Then T is of strong type (p,p) for every $1<p<q$ that is

$$||Tf||_p \leq C\,||f||_p$$

for every $f \in L^p(\mathbf{R}^n)$.

Proof. **Case $q<\infty$.** Let $f = f_1 + f_2$ where as before

$$f_1 = f\chi_{\{|f|\leq\lambda\}} \quad \text{and} \quad f_2 = f\chi_{\{|f|>\lambda\}}$$

and recall that $f_1 \in L^q$ and $f_2 \in L^1$. Subadditivity implies

$$|Tf| \leq |Tf_1| + |Tf_2|$$

for a.e. $x \in \mathbf{R}^n$. Thus

$$\begin{aligned} m(\{x\in\mathbf{R}^n : |Tf(x)|>\lambda\}) &\leq m(\{x\in\mathbf{R}^n : |Tf_1(x)|>\lambda/2\}) \\ &\quad + m(\{x\in\mathbf{R}^n : |Tf_2(x)|>\lambda/2\}) \\ &\leq \Big(\frac{C_1}{\lambda/2}\,||f_1||_q\Big)^q + \frac{C_2}{\lambda/2}\,||f_2||_1 \\ &\leq \frac{(2C_1)^q}{\lambda^q}\int_{\{x\in\mathbf{R}^n : |f(x)|\leq\lambda\}} |f(x)|^q \,\mathrm{d}x \\ &\quad + \frac{2C_2}{\lambda}\int_{\{x\in\mathbf{R}^n : |f(x)|>\lambda\}} |f(x)|\,\mathrm{d}x. \end{aligned}$$

Then by Lemma 2.9, it follows that

$$\begin{aligned} \int_{\mathbf{R}^n} |Tf|^p \,\mathrm{d}x &= p\int_0^\infty \lambda^{p-1} m(\{x\in\mathbf{R}^n : |Tf(x)|>\lambda)\,\mathrm{d}\lambda \\ &\leq (2C_1)^q p\int_0^\infty \lambda^{p-q-1}\int_{\{x\in\mathbf{R}^n : |f(x)|\leq\lambda\}} |f(x)|^q \,\mathrm{d}x\,\mathrm{d}\lambda \\ &\quad + 2pC_2\int_0^\infty \lambda^{p-2}\int_{\{x\in\mathbf{R}^n : |f(x)|>\lambda\}} |f(x)|\,\mathrm{d}x\,\mathrm{d}\lambda. \end{aligned}$$

Further by Fubini's theorem

$$\begin{aligned} \int_0^\infty \lambda^{p-q-1}\int_{\{x\in\mathbf{R}^n : |f(x)|\leq\lambda\}} |f(x)|^q \,\mathrm{d}x\,\mathrm{d}\lambda &= \int_{\mathbf{R}^n} |f(x)|^q \int_{|f(x)|}^\infty \lambda^{p-q-1}\,\mathrm{d}\lambda\,\mathrm{d}x \\ &= \frac{1}{q-p}\int_{\mathbf{R}^n} |f(x)|^q\,|f(x)|^{p-q}\,\mathrm{d}x \\ &= \frac{1}{q-p}\int_{\mathbf{R}^n} |f(x)|^p\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \lambda^{p-2} \int_{\{x\in\mathbf{R}^n \,:\, |f(x)|>\lambda\}} |f(x)| \,\mathrm{d}x \,\mathrm{d}\lambda &= \int_{\mathbf{R}^n} |f(x)| \int_0^{|f(x)|} \lambda^{p-2} \,\mathrm{d}\lambda \,\mathrm{d}x \\ &= \frac{1}{p-1}\int_{\mathbf{R}^n} |f(x)|^{p-1} |f(x)| \,\mathrm{d}x \\ &= \frac{1}{p-1}\int_{\mathbf{R}^n} |f(x)|^{p} \,\mathrm{d}x.\end{aligned}$$

Thus we arrive at

$$||Tf||_p^p \le p\Big(\frac{2C_2}{p-1} + \frac{(2C_1)^q}{q-p}\Big) ||f||_p^p .$$

Case $q=\infty$. Suppose that

$$||Tg||_\infty \le C_2 ||g||_\infty$$

for every $g \in L^\infty(\mathbf{R}^n)$. We again split $f \in L^p(\mathbf{R}^n)$ as $f = f_1 + f_2$ where

$$f_1 = f\chi_{\{|f|\le\lambda/(2C_2)\}} \quad \text{and} \quad f_2 = f\chi_{\{|f|>\lambda/(2C_2)\}}$$

and by Lemma 2.20, $f_1 \in L^\infty$ and $f_2 \in L^1$. We have a.e.

