HARMONIC ANALYSIS, 4. EXERCISE

1. Show that if $m = \operatorname{ess\,inf}_{x\in\mathbf{R}^n} |f(x)| < \infty$, then
$$|f(x)| \geq m \text{ a.e. } x \in \mathbf{R}^n.$$
Further, let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Check that the previous result implies that for each $\varepsilon > 0$ there exists a set $E_\varepsilon \subset Q$ such that $m(E_\varepsilon) > 0$ and
$$|f(x)| < \operatorname*{ess\,inf}_{y\in Q} |f(y)| + \varepsilon$$
for every $x \in E_\varepsilon$.
2. Suppose that $f \in L^1(\mathbf{R}^n)$ and that
$$\int_{\mathbf{R}^n} f(x)g(x)\,\mathrm{d}x = 0$$
for every $g \in C_0(\mathbf{R}^n)$. Show that $f(x) = 0$ a.e. $x \in \mathbf{R}^n$. Hint: Consider ϕ_ε.
3. Let $\Omega \subset \mathbf{R}^n$ be an open set with the following property: there is $\gamma,\ 0 < \gamma < 1$ such that
$$m(B(x,r) \cap \Omega) \geq \gamma m(B(x,r))$$
for every $x \in \partial\Omega$ and $r > 0$. Show that $m(\partial\Omega) = 0$.
4. Show that $w \in A_p \Leftrightarrow w^{1-p'} \in A_{p'}$.
5. Show that if $w_1, w_0 \in A_1$ then $w_0 w_1^{1-p} \in A_p$.
6. Show for a doubling measure that there are constants $C, N > 0$ such that
$$\frac{\mu(Q(x,L))}{\mu(Q(x,l))} \leq C\Big(\frac{L}{l}\Big)^N,$$
for every $x \in \mathbf{R}^n$ and $0 < l < L < \infty$. What is N for the Lebesgue measure?

Date: 30.9.2010, deadline 14.10.2010.