi \in L^1(\mathbf{R}^n)$ *and recall that* $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$*. Then*

(i)

$$\int_{\mathbf{R}^n} \phi_\varepsilon(x)\,\mathrm{d}x = \int_{\mathbf{R}^n} \phi(x)\,\mathrm{d}x$$

for every $\varepsilon > 0$*.*

(ii)

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^n\setminus B(0,r)} |\phi_\varepsilon(x)|\,\mathrm{d}x = 0$$

for every $r > 0$*.*

Proof. (i) Change of variables, see above.

(ii) We calculate

$$\begin{aligned}\int_{\mathbf{R}^n\setminus B(0,r)} |\phi_\varepsilon(x)|\,\mathrm{d}x &= \frac{1}{\varepsilon^n}\int_{\mathbf{R}^n\setminus B(0,r)} |\phi(x/\varepsilon)|\,\mathrm{d}x \\ &\overset{y=\frac{x}{\varepsilon},\ \mathrm{d}x=\varepsilon^n\,\mathrm{d}y}{=} \int_{\mathbf{R}^n\setminus B(0,r/\varepsilon)} \phi(y)\,\mathrm{d}y \\ &= \int_{\mathbf{R}^n} \phi(y)\chi_{\mathbf{R}^n\setminus B(0,r/\varepsilon)}\,\mathrm{d}y \to 0\end{aligned}$$

as $\varepsilon \to 0$ by Lebesgue's dominated convergence theorem. □

Theorem 3.7. *Let $\phi \in L^1(\mathbf{R}^n)$,*

$$a = \int_{\mathbf{R}^n} \phi(x)\,\mathrm{d}x$$

and $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$. Then

$$||\phi_\varepsilon * f - af||_p \to 0$$

as $\varepsilon \to 0$.

Notice that the statement is invalid if $p = \infty$.

Proof. We will work out the details below, but the idea in the proof is that by using the definition of the convolution together with Hölder's inequlity and Fubini's theorem, we obtain

$$\begin{aligned}&\int_{\mathbf{R}^n} |(f*\phi_\varepsilon)(x) - af(x)|^p\,\mathrm{d}x \\ &\quad\le ||\phi||_1^{p/p'} \int_{\mathbf{R}^n} |\phi_\varepsilon(y)| \Big(\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p\,\mathrm{d}x\Big)\,\mathrm{d}y \\ &\quad= ||\phi||_1^{p/p'} \int_{B(0,r)} |\phi_\varepsilon(y)| \Big(\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p\,\mathrm{d}x\Big)\,\mathrm{d}y \\ &\qquad+ ||\phi||_1^{p/p'} \int_{\mathbf{R}^n\setminus B(0,r)} |\phi_\varepsilon(y)| \Big(\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p\,\mathrm{d}x\Big)\,\mathrm{d}y \\ &\quad= I_1 + I_2,\end{aligned} \tag{3.8}$$

where $1/p + 1/p' = 1$. The first term on the right hand side, I_1, is small when r is small because intuitively then $f(x-y)$ only differs little from $f(x)$. On the other hand, the second integral, I_2, is small for small enough $\varepsilon > 0$ for any r because ϕ_ε gets more and more concentrated. 16.9.2010

Next we work out the details. By the previous lemma

$$af(x) = f(x)\int_{\mathbf{R}^n} \phi(y)\,\mathrm{d}y = \int_{\mathbf{R}^n} f(x)\phi_\varepsilon(y)\,\mathrm{d}y.$$