$$|Tf_1(x)| \le ||Tf_1||_\infty \le C_2 ||f_1||_\infty \le C_2 \frac{\lambda}{2C_2} = \frac{\lambda}{2}.$$

Thus

$$\begin{aligned} m(\{x\in\mathbf{R}^n \,:\, |Tf(x)|>\lambda\}) \le \underbrace{m(\{x\in\mathbf{R}^n \,:\, |Tf_1(x)|>\lambda/2\})}_{=0} \\ + m(\{x\in\mathbf{R}^n \,:\, |Tf_2(x)|>\lambda/2\}).\end{aligned}$$

It follows that

$$\begin{aligned} m(\{x\in\mathbf{R}^n \,:\, |Tf(x)|>\lambda\}) &\le m(\{x\in\mathbf{R}^n \,:\, |Tf_2(x)|>\lambda/2\}) \\ &\overset{\text{weak }(1,1)}{\le} \frac{C_1}{\lambda/2}\int_{\mathbf{R}^n} |f_2(x)| \,\mathrm{d}x \\ &= \frac{2C_1}{\lambda}\int_{\{x\in\mathbf{R}^n \,:\, |f(x)|>\lambda/(2C_2)\}} |f(x)| \,\mathrm{d}x.\end{aligned}$$

Then by using Lemma 2.9 again, we see that

$$\begin{aligned}\int_{\mathbf{R}^n} |Tf(x)|^p \,\mathrm{d}x &= p\int_0^\infty \lambda^{p-1} m(\{x\in\mathbf{R}^n \,:\, |Tf(x)|>\lambda\}) \,\mathrm{d}\lambda \\ &\le 2C_1 p \int_0^\infty \lambda^{p-2} \int_{\{x\in\mathbf{R}^n \,:\, |f(x)|>\lambda/(2C_2)\}} |f(x)| \,\mathrm{d}x \,\mathrm{d}\lambda \\ &\overset{\text{Fubini}}{=} 2^p C_2^{p-1} C_1 \frac{p}{p-1}\int_{\mathbf{R}^n} |f(x)|^p \,\mathrm{d}x.\end{aligned}$$

□

Example 2.22 (Proof of the Sobolev's inequality via the maximal function)**.** *Suppose that* $u \in C_0^\infty(\mathbf{R}^n)$*. We immediately have*

$$u(x) = -\int_0^\infty \frac{\partial}{\partial r} u(x + r\omega)\,\mathrm{d}r,$$

where $\omega \in \partial B(0,1)$*. Integrating this over the whole unit sphere*

$$\begin{aligned}
\omega_{n-1} u(x) &= \int_{\partial B(0,1)} u(x)\,\mathrm{d}S(\omega) \\
&= -\int_{\partial B(0,1)} \int_0^\infty \frac{\partial}{\partial r} u(x + r\omega)\,\mathrm{d}r\,\mathrm{d}S(\omega) \\
&= -\int_{\partial B(0,1)} \int_0^\infty \nabla u(x + r\omega) \cdot \omega\,\mathrm{d}r\,\mathrm{d}S(\omega) \\
&= -\int_0^\infty \int_{\partial B(0,1)} \nabla u(x + r\omega) \cdot \omega\,\mathrm{d}S(\omega)\,\mathrm{d}r
\end{aligned}$$

and changing variables so that $y = x + r\omega$, $\mathrm{d}S(y) = r^{n-1}\,\mathrm{d}S(\omega)$, $\omega = (y-x)/\left|y-x\right|, r = \left|y - x\right|$ *we get*

$$\omega_{n-1} u(x) = -\int_0^\infty \int_{\partial B(0,r)} \nabla u(y) \cdot \frac{y-x}{\left|y-x\right|^n}\,\mathrm{d}S(y)\,\mathrm{d}r$$

so that

$$u(x) = -\frac{1}{\omega_{n-1}} \int_{\mathbf{R}^n} \frac{\nabla u(y) \cdot (x-y)}{\left|x-y\right|^n}\,\mathrm{d}y.$$

Further

$$\left|u(x)\right| \leq \frac{1}{\omega_{n-1}} \int_{\mathbf{R}^n} \frac{\left|\nabla u(y)\right|}{\left|x-y\right|^{n-1}}\,\mathrm{d}y$$

which is so called Riesz potential. We split this into a bad part and a good part as $\int_{\mathbf{R}^n} = \int_{B(x,r)} + \int_{\mathbf{R}^n \setminus B(x,r)}$*. By estimating the bad part over the sets* $B(x, 2^{-i}r) \setminus B(x, 2^{-i-1}r)$ *as*

