HARMONIC ANALYSIS, 3. EXERCISE

1. Show by a formal calculation that for $f, g, h \in L^1(\mathbf{R}^n)$
   (i) $f * g = g * f$
   (ii) $f * (g * h) = (f * g) * h$.
2. Let
$$f = \chi_{(-1,1)} \qquad g = \chi_{(-\varepsilon,\varepsilon)}.$$
   with $0 < \varepsilon < 1$. Calculate $(f * g)(x)$ and draw a picture.
3. Let $f, g \in C_0(\mathbf{R}^n)$. Show that $f * g \in C_0(\mathbf{R}^n)$.
4. Let $f \in L^p(\mathbf{R}^n)$ and $g \in L^{p'}(\mathbf{R}^n)$, $1 < p < \infty$, $1/p + 1/p' = 1$. Show that $f * g \in L^\infty(\mathbf{R}^n)$ and that $f * g$ is uniformly continuous. Hint: Hölder, the proof of Theorem 3.7 (where we estimated $\int |f(x-y) - f(x)|^p \, \mathrm{d}x$).
5. Show that there is no a function $g \in L^1(\mathbf{R}^n)$ such that $f * g = f$ for every $f \in L^1(\mathbf{R}^n)$. Hint: Problem 4.
6. Let $P_t(x),\ t > 0, x \in \mathbf{R}^n$ be a Poisson kernel. Show that
   (i) $P_t \in L^1(\mathbf{R}^n)$ for $t > 0$.
   (ii) $(x, t) \mapsto P_t(x)$ is harmonic (i.e. $\Delta P_t(x) = 0$) in $\mathbf{R}^{n+1}_+$.

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