$$\begin{aligned}
\int_{B(x,r)} \frac{\left|\nabla u(y)\right|}{\left|x-y\right|^{n-1}}\,\mathrm{d}y &= \sum_{i=0}^\infty \int_{B(x,2^{-i}r)\setminus B(x,2^{-i-1}r)} \frac{\left|\nabla u(y)\right|}{\left|x-y\right|^{n-1}}\,\mathrm{d}y \\
&\leq \sum_{i=0}^\infty \int_{B(x,2^{-i}r)\setminus B(x,2^{-i-1}r)} \frac{\left|\nabla u(y)\right|}{(2^{-i-1}r)^{n-1}}\,\mathrm{d}y \\
&\leq \sum_{i=0}^\infty \frac{2^{-i}r}{2^{-i}r} \int_{B(x,2^{-i}r)} 2^{n-1} \frac{\left|\nabla u(y)\right|}{(2^{-i}r)^{n-1}}\,\mathrm{d}y \\
&\leq C \sum_{i=0}^\infty 2^{n-1} 2^{-i} r ⨍_{B(x,2^{-i}r)} \left|\nabla u(y)\right|\,\mathrm{d}y \\
&\leq C 2^{n-1} r M\left|\nabla u\right|(x) \sum_{i=0}^\infty 2^{-i}
\end{aligned}$$

we get

$$\int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \le CrM\,|\nabla u|\,(x). \tag{2.23}$$

On the other hand, for the good part we use Hölder's inequality with the powers p *and* $p/(p-1)$ *, where* $p < n$*, as*

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \\ &\le \Big(\int_{\mathbf{R}^n\setminus B(x,r)} |\nabla u(y)|^p\,\mathrm{d}y\Big)^{1/p}\Big(\int_{\mathbf{R}^n\setminus B(x,r)} |x-y|^{(1-n)p/(p-1)}\,\mathrm{d}y\Big)^{(p-1)/p}.\end{aligned}$$

Then we calculate

$$\begin{aligned}&\Big(\int_{\mathbf{R}^n\setminus B(x,r)} |x-y|^{(1-n)p/(p-1)}\,\mathrm{d}y\Big)^{(p-1)/p} \\ &= \Big(\int_r^\infty \omega_{n-1}\rho^{n-1}\rho^{(1-n)p/(p-1)}\,\mathrm{d}\rho\Big)^{(p-1)/p} \\ &= \Big(\omega_{n-1}\int_r^\infty \rho^{(1-n)/(p-1)}\,\mathrm{d}\rho\Big)^{(p-1)/p} = \Big(\omega_{n-1}\int_r^\infty \rho^{-1+(p-n)/(p-1)}\,\mathrm{d}\rho\Big)^{(p-1)/p}.\end{aligned}$$

Combining the previous calculations, we get 14.9.2010

$$\int_{\mathbf{R}^n\setminus B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \le C\,||\nabla u||_p\,r^{1-\frac{n}{p}}, \tag{2.24}$$

with $p < n$*. Choosing* $r = \Big(||\nabla u||_p/(M\,|\nabla u|\,(x))\Big)^{p/n}$ *as well as combining the estimates* (2.23) *and* (2.24)*, we get*

$$\begin{aligned}|u(x)| &\le C\int_{\mathbf{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \\ &\le C\,||\nabla u||_p^{p/n}\,M\,|\nabla u|\,(x)^{(n-p)/n}.\end{aligned}$$

Then we take the power[1] $np/(n-p)$ *on both sides and end up with*

$$|u(x)|^{np/(n-p)} \le C\,||\nabla u||_p^{p^2/(n-p)}\,M\,|\nabla u|\,(x)^p.$$

By recalling Hardy-Littlewood II, we obtain

$$\begin{aligned}\int_{\mathbf{R}^n} |u(x)|^{np/(n-p)}\,\mathrm{d}x &\le C\,||\nabla u||_p^{p^2/(n-p)}\int_{\mathbf{R}^n} M\,|\nabla u|\,(x)^p\,\mathrm{d}x \\ &\le C\,||\nabla u||_p^{p^2/(n-p)}\,||\nabla u||_p^p \le C\,||\nabla u||_p^{np/(n-p)}.\end{aligned}$$

This is so called Sobolev's inequality

$$\Big(\int_{\mathbf{R}^n} |u(x)|^{p^*}\,\mathrm{d}x\Big)^{1/p^*} \le C\Big(\int_{\mathbf{R}^n} |\nabla u(x)|^p\,\mathrm{d}x\Big)^{1/p},$$

which holds for every $u \in C_0^\infty(\mathbf{R}^n)$ *and* $p < n$*.*

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[1] This is sometimes denoted by $p^* = np/(n-p)$ and called a Sobolev conjugate. It satisfies $1/p + 1/p^* = 1/n